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Analyzing a generalized Loop subdivision scheme

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Abstract

In this paper a class of subdivision schemes generalizing the algorithm of Loop is presented. The stencils have the same support as those from the algorithm of Loop, but allow a variety of weights. By varying the weights a class of C^1 regular subdivision schemes is obtained. This class includes the algorithm of Loop and the midpoint schemes of order one and two for triangular nets. The proof of C^1 regularity of the limit surface for arbitrary triangular nets is provided for any choice of feasible weights.

The purpose of this generalization of the subdivision algorithm of Loop is to demonstrate the capabilities of the applied analysis technique. Since this class includes schemes that do not generalize box spline subdivision, the analysis of the characteristic map is done with a technique that does not need an explicit piecewise polynomial representation. This technique is computationally simple and can be used to analyze classes of subdivision schemes. It extends previously presented techniques based on geometric criteria.

AMS Subject Classifications: 65D17, 65D18.

Keywords: Subdivision, loop, characteristic map.

1. Introduction

In the past subdivision schemes have often been developed as generalizations of subdivision schemes for tensor-product B-spline or box spline representations [2], [11]. Their popularity is partly due to the fact that the B-spline or box spline representation of the characteristic map can be used to prove regularity and injectivity [14], [19] and is therefore the key for the proof of C^1 regularity of the limit surface. For interpolatory schemes [6], [10] or schemes that focus on geometric properties, no explicit representation of the basis functions is known. Different techniques have been developed to handle such algorithms. These techniques are either based on massive numerical computations and, hence, complicated to compute [21], [24] or on visual inspection and, hence, may lead to mistakes.

In [20], a technique is introduced that allows analyzing the characteristic map even if an explicit representation of the basis is not available. It requires that the first divided difference schemes use strict convex combinations and that the first difference schemes for a given direction combines only differences from the same direction. Then geometric criteria suffice to prove regularity and injectivity of the characteristic map for any valence.

The goal of this paper is to explore and extend the capability of this technique and to define a class of subdivision algorithms that fulfills the conditions necessary for the applicability of the technique. Since triangular representations are very popular and since the stencils should be as small as possible, the stencils of the subdivision algorithm of Loop [11] are generalized. Of course, the subdivision algorithm of Loop has been modified in many aspects, see, e.g. [7], [12], [13] [16], [23]. These modifications are done to improve the shape of the limit surfaces. The modifications presented in this paper are done to find out to what extend arbitrary weights can be used to generate C^1 regular limit surfaces. This generalization of the algorithm of Loop is used to demonstrate that the presented analysis technique can be applied to a whole class of subdivision algorithms containing a variety of different schemes.

In [7], modifications of the subdivision algorithm of Loop are presented without a rigorous analysis that are similar to the modifications suggested in this paper. The modifications done in this paper are more general and the complete and rigorous analysis of this class of subdivision schemes is presented here.

In Sect. 2, the subdivision rules for quartic box splines over the three directional grid are generalized and a feasible set of parameters is computed that leads to C^1 surfaces that are not box spline surfaces. These rules are extended to triangular nets with vertices of arbitrary valence in Sect. 3. Finally, in Sect. 4 the characteristic map of the resulting subdivision algorithm is analyzed using a new and extended technique that is based on geometric criteria.

2. Generalized box spline subdivision on regular triangular nets

The analysis of subdivision algorithms on regular triangular nets is well understood and published with many examples, see, e.g. [3], [5], [22]. Although, many aspects of the analysis in the regular setting presented in this section are covered by the above references, this sections is included to make the paper complete and self-contained.

A regular triangular net $C = [\mathbf{c_i}]_{\mathbf{i}\in\mathbb{Z}^2}$ is given by a set of points $\mathbf{c_i}$ which are neighboring with respect to the directions $\mathbf{e}_1 = [1, 0]$, $\mathbf{e}_2 = [0, 1]$ and $\mathbf{e}_3 = \mathbf{e}_1 + \mathbf{e}_2$ as shown in Fig. 1.

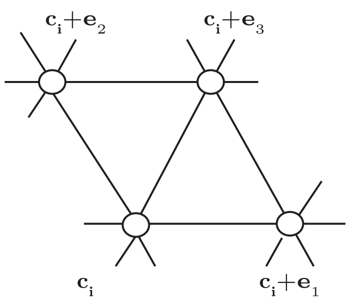

Fig. 1. The three directional grid

Application of the subdivision algorithm for quartic box splines to $C^0 := C$ generates a sequence of triangular nets $C^l = [\mathbf{c}_\mathbf{i}^l]_{\mathbf{i}\in\mathbb{Z}^2}, l \geq 0$. This sequence converges to the quartic box splines surface defined by the control net C, see [1].

To generalize the subdivision algorithm for quartic box splines we take the corresponding stencils and replace the weights with variables.

Since symmetric schemes are geometrically meaningful and make analysis easier, we choose the weights as shown in Fig. 2. For affine invariance the weights of each stencil must sum to 1 and for the convex hull property all weights must be positive. This yields

$$a, b, c, d > 0, \tag{1}$$

$$1 - 6a = b, \tag{2}$$

$$d + c = 1/2. \tag{3}$$

Denote by $\nabla_k \mathbf{c}_\mathbf{i}^l = \mathbf{c}_\mathbf{i}^l - \mathbf{c}_{\mathbf{i}-\mathbf{e}_k}^l, k = 1, 2, 3$. Sufficient conditions on a, b, c, d can be formulated to guarantee the existence of difference schemes S_k, that map differences $\nabla_k \mathbf{c}_\mathbf{i}^l$ to differences $\nabla_k \mathbf{c}_\mathbf{j}^{l+1}, \mathbf{i}, \mathbf{j} \in \mathbb{Z}^2$. To compute a difference $\nabla_k \mathbf{c}_{2\mathbf{j}}^{l+1}$ (black arrow in Fig. 3 (left)) only from differences $\nabla_k \mathbf{c}_\mathbf{i}^l$ (dashed arrows in Fig. 3 (left)), S_k must satisfy $a = d - a$ and $-(c - b - a) = c - a$ or equivalently

$$d = 2a, \tag{4}$$

$$c = a + 1/2b. \tag{5}$$

The construction for $\nabla_k \mathbf{c}_{2\mathbf{j}-\mathbf{e}_1}^{l+1}$ also yields (4) and (5). To compute a difference $\nabla_k \mathbf{c}_{2\mathbf{j}+\mathbf{e}_2}^{l+1}$ (black arrow in Fig. 3 (right)) only from differences $\nabla_k \mathbf{c}_\mathbf{i}^l$ (dashed arrows in Fig. 3 (right)), S_k must satisfy no additional condition, because the stencil is symmetric. The same is true for $\nabla_k \mathbf{c}_{2\mathbf{j}-\mathbf{e}_3}^{l+1}$. Thus, S_1 is represented by

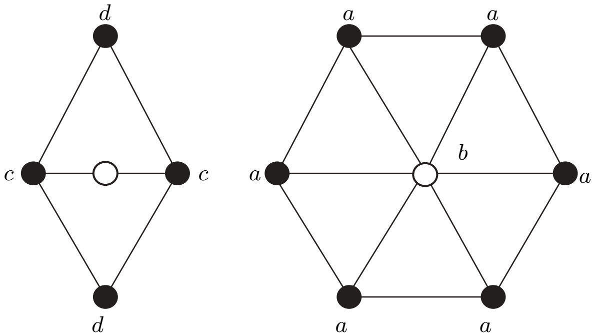

Fig. 2. Generalized box spline subdivision masks

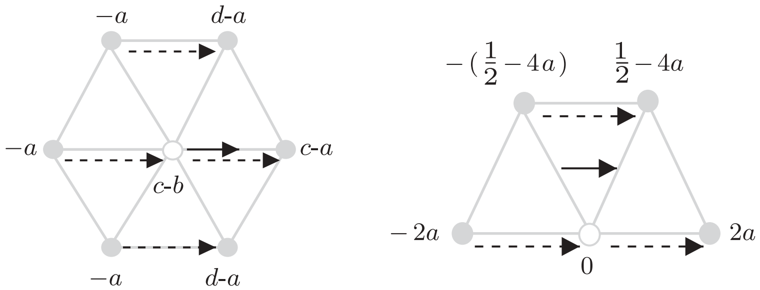

Fig. 3. Construction of a difference $\nabla_k \mathbf{c}_{2\mathbf{j}}^{l+1}$ (left) and $\nabla_k \mathbf{c}_{2\mathbf{j}+\mathbf{e}_2}^{l+1}$ (right)

$$\begin{aligned}
\nabla_1 \mathbf{c}_{2\mathbf{i}+\mathbf{v}}^{l+1} &= (1/2-3a)\cdot\nabla_1\mathbf{c}_{\mathbf{i}+\mathbf{v}}^{l} + a\cdot(\nabla_1\mathbf{c}_{\mathbf{i}-\mathbf{v}-\mathbf{e}_1}^{l} + \nabla_1\mathbf{c}_{\mathbf{i}+\mathbf{e}_2}^{l} + \nabla_1\mathbf{c}_{\mathbf{i}-\mathbf{e}_3}^{l})\\
&\qquad \text{for } \mathbf{v}\in\{\mathbf{0},-\mathbf{e}_1\} \text{ and}\\
\nabla_1 \mathbf{c}_{2\mathbf{i}+\mathbf{v}}^{l+1} &= (1/2-4a)\cdot\nabla_1\mathbf{c}_{\mathbf{i}+\mathbf{v}}^{l} + 2a\cdot(\nabla_1\mathbf{c}_{\mathbf{i}}^{l} + \nabla_1\mathbf{c}_{\mathbf{i}-\mathbf{e}_1}^{l})\\
&\qquad \text{for } \mathbf{v}\in\{\mathbf{e}_2,-\mathbf{e}_3\}
\end{aligned}$$

and analogous rules for S_2 and S_3. The weights of the difference schemes S_k, $k = 1, 2, 3$, sum to $1/2$. They are strictly positive if $a > 0$, $1/2-3a > 0$ and $1/2-4a > 0$, i.e.,

$$a \in (0, 1/8). \tag{6}$$

Note that (2), (4) and (5) imply (3). Therefore, (1)–(5) fix the weights b, c, d and lead to stencils similar to the ones proposed in [7]. Thus, for a subdivision scheme with stencils as in Fig. 4 condition (6) guarantees convergence, affine invariance and the convex hull property.

For C^1 regularity second differences must be checked, since convergence of the divided second differences is sufficient for C^1 regularity [3], [5]. The construction of $\nabla_k^2 \mathbf{c}_{\mathbf{j}}^{l+1}$ from $\nabla_k^2 \mathbf{c}_{\mathbf{i}}^{l}$ is done analogously to the construction of $\nabla_k \mathbf{c}_{\mathbf{j}}^{l+1}$ above. It yields for $k = 1$ the rules

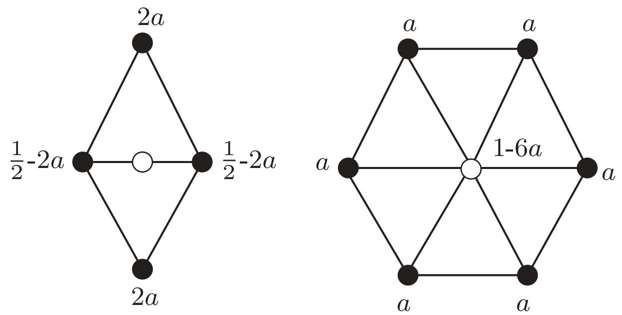

Fig. 4. The stencils for generalized box spline subdivision

$$\begin{aligned}
\nabla_1^2 \mathbf{c}_{2\mathbf{i}}^{l+1} &= a \cdot \left(\nabla_1^2 \mathbf{c}_{\mathbf{i}-\mathbf{e}_3}^{l} + \nabla_1^2 \mathbf{c}_{\mathbf{i}+\mathbf{e}_2}^{l} + \nabla_1^2 \mathbf{c}_{\mathbf{i}}^{l} + \nabla_1^2 \mathbf{c}_{\mathbf{i}-\mathbf{e}_1}^{l} \right), \\
\nabla_1^2 \mathbf{c}_{2\mathbf{i}-\mathbf{e}_1}^{l+1} &= (1/2 - 4a) \cdot \nabla_1^2 \mathbf{c}_{\mathbf{i}}^{l}, \\
\nabla_1^2 \mathbf{c}_{2\mathbf{i}+\mathbf{e}_2}^{l+1} &= 2a \left(\cdot \nabla_1^2 \mathbf{c}_{\mathbf{i}-\mathbf{e}_1}^{l} + \nabla_1^2 \mathbf{c}_{\mathbf{i}+\mathbf{e}_2}^{l} \right) + (1/2 - 8a) \cdot \left(\nabla_1 \mathbf{c}_{\mathbf{i}}^{l} - \nabla_1 \mathbf{c}_{\mathbf{i}+\mathbf{e}_2}^{l} \right), \\
\nabla_1^2 \mathbf{c}_{2\mathbf{i}+\mathbf{e}_3}^{l+1} &= 2a \left(\cdot \nabla_1^2 \mathbf{c}_{\mathbf{i}}^{l} + \nabla_1^2 \mathbf{c}_{\mathbf{i}+\mathbf{e}_2}^{l} \right) + (1/2 - 8a) \cdot \left(\nabla_1 \mathbf{c}_{\mathbf{i}+\mathbf{e}_3}^{l} - \nabla_1 \mathbf{c}_{\mathbf{i}}^{l} \right)
\end{aligned}$$

and analogous rules for $k = 2, 3$. The conditions for convergence of the second divided differences are $2 \cdot |4a| < 1$, $2 \cdot |1/2 - 4a| < 1$ and $2 \cdot (|4a| + |1/2 - 8a|) < 1$. These conditions are true for

$$a \in (0, 1/12). \tag{7}$$

Therefore, using the stencils shown in Fig. 4 condition (7) guarantees C^1-regularity of the subdivision surface for regular triangular nets.

3. Analyzing the subdivision matrix

In order to expand the class of subdivision algorithms to triangular nets with vertices of valence $n \neq 6$ the stencil can be generalized as in Fig. 5. Note that in this setting $\beta = a$ has to be chosen for $n = 6$.

For the analysis of C^1 regularity of the subdivision surface at extraordinary points it is necessary to analyze the subdivision matrix [4]. For the algorithm of Loop three rings of points around an irregular vertex define a complete surface ring. The generalized stencils have the same support as the stencils of the algorithm of Loop. Therefore, three rings of points around an irregular vertex also suffice for the generalized stencils, since the support of the nodal functions does not change [8].

With the usual labeling of the control points segment-wise from inner to outer rings of control points the subdivision matrix A is block-circulant. Thus, A is similar to a block-diagonal matrix $\widehat{A} = \mathrm{diag}(\widehat{A}_0, \ldots, \widehat{A}_{n-1})$, which results from discrete Fourier transformation. The blocks $\widehat{A}_i$, $i = 0, \ldots, n-1$, are given by

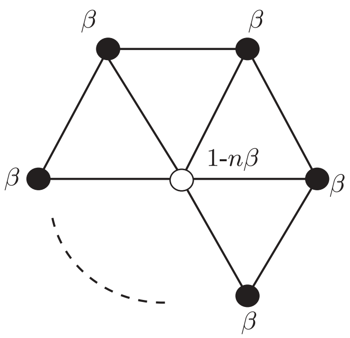

Fig. 5. Stencil for the generalized algorithm of Loop at a vertex of valence n

$$
\widehat{A}_i = \left[\begin{array}{c|c|cc|ccc}
(1-n\beta)\delta_{i,0} & n\beta\delta_{i,0} & & & & & \\
(\frac{1}{2}-2a)\delta_{i,0} & (\frac{1}{2}-2a)+4ac_i & & & & & \\
a\delta_{i,0} & (1-6a)+2ac_i & a & (1+\omega^i)a & & & \\
2a\delta_{i,0} & (1+\omega^i)(\frac{1}{2}-2a) & 0 & 2a & & & \\
0 & \frac{1}{2}-2a & \frac{1}{2}-2a & (1+\omega^i)2a & 0 & 0 & 0 \\
0 & \frac{1}{2}-2a+2a\omega^{-i} & 2a & \frac{1}{2}-2a & 0 & 0 & 0 \\
0 & 2a+(\frac{1}{2}-2a)\omega^{-i} & 2a\omega^{-i} & \frac{1}{2}-2a & 0 & 0 & 0
\end{array}\right],
$$

where $\omega := \exp(2\pi\sqrt{-1}/n)$, $c_i := \cos(2\pi i/n)$ and the Kronecker symbol $\delta_{i,0}$. From $\widehat{A}_i$ the eigenvalues of A can be computed as

- the eigenvalue 1 from $\widehat{A}_0$,
- the eigenvalue $\mu_\beta := 1/2 + 2a - n\beta$ from $\widehat{A}_0$,
- the eigenvalues $\mu_i := 1/2 - 2a + 4ac_i$ from $\widehat{A}_i$, $i = 1, \ldots, n-1$,
- the n-fold eigenvalue $2a$ from $\widehat{A}_i$, $i = 0, \ldots, n-1$,
- the n-fold eigenvalue a from $\widehat{A}_i$, $i = 0, \ldots, n-1$,
- the $(4n-1)$-fold eigenvalue 0 from $\widehat{A}_i$, $i = 0, \ldots, n-1$.

Since the largest eigenvalue of A is the single eigenvalue 1 with corresponding eigenvector $[1 \ldots 1]^t$ the subdivision scheme is guaranteed to converge to a unique limit point. Furthermore, it can be observed that $\mu_i = \mu_{n-i}$ for $i = 1, \ldots, \lfloor n/2 \rfloor$ and $\mu_1 > \mu_i$ for $i = 2, \ldots, \lfloor n/2 \rfloor$. The sub-dominant eigenvalue has algebraic and geometric multiplicity two and equals μ_1 if

$$
\begin{aligned}
\mu_1 &= 1/2 - 2a + 4ac_1 > 1/8 && \text{and} \\
\mu_1 &= 1/2 - 2a + 4ac_1 > 1/2 + 2a - n\beta = \mu_\beta.
\end{aligned}
$$

These conditions hold for $n \geq 3$ if condition (7) and

$$
\beta > 4a(1 - c_1)/n \tag{8}
$$

are fulfilled.

Although not necessary for C^1 regularity, a reasonable choice is $\beta = 4a(1 - c_2)/n$ since this results in a triple subsub-dominant eigenvalue from $\widehat{A}_i$, $i = 0, 2, n-2$. This is a necessary condition for facilitating limit surfaces with arbitrary shape [9], [15]. This also implies $\beta = a$ for $n = 6$, the convex hull property and condition (8) for $n \geq 3$.

4. Analyzing the characteristic map

On conditions (7) and (8), regularity and injectivity of the characteristic map lead to a C^1 regular limit surface for almost all initial control nets [17]. The characteristic map $\mathbf{x}$ is the planar spline ring defined by the net $[\mathbf{v}_1, \mathbf{v}_{n-1}]$, where $\mathbf{v}_1, \mathbf{v}_{n-1}$ are the eigenvectors of A for μ_1, μ_{n-1}, respectively. For a symmetric scheme the characteristic map consists of n rotationally symmetric segments $\mathbf{x}_i$, $i = 0, \ldots, n-1$, and can be normalized such that $\mathbf{x}_0$ is symmetric with respect to the 1-axis [14]. The labeling of the control points $\mathbf{c}_i$ of the control net X of the segment $\mathbf{x}_0$ of the characteristic map is given by

$$X = \begin{array}{ccccc} & & \mathbf{c}_8 & & \\ & \mathbf{c}_4 & & \mathbf{c}_{13} & \\ \mathbf{c}_1 & & \mathbf{c}_9 & & \\ & \mathbf{c}_5 & & \mathbf{c}_{14} & \\ \mathbf{c}_2 & & \mathbf{c}_{10} & & \\ & \mathbf{c}_6 & & \mathbf{c}_{15} & \\ \mathbf{c}_3 & & \mathbf{c}_{11} & & \\ & \mathbf{c}_7 & & \mathbf{c}_{16} & \\ & & \mathbf{c}_{12} & & \end{array}.$$

Note that X is mirror symmetric with respect to the 1-axis. The points $\mathbf{c}_i, i = 1, \ldots, 16$, can be computed using a computer algebra package such as Maple and are given by

$$\mathbf{c}_1 = \begin{bmatrix} 2\alpha\beta\gamma\delta(-1+2c_1) \\ \beta\delta s(1+(2-8a)c_1+8ac_2) \end{bmatrix}, \qquad \mathbf{c}_4 = \begin{bmatrix} \alpha\beta\delta(4-16a)c_1 \\ \alpha\beta\delta s(4-16a) \end{bmatrix},$$

$$\mathbf{c}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \mathbf{c}_5 = \begin{bmatrix} 2\alpha\beta\gamma\delta \\ \beta\gamma\delta s \end{bmatrix},$$

$$\mathbf{c}_8 = \begin{bmatrix} 2\alpha(1-6a+96a^2-576a^3+(2-4a-128a^2+896a^3)c_1+(8a+32a^2-512a^3)c_2+128a^3c_3) \\ s(3-26a+112a^2-128a^3+(2+12a-128a^2+256a^3)c_1+(8a+32a^2-256a^3)c_2+128a^3c_3) \end{bmatrix},$$

$$\mathbf{c}_9 = \begin{bmatrix} 2\alpha\delta(2-24a+96a^2+(24a-144a^2)c_1+16a^2c_2) \\ \delta s(2-24a+96a^2+(24a-144a^2)c_1+16a^2c_2) \end{bmatrix}, \qquad \mathbf{c}_{10} = \begin{bmatrix} \alpha\beta\delta(4-16a) \\ 0 \end{bmatrix},$$

$$\mathbf{c}_{13} = \begin{bmatrix} 2\alpha(3-42a+280a^2-640a^3+(48a-432a^2+960a^3)c_1+(80a^2-320a^3)c_2) \\ s(3-42a+280a^2-640a^3+(48a-432a^2+960a^3)c_1+(80a^2-320a^3)c_2) \end{bmatrix},$$

$$\mathbf{c}_{14} = \begin{bmatrix} 2\alpha(3-34a+112a^2+192a^3+(32a-144a^2-320a^3)c_1+(32a^2+64a^3)c_2) \\ s(1-14a+96a^2-256a^3+(16a-144a^2+320a^3)c_1+(32a^2-192a^3)c_2) \end{bmatrix},$$

with $s = \sin(2\pi/n), \alpha = 1/2 + 1/2c_1 > 0, \beta = (1-6a+8ac_1), \gamma = (1-8a+8ac_1)$ and $\delta = (1-4a+8ac_1)$.

The divided ∇_k-differences of X converge towards directional derivatives of the characteristic map with respect to $\mathbf{e}_k$ for $k = 1, 2, 3$. The ∇_k-differences of X lie within pointed cones B_k, which are characterized by a set of vectors

$$I_c = \left\{ \sum_i \alpha_i \mathbf{v}_i : \alpha_i \geq 0, \mathbf{v}_i \in I \text{ for all } i \right\}$$

such that $B_k \subseteq \{\mathbf{l}_k, \mathbf{u}_k\}_c$ where $\mathbf{l}_k$ and $\mathbf{u}_k$ are ordered in a rotationally positive sense and $\mathbf{l}_k^m$ and $\mathbf{u}_k^m$ denote the m-th coordinates of $\mathbf{l}_k$ and $\mathbf{u}_k$ for $m = 1, 2$, respectively. Note that B_1 and B_3 are symmetric, i.e., $\mathbf{l}_1^1 = \mathbf{u}_3^1, \mathbf{l}_1^2 = -\mathbf{u}_3^2$ and $\mathbf{u}_1^1 = \mathbf{l}_3^1, \mathbf{u}_1^2 = -\mathbf{l}_3^2$, and that B_2 is symmetric, i.e., $\mathbf{u}_2^2 = \mathbf{l}_2^1$ and $\mathbf{u}_2^2 = -\mathbf{l}_2^2$. The cones provide a criterion for the characteristic map to be regular and injective [20]:

Theorem 1: *For a symmetric subdivision scheme the characteristic map is regular and injective, if*

(1) *the divided* ∇_1- *and* ∇_3-*difference schemes use only strict convex combinations,*
(2) *none of the* ∇_3-*differences of* X *vanish and*
(3) *the cone* B_3 *satisfies* $\mathbf{l}_3^m, \mathbf{u}_3^m \geq 0$ *for* $m = 1, 2$.

Condition *1* is implied by (7). Because no two points of X coincide, condition *2* is also satisfied. Condition *3* is verified by checking the coordinates of all ∇_3-differences of X.

Denote by $\mathbf{d}^m_{\mu,\nu}$ the m-th coordinate of the difference $\mathbf{c}_\mu - \mathbf{c}_\nu$ for $m = 1, 2$, which can be written as trigonometric polynomial

$$\mathbf{d}^m_{\mu,\nu} = c^m_{\mu,\nu} \cdot \sum_{i=0}^{3} p^{m,i}_{\mu,\nu}(a) \cdot c_1^i, \qquad m = 1, 2. \tag{9}$$

Here, $c^m_{\mu,\nu}$ are positive constants for all μ, ν, m and $n \geq 4$ (see Appendix A). The coefficients $p^{m,i}_{\mu,\nu}(a)$ are polynomials in a with real-valued coefficients depending on μ and ν. Since $p^{m,0}_{\mu,\nu}(a) > 0$ for all μ, ν (see Appendix A) and $c_1 \geq 0$ for $n \geq 4$, $\mathbf{d}^m_{\mu,\nu} > 0$ if

$$p^{m,0}_{\mu,\nu}(a) + \sum_{p^{m,i}_{\mu,\nu}(a)<0} p^{m,i}_{\mu,\nu}(a) > 0. \tag{10}$$

This argument can be directly applied to most of the differences. This becomes especially simple if there are no negative coefficients $p^{m,i}_{\mu,\nu}(a)$. Where the estimation by (10) of a difference $\mathbf{d}^m_{\mu,\nu}$ fails a more precise estimation must be used, e.g., the Bézier representation of $\mathbf{d}^m_{\mu,\nu}$ as a bi-variate polynomial in a and c_1 (see Appendix A). This yields Condition *3* for all relevant $\mathbf{d}^m_{\mu,\nu}$ if $a \leq 1/12$ except for $\mathbf{d}^2_{4,1}$ which implies the condition

$$a \leq 1/16c_1^2. \tag{11}$$

Thus, (7) and (11), i.e., $a \in (0, 1/16c_1^2] \subset (0, 1/12)$, are sufficient to guarantee the conditions of Theorem 1.

For $a \in (1/16c_1^2, 1/12)$ Theorem 1 cannot be applied, because $\mathbf{l}_3^2 < 0$. The proof of Theorem 1 in [20] uses only the fact that there exist two cones of directional differences that satisfy the conditions of the theorem. So, instead of using the cones B_3 and its symmetric counterpart B_1 the theorem remains true for the cones

$$\widetilde{B}_3 := \alpha B_3 + (1-\alpha)B_2 \subseteq \{\alpha\mathbf{l}_3 + (1-\alpha)\mathbf{l}_2, \alpha\mathbf{u}_3 + (1-\alpha)\mathbf{u}_2\}_c =: \{\widetilde{\mathbf{l}}_3, \widetilde{\mathbf{u}}_3\}_c$$

and its symmetric counterpart $\widetilde{B}_1$ for $\alpha \geq 0$. In order for $\widetilde{B}_3$ to satisfy Condition *3* of Theorem 1 there must exist a $\alpha \geq 0$ such that $\widetilde{\mathbf{u}}_3^m, \widetilde{\mathbf{l}}_3^m \geq 0$ for $m = 1, 2$. This is induced by $\mathbf{l}_2^1, \mathbf{l}_2^2, \mathbf{l}_3^1, \mathbf{u}_3^2 \geq 0$ and $\mathbf{l}_3^2 < 0$ and

$$\mathbf{l}_2^2\mathbf{u}_3^1 + \mathbf{l}_2^1\mathbf{l}_3^2 < 0. \tag{12}$$

Theorem 1′: *For a symmetric subdivision scheme the characteristic map is regular and injective, if*

(1′) *the divided ∇_k-difference schemes use only strict convex combinations for $k = 1, 2, 3$,*
(2) *none of the ∇_3-differences of X vanish and*
(3′) *the cones B_2 and B_3 satisfy $\mathbf{l}_2^1, \mathbf{l}_2^2, \mathbf{l}_3^1, \mathbf{u}_3^2 \geq 0$ and $\mathbf{l}_3^2 < 0$ and (12).*

For $a \in (1/16c_1^2, 1/12)$ it remains to check condition *3′*. where only $\mathbf{l}_2^m \geq 0, m = 1, 2$, and (12) are left to prove. Because B_2 is symmetric and contains vectors not parallel to $\mathbf{e}_2$ condition $\mathbf{l}_2^1 > 0$ is guaranteed. The remaining conditions $\mathbf{l}_2^2 \geq 0$ and (12) can be checked in the same way Condition *3* was checked (see Appendix A and B). In summary, this proves the following theorem.

Theorem 2: *The generalized algorithm of Loop for triangular nets with $a \in (0, 1/12)$ generates for almost all initial control nets regular surfaces with continuous normal everywhere.*

Besides C^1 regularity some geometric properties can be observed.

Remark 3: *Since the corresponding left eigenvectors for the eigenvalues μ_1 and μ_{n-1} do not depend on a, every scheme produces the same directions $\mathbf{d}_1$ and $\mathbf{d}_2$ spanning the tangent plane.*

Remark 4: *The limit point depends on a, so the tangent planes of the generalized Loop subdivision schemes at irregular vertices are parallel for a given initial control nets. The stencil for calculating the limit value of an extraordinary point is the same as the stencil for computing the value on the next level shown in Fig. 5 where β replaced by $(n + (1/2 - 2a)/\beta)^{-1}$.*

Remark 5: *The sub-dominant eigenvalues μ_1, μ_{n-1} and μ_β depend on a and tend towards $1/2$ if a goes to 0. Therefore, polar artifacts are reduced by decreasing a [18].*

Remark 6: *For the choice of a and β as in Sect. 3 yielding a limit surface with bounded Gauss curvature of arbitrary sign Theorem 2 guarantees C^1-regularity only for $n = 4, \ldots, 7$.*

5. Conclusion

In this paper, a generalization of the algorithm of Loop is presented. It allows to chose a free parameter arbitrarily from the interval $a \in (0, 1/12)$ which might be used for optimization. This algorithm generates C^1-regular limit surfaces for triangular nets with irregular vertices of valence $n \geq 4$ for all $a \in (0, 1/12)$. In order to prove this fact the known geometric smoothness criteria have been extended to a more general set of directional derivatives that are guaranteed to be linear independent.

Appendix

A The differences $\mathrm{d}^{\mathrm{m}}_{\mu,\nu}$

Here, the estimation technique described in Sect. 4 for first and second coordinates of the differences is demonstrated. Using (10), the 22 differences $\mathbf{d}^1_{14,10}$, $\mathbf{d}^1_{13,9}$, $\mathbf{d}^1_{10,6}$, $\mathbf{d}^1_{9,5}$, $\mathbf{d}^1_{6,3}$, $\mathbf{d}^1_{5,2}$, $\mathbf{d}^2_{16,12}$, $\mathbf{d}^2_{15,11}$, $\mathbf{d}^2_{14,10}$, $\mathbf{d}^2_{13,9}$, $\mathbf{d}^2_{11,7}$, $\mathbf{d}^2_{10,6}$, $\mathbf{d}^2_{9,5}$, $\mathbf{d}^2_{6,3}$, $\mathbf{d}^2_{5,2}$, $\mathbf{d}^2_{2,3}$, $\mathbf{d}^2_{5,6}$, $\mathbf{d}^2_{6,7}$, $\mathbf{d}^2_{10,11}$, $\mathbf{d}^2_{11,12}$, $\mathbf{d}^2_{14,15}$, $\mathbf{d}^2_{15,16}$ can be shown to be positive if $a \in (0, 1/12)$ and $n \geq 4$. For example, the difference

$$\mathbf{d}^2_{13,9} = s((1-10a+24a^2)+(8a-256a^3)c_1+(-64a^2+640a^3)c_1^2+(-256a^3)c_1^3)$$

has coefficients $p^{1,i}_{11,16}(a)$ which fulfill (10) for $a \in (0, 1/12)$. The differences $\mathbf{d}^1_{16,12}$, $\mathbf{d}^1_{15,11}$, $\mathbf{d}^1_{11,7}$, $\mathbf{d}^1_{4,1}$, $\mathbf{d}^1_{8,4}$, $\mathbf{d}^2_{8,4}$ need a more careful estimation, e.g., based on the Bézier representation in a and c_1:

$$\mathbf{d}^1_{4,1} = (2-16a)B^2_0(c_1) + 2B^2_1(c_1) + (2-16a)B^2_2(c_1).$$

These three differences are positive if $a \in (0, 1/12)$ and $n \geq 4$. There is now one difference left that needs special attention

$$\mathbf{d}^2_{4,1} = (1-16ac_1^2)\beta\delta s.$$

This term is only positive if $a \leq 1/16c_1^2$ and further restricts the interval of valid choices for a to $a \in (0, 1/16]$ for arbitrary valences n. For valences $n \leq 12$ the weight $a \in (0, 1/12)$ is permitted.

B Checking condition (12)

Appendix A shows that $\mathbf{l}^2_3 = \mathbf{d}^2_{4,1}$. Therefore, condition (12) has to be checked for all ∇_2-differences with positive first coordinate and ∇_3-differences with positive second coordinate. These 50 checks can be done using the estimation techniques described above, e.g. $\mathbf{d}^2_{11,12}\mathbf{d}^1_{6,3}+\mathbf{d}^1_{11,12}\mathbf{d}^2_{4,1}$ has over $[1/2, 1]\times[1/16, 1/12]$ the Bézier coefficients (up to a positive scaling)

$$\begin{bmatrix} 178848 & 169992 & 161856 & 154368 & 147456 \\ 156330 & 150282 & 144576 & 139200 & 134144 \\ 118692 & 115812 & 112784 & 109664 & 106496 \\ 65448 & 66501 & 66864 & 66672 & 66048 \\ 0 & 7128 & 13248 & 18624 & 23552 \end{bmatrix},$$

and yield that (12) is satisfied for all $a \in (1/16c_1^2, 1/12)$ and $n \geq 4$.

References

[1] Boor, C.d., Höllig, K., Riemenschneider, S.: Box splines. Springer 1993.
[2] Catmull, E., Clark, J.: Recursive generated B-spline surfaces on arbitrary topological meshes. Computer Aided Design *10*, 350–355 (1978)
[3] Cavaretta, A., Dahmen, W., Michelli, C.: Stationary subdivision. Memoirs of the AMS, 1991.
[4] Doo, D., Sabin, M.: Behavior of recursive division surfaces near extraordinary points. Computer-Aided Design *10*, 356–360 (1978).
[5] Dyn, N.: Subdivision schemes in CAGD. In: Advances in Numerical Analysis (Light, W., ed.), Vol. II: Wavelets, Subdivision Algorithms, and Radial Basis Functions, pp. 37–104 (1992).
[6] Dyn, N., Gregory, J., Levin, D.: A butterfly subdivision scheme for surface interpolation with tension control. ACM Trans. Graphics 160–169 (1990).
[7] Holt, F.: Towards a curvature continuous stationary subdivision algorithm. Z. Angew. Math. Mech., *76*, 423–424 (1995).
[8] Ivrissimtzis, I., Sabin, M., Dodgson, N.: On the support of recursive subdivision. ACM Trans. Graphics *23*(4), 1043–1060 (2004).
[9] Karciauscas, K., Peters, J., Reif, U.: Shape characterization of subdivision surfaces – case studies. Comput. Aided Geom. Design *21*(6), 601–614 (2004).
[10] Kobbelt, L.: Interpolatory subdivision on open quadrilateral nets with arbitrary topology. In: Eurographics ’96 Conference, pp. 409–420 (1996).
[11] Loop, C.: Smooth subdivision surfaces based on triangles. Master’s Thesis, University of Utah, 1987.
[12] Loop, C.: Bounded curvature triangle mesh subdivision with the convex hull property. The Visual Computer *18*(5–6), 316–325 (2002).
[13] Loop, C.: Smooth ternary subdivision of triangle meshes. In: Curve and Surface Fitting (Cohen, A., Merrien, J.-L., Schumaker, L., eds), pp. 295–302 (2003).
[14] Peters, J., Reif, U.: Analysis of algorithms generalizing B-spline subdivision. SIAM J. Numer. Anal. *35*, 728–748 (1998).
[15] Peters, J., Reif, U.: Shape characterization of subdivision surfaces – basic principles. Comput. Aided Geom. Design *21*(6), 585–599 (2004).
[16] Prautzsch, H., Umlauf, G.: A G^1 and G^2 subdivision scheme for triangular nets. Int. J. Shape Model *6*(1), 21–35 (2000).
[17] Reif, U.: A unified approach to subdivision algorithms near extraordinary vertices. Comput. Aided Geom. Design *12*, 153–174 (1995).
[18] Sabin, M., Barthe, L.: Artifacts in recursive subdivision surfaces. In: Curve and Surface Fitting (Cohen, A., Merrien, J.-L., Schumaker, L., eds), pp. 353–362 (2003).
[19] Umlauf, G.: Analyzing the characteristic map of triangular subdivision schemes. Constr. Approx. *16*, 145–155 (2000).
[20] Umlauf, G.: A technique for verifying the smoothness of subdivision schemes. In: Geometric Modeling and Computing (Lucian, M., Neamtu, M., eds), pp. 513–521, Seattle 2004.
[21] Velho, L., Zorin, D.: 4-8 subdivision. Comput. Aided Geom. Design *18*, 397–427 (2001).
[22] Warren, J., Weimer, H.: Subdivision methods for geometric design. Morgan Kaufmann 2002.
[23] Zorin, D.: Stationary subdivision and multiresolution surface representations. PhD thesis, California Institute of Technology, Pasadena 1998.
[24] Zorin, D.: Smoothness of subdivision on irregular meshes. Constr. Approx. *16*, 359–397 (2000).

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