A G^2–subdivision algorithm[1]

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Abstract

In this paper we present a method to optimize the smoothness order of subdivision algorithms generating surfaces of arbitrary topology. In particular we construct a subdivision algorithm with some negative weights producing G^2–surfaces. These surfaces are piecewise bicubic and are flat at their extraordinary points. The underlying ideas can also be used to improve the smoothness order of subdivision algorithms for surfaces of higher degree or triangular nets.

Keywords: Catmull/Clark algorithm, subdivision, extraordinary points.

1 Introduction

Subdivision algorithms were first introduced to CAGD by Chaikin [1974] and Lane & Riesenfeld [1980] for the generation of curves and tensor product surfaces. Efforts to understand and analyze these and other subdivision algorithms lead to the general class of stationary subdivision algorithms [Micchelli & Prautzsch '87, Prautzsch '91, Dyn & Levin '92]. These algorithms which operate on regular control nets are a well–understood and rife tool today [Cavaretta et al. '91].

However, most types of surfaces cannot be generated from regular control nets as for example closed surfaces of genus 0. Therefore stationary subdivision algorithms were modified to be applicable to arbitrary control nets with irregular vertices or irregular meshes. [Catmull & Clark '78, Doo & Sabin '78,

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Loop '87, Dyn et al. '90, Qu '90] All these algorithms base on a simple common concept: given a control net a new net is computed by simple affine combinations of the given vertices. Iterating this process leads to a sequence of ever denser nets which converge to a continuous surface. The main problem for these algorithms is to analyze the smoothness of this limiting surface. First attempts to solve these questions were already made by Doo & Sabin [1978]. Loop [1987] and Ball & Storry [1988] refined their ideas. But in 1993 Reif [1993] showed that all former proofs were incomplete. Recently Reif [1995] and Prautzsch [1998] could construct conditions that guarantee the convergence of subdivision algorithms over irregular meshes to G^1– and G^k–surfaces respectively. These conditions are the key to analyze existing subdivision, algorithms and to improve the smoothness order of their limiting surface.

2 The Catmull/Clark Algorithm

We will show how the general conditions can be applied for the example of the Catmull/Clark algorithm. We investigate its smoothness order and modify the algorithm so as to increase its smoothness order.

First we describe the Catmull/Clark algorithm. Given a quadrilateral net $\mathcal{N}_0$ it produces a new quadrilateral net $\mathcal{N}_1$ whose vertices are classified as M–, E–, and V–vertices. Averaging the four vertices of each mesh in $\mathcal{N}_0$ gives the M–vertices. Averaging the midpoint of each edge common to two adjacent meshes in $\mathcal{N}_0$ with the M–vertices of the meshes gives the E–Vertices. Finally, computing a weighted average of the $2n+1$ vertices of all meshes in $\mathcal{N}_0$ with a common interior vertex gives the V–vertices of $\mathcal{N}_1$. The weights are given by the masks in Figure 1.

Connecting each E–vertex with the M– and V–vertices corresponding to the two pairs of meshes and vertices defining both the underlying edge gives the new net $\mathcal{N}_1$.

By the same procedure a new net $\mathcal{N}_2$ is then obtained from $\mathcal{N}_1$ and so on.

Note that the new nets $\mathcal{N}_i$, $i \geq 1$, contain only four–sided meshes. Therefore the only extraordinary vertices of a net $\mathcal{N}_i$, $i \geq 2$, are V–vertices which correspond to an extraordinary vertex of $\mathcal{N}_{i-1}$. Consequently, all nets $\mathcal{N}_i$, $i \geq 1$ have the same number of extraordinary vertices and these vertices are

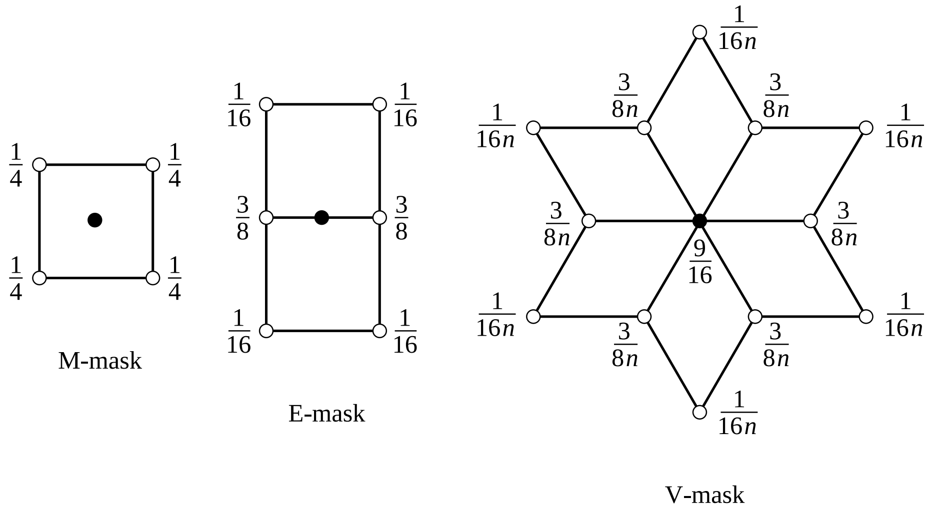

Figure 1: The masks of the Catmull/Clark algorithm.

the more isolated the higher n is.

If $\mathcal{N}_0$ is a regular quadrilateral net, then the Catmull/Clark algorithm is simply the subdivision algorithm for the bicubic box splines over a rectangular grid. Thus each regular part of 4×4 vertices of any control net $\mathcal{N}_i$ defines a bicubic patch of the limiting surface where the 4×4 vertices are the usual spline control points.

The only interesting parts of the limiting surface are those which are determined by (sub)nets consisting of one extraordinary vertex surrounded by 3 rings of quadrilateral meshes as shown in Figure 2. Let $\mathcal{N}_0$ denote such a (sub)net and let $\mathbf{p}_1, \ldots, \mathbf{p}_m$ be its vertices. Refining $\mathcal{N}_0$ gives a net $\mathcal{N}_1$ with 6 rings of quadrilateral meshes around an extraordinary vertex. Discarding the 3 outer rings of control points results in a net $\mathcal{N}_1$ with the same number of vertices and connectedness as the initial one. Let $\mathbf{q}_1, \ldots, \mathbf{q}_m$ be its vertices. Then there is an $m \times m$ matrix A such that

$$[\mathbf{q}_1 \ldots \mathbf{q}_m]^t = A[\mathbf{p}_1 \ldots \mathbf{p}_m]^t.$$

For later reference consider the surface $\mathbf{s}_0$ defined by the net $\mathcal{N}_0$ (and the Catmull/Clark algorithm). It is also given by $\mathcal{N}_1$, whereas the subnet

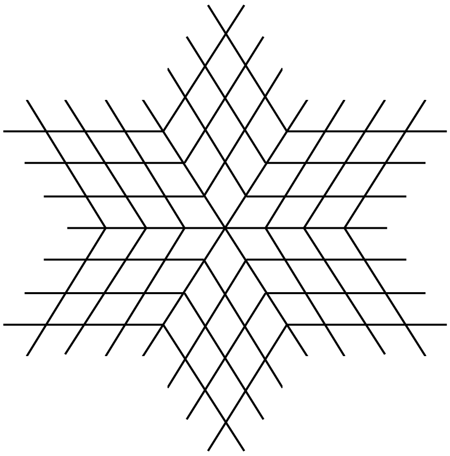

Figure 2: A control net with an extraordinary vertex of valence 6.

$\mathbf{q}_1, \ldots, \mathbf{q}_m$ defines only a part $\mathbf{s}_1$ of $\mathbf{s}_0$. The surface ring obtained from $\mathbf{s}_0$ by taking $\mathbf{s}_1$ away will be called the *first surface ring associated with* $\mathcal{N}_0$.

Next consider the class of all subdivision algorithms obtained from the Catmull/Clark algorithm by changing the matrix A. The spectral properties of A determine whether such an algorithm generates regular C^k–manifolds, i.e. G^k–surface using CAGD terminology. Specializing the results in [Prautzsch '98] we have

Theorem 2.1 *Let* $1, \lambda, \lambda, \mu, \ldots, \zeta$ *be the* m *(possibly complex) eigenvalues of* A *where* $1 > |\lambda| > |\mu| \geq \ldots \geq |\zeta|$ *and assume two eigenvectors* $\mathbf{c}$ *and* $\mathbf{d}$ *associated with* λ*. If the first surface ring of the net given by* $[\mathbf{c}_1 \ldots \mathbf{c}_m]^t = [\mathbf{c}\,\mathbf{d}]^t$ *is regular without self–intersections and*

$$|\lambda|^k > |\mu|, \quad k = 1, 2,$$

then the limiting surface is a G^k*–surface for almost all initial nets* $\mathcal{N}_0$*. Further, the condition* $|\lambda|^k > |\mu|$ *is also necessary.*

Remark 2.2 *If the limiting surface is a* G^2*–surface, then it has a flat point corresponding to the extraordinary point of* $\mathcal{N}_0$*. In general it is not possible to generate* G^k*–surfaces by subdivision algorithms consisting of poly-*

nomial patches of total bidegree $k + 1$ without flat points, see [Reif '96, Prautzsch & Reif '97].

Remark 2.3 *Reif was the first to find out the importance of the first surface ring of* $[\mathbf{c}\,\mathbf{d}]$. *He called it a* characteristic map *of A [Reif '95].*

3 A modification of the Catmull–Clark–Algorithm

The matrix A satisfies for any n the G^1–conditions [Peters & Reif '97], but not the G^2–conditions of the Theorem above [Catmull & Clark '78, Ball & Storry '88]. Moreover, all matrices obtained from a Catmull/Clark matrix A by changing only the weights of the V–masks in Figure 1 do not satisfy the G^2–conditions [Umlauf '96, Ball & Storry '88].

Still in order to obtain a G^2–algorithm we just diagonalize the matrices $A = V \Lambda V^{-1}$, where $\Lambda = \text{diag}(1, \lambda, \lambda, \mu, \ldots, \zeta)$, change the modal matrix Λ into $\Lambda' = \text{diag}(1, \lambda, \lambda, \mu', \ldots, \zeta')$ and compute the modified subdivision matrix $A' = V \Lambda' V^{-1}$.

Lemma 3.1 *The matrices A and A' have the same characteristic maps.*

Proof The eigenvectors associated with λ are the same for A and A'. They define a planar control net $\mathcal{N}_0$. Subdividing $\mathcal{N}_0$ by the Catmull/Clark algorithm and also by the modification results both times in the same sequence of nets $\mathcal{N}_i$. The extraordinary vertex and its three surrounding rings of control points in $\mathcal{N}_i$ are scaled versions of $\mathcal{N}_0$. The other control points of $\mathcal{N}_i$ are computed by the subdivision rules for regular nets. Thus the Catmull/Clark algorithm and its modification applied to $\mathcal{N}_0$ produce the same surface in the limit. □

Conclusion 3.2 *As a consequence of Theorem 2.1 and Lemma 3.1 the modified algorithm produces surfaces which are tangent plane and curvature continuous even at their extraordinary points.*

The symmetry of the Catmull/Clark algorithm masks with respect to the extraordinary point corresponds to a block circulant structure of A [Doo & Sabin '78]. Changing Λ does not affect the block circulance which means that the masks given by A' are again symmetric with respect to the extraordinary point.

Because of the block circulance one can use a block Fourier transformation to bring A into block–diagonal form. Then it suffices to further diagonalize only the blocks belonging to the eigenvalues with modulus in $(\lambda, \lambda^2]$. Changing all eigenvalues of A with modulus in $(\lambda, \lambda^2]$ to 0.3 which is always less than λ^2 one obtains the masks of Figure 3 and Table 4. We listed only the most interesting masks for extraordinary vertices of valence $n = 5, 6, 7, 8, 9$ and 10. For $n = 3$ the M– and E–masks are not changed.

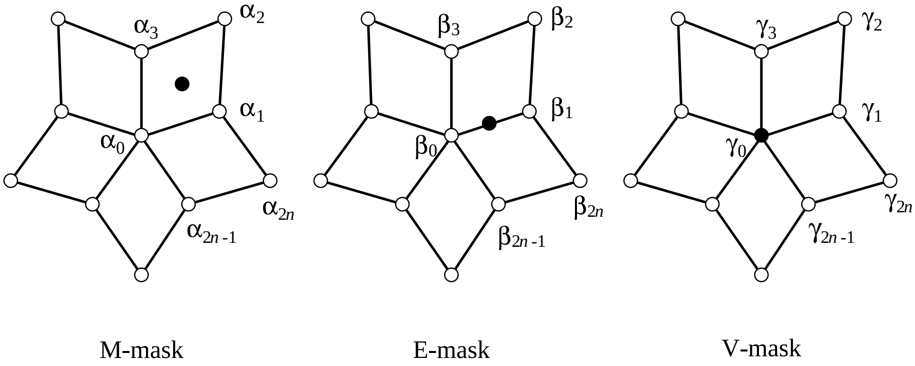

Figure 3: The modified masks for $n = 5$. The weights are listed in Table 4.

Figure 6 shows an example. The surface on the left was produced by the Catmull/Clark algorithm and the surface on the right by the above modification from the control net shown in Figure 5. The surfaces are shown with a visualization of their Gaussian curvature. This curvature is not a discrete approximation obtained from the subdivided control net. We used the piecewise bicubic parametrization of the surface to compute the Gaussian curvature.

Remark 3.3 *Similar to the above modification, one can modify the subdivision algorithms by Loop and Qu so as to obtain G^2– and G^3–algorithms, respectively. For details see [Umlauf '96, Prautzsch & Umlauf '98].*

	$n = 5$		$n = 6$		$n = 7$	
	$\alpha_i =$	$\beta_i =$	$\alpha_i =$	$\beta_i =$	$\alpha_i =$	$\beta_i =$
					0.250000	0.375000
			0.250000	0.375000	0.222472	0.344994
0	0.250000	0.375000	0.232198	0.352200	0.233758	0.055618
	0.245100	0.365736	0.236100	0.058049	0.222472	0.069176
	0.243203	0.061275	0.232198	0.073899	0.003614	0.009944
	0.245100	0.069980	0.006949	0.008900	0.039778	0.027033
	0.005498	0.003206	0.035603	0.011399	0.014633	0.002456
i	0.012826	−0.002857	0.006949	−0.004450	0.009824	−0.018707
	−0.002100	−0.003963	−0.017801	−0.022799	−0.010126	−0.011037
	−0.015854	−0.002857	−0.013899	−0.004450	−0.044151	−0.018707
	−0.002100	0.003206	−0.017801	0.011399	−0.010126	0.002456
$2n$	0.012826	0.069980	0.006949	0.008900	0.009824	0.027033
	0.005498	0.061275	0.035603	0.073899	0.014633	0.009944
			0.006949	0.058049	0.039778	0.069176
					0.003614	0.055618

	$n = 8$		$n = 9$		$n = 10$	
	$\alpha_i =$	$\beta_i =$	$\alpha_i =$	$\beta_i =$	$\alpha_i =$	$\beta_i =$
					0.250000	0.375000
			0.250000	0.375000	0.195996	0.321865
	0.250000	0.375000	0.201818	0.325340	0.223867	0.048999
	0.216666	0.341666	0.224431	0.050454	0.195996	0.057675
	0.233333	0.054166	0.201818	0.064116	−0.001581	0.010257
0	0.216666	0.062500	0.001802	0.011859	0.041030	0.042986
	0.000000	0.008333	0.047437	0.039981	0.021141	0.011587
	0.033333	0.033333	0.019952	0.008169	0.046349	0.012630
	0.016666	0.008333	0.032677	0.002029	0.004139	−0.003096
	0.033333	0.000000	−0.001114	−0.005024	−0.012384	−0.01641
i	0.000000	−0.008333	−0.020099	−0.018797	−0.008075	−0.005247
	−0.033333	−0.033333	−0.007855	−0.005917	−0.020991	−0.015611
	−0.016666	−0.008333	−0.023668	−0.018797	−0.005116	−0.005247
	−0.033333	0.000000	−0.007855	−0.005024	−0.020991	−0.016419
$2n$	0.000000	0.008333	−0.020099	0.002029	−0.008075	−0.003096
	0.033333	0.033333	−0.001114	0.008169	−0.012384	0.012630
	0.016666	0.008333	0.032677	0.039981	0.004139	0.011587
	0.033333	0.062500	0.019952	0.011859	0.046349	0.042986
	0.000000	0.054166	0.047437	0.064116	0.021141	0.010257
			0.001802	0.050454	0.041030	0.057675
					−0.001581	0.048999

$$\gamma_0 = \tfrac{5}{12}, \quad \gamma_{2i-1} = \tfrac{1}{6}, \quad \gamma_{2i} = \tfrac{1}{36}, \quad i = 1, \ldots, n, \text{ for } n = 3.$$

$$\gamma_0 = \tfrac{1}{4}, \quad \gamma_{2i-1} = \tfrac{1}{2n}, \quad \gamma_{2i} = \tfrac{1}{4n}, \quad i = 1, \ldots, n, \text{ for } n \geq 5.$$

Table 4: The weights of the modified masks for $n = 5, 6, 7, 8, 9$ and 10.

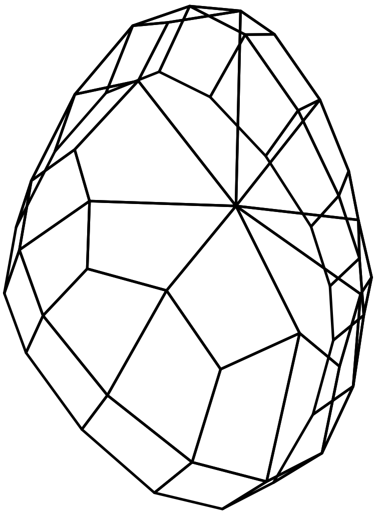

Figure 5: The control net used for Figure 6.

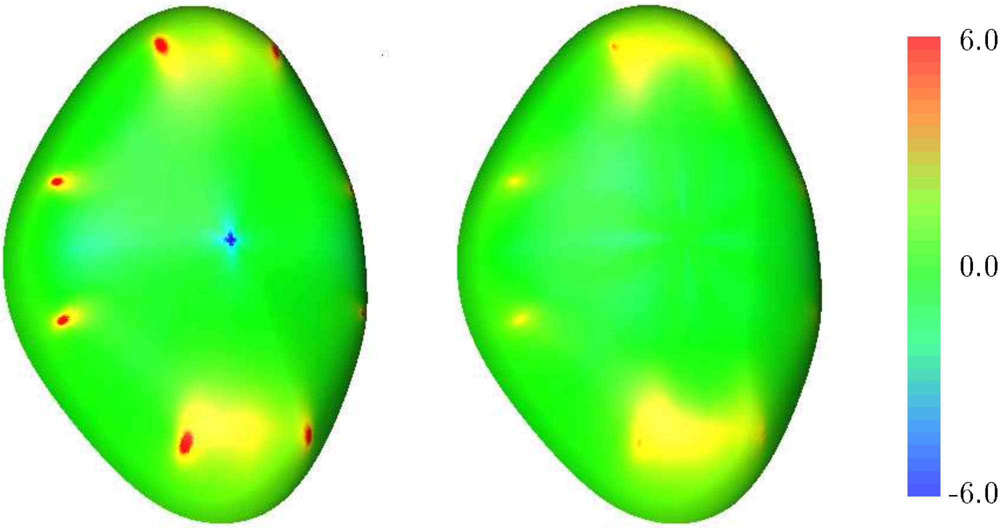

Figure 6: The surfaces produced by the Catmull/Clark algorithm (left) and its modification (right).

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