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Abstract

In recent years portfolio management began to integrate constraints which are necessary to achieve an efficient portfolio selection. Cardinality constraints in the context of benchmark tracking, threshold restrictions or other restrictions to meet legal requirements are some examples. Linear models can solve the resulting mixed integer optimization problems using branch&bound algorithms, while quadratic models, like the well known mean-variance-approach, depend on heuristics.

Heuristics do not offer information about the quality of the solution. Therefore the performance of two linear approaches, the mean-absolute-deviation- and the mean-target-shortfall-probability-mode, were compared with the mean-variance-model. For the comparison some thousand portfolios were computed out of stocks of the Tokyo Stock Exchange. This empirical simulation showed, that the mean-absolute-deviation-portfolios produced in general good results compared with mean-variance-portfolios, while mean-target-shortfall-probability-portfolios can only be recommended under certain conditions.

The analysis of the influence of the skewness of the return distributions demonstrated that skewness seem not to be always diversified away by 6 or 7 securities in the portfolios. Further more the number of periods was estimated which should be used in linear models to get robust results like quadratic optimization approaches.

Key-Words: Linear portfolio optimization, empirical simulation, target-shortfall-probability, portfolio optimization with constraints, Konno H. and Yamazaki H.

1. Introduction

Modern Portfolio Management uses quadratic optimization since its foundation in the year 1952 by H. Markowitz. Since this moment alternative linear models[2] were discussed but without remarkable applications in fund management. From the theoretical point of view based on the correctness of assumptions (e.g. normal distributed returns or risk-averse investors) or absence of constraints (e.g. cardinality-constraints in benchmark-tracking or transaction expenses), it is the accurate instrument to find efficient portfolios.

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[2] Linear models often used alternative measure of risk like the following examples: Philippatos G. C. and Wilson CH. J. (1972) with risk-measure entropy, Shalit H. and Yitzhaki S. (1984) with risk measure Gini coefficient, Konno H. and Yamazaki H. (1991) like Feinstein C. D. and Thapa M. N. (1993) with absolute deviation as risk measure.

For fussy researchers assumptions like normal distributed returns are always violated. The justification could be, that the loss but not the profit of an investment is always limited to at most 100%. Apart from that nitpicking argument, empirical data show often strong skewed distributed returns. At least 70% of the stocks on the Tokyo Stock Exchange exhibit too large skewness for the normal distribution.[3] Some researchers found, that positive skewness is diversified away by six or seven assets.[4] To avoid skewness it seem not to be sufficient to use some securities in portfolio. If the returns remain to be skewed, the orientation in the variance of a portfolio would have the consequence, that assets which offer sometimes very high returns (positive skewness) will not be selected in a portfolio, due to the "risk" having sometimes high returns. The risk measure "absolute deviation" do not overrate high returns like the risk measure "variance". For downside risk measure like e.g. "target shortfall probabilities" high returns do not matter.

Have linear models advantages in performance if assumptions like the normal distribution are violated? The answer for that question was searched by empirical simulation. Empirical simulation can find differences between models under real conditions. Like sampling, empirical simulation is always restricted to the selected data (e.g. a certain market and time period). The results may not be representative for other markets or time periods. The sample size of some thousand computed portfolios was selected sufficiently big for significant results and small enough to avoid repetition of selected portfolios.

Two models were tested by empirical simulation: The Mean–Absolute-Deviation-model[5] (M-AD-model) and the Mean–Target-Shortfall-Probability model[6] (M-TSP-model). The performance of the two approaches was measured relative to that of the Mean–Variance-portfolio (M-V-portfolio).

The well known M-V-model minimizes the variance of a portfolio, selected out of n assets with weightings x_i (i = 1, ..., n) under the condition, that the expected return of the portfolio will be μ. This expected return is computed out of the expected returns μ_i (i = 1, ..., n) of the securities. The objective function

$$\sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \,\mathrm{cov}_{ij} \qquad (1)$$

with covariance cov_{ij} ($i = 1, ..., n$; $j = 1, ..., n$) is constrained by

$$\sum_{i=1}^{n} x_i = 1, \quad \text{with } x_i \geq 0, \; (i = 1, \ldots, n) \qquad (2)$$

and

$$\sum_{i=1}^{n} \mu_i x_i \geq \mu. \qquad (3)$$

[3] Kariya T., Tsukuda Y., Maru J. (1989) showed skewness for the early 1970s to late 1980s. Now, the data of the following two decades were used for the empirical simulation. In most cases the observed skewness was positive. The return of some assets exhibited a skewness greater than 2.
[4] see Simkowitz M. A. and others (1978), Duvall R. and others (1981), Kane A. (1982).
[5] see Konno H. and Yamazaki H. (1991), Feinstein C. D. and Thapa M. N. (1993).
[6] see Engesser K., Schubert L., Woog M. (1997), Schubert L. (2002).

In the optimization process, the M-V-model works only with parameters μ_i and cov_{ij} of the return distributions. The both other models use the return data directly applying stochastic programming. Therefore it is possible to respect more the characteristics of other distribution.

2. M–AD-portfolios versus M-V- portfolios

The M-AD–portfolio which was applied in the empirical simulation is a reformation of the model of Konno and Yamazaki (1991) by Feinstein and Thapa (1993)[7]. If the returns r_i of the security *i* (*i*=1, ..., *n*) are multivariate normally distributed, then minimizing the AD is equivalent to minimizing the variance[8]. The objective function consists of the slack variables $w_t \geq 0$ and $v_t \geq 0$ ($t = 1, ..., T$) which represent the absolute deviation.

Minimize

$$\frac{1}{T}\sum_{t=1}^{T} w_t + v_t \tag{4}$$

subject to the constraints (2), (3) and

$$-w_t + v_t + \sum_{i=1}^{n} (r_{it} - \mu_i)\, x_i = 0, \quad (t = 1, \ldots, T). \tag{5}$$

Some years ago the M-AD–portfolio was tested with stocks from the NIKKEI 225 and NIKKEI 50 by Konno y Yamazaki[9]. The results were concerned to the position of the efficient frontier of the M-AD-portfolios in the mean-variance space, the CPU time to compute the different portfolios and some examples for performance differences. Three set of data were the base for the selection of three portfolios. The portfolios were optimized for different μ to find the efficient frontier. Each set was characterized by 60 monthly returns, which were selected out of the years 1981-1987.

The number of portfolios in the test of Konno and Yamazaki were to small for to judge the performance of the model. The characterization of the return distribution by 60 periods were small too. Deviation between the models were explained by non-normality of the data without separating the effect of e.g. skewness.

Therefore the following test was constructed to find answers to the questions: How many time periods are necessary using stochastic programming to get robust results? How strong is the influence of skewness to the realized return in the post period? Is the performance difference depending on the economic business cycle?

2.1 Data and software program

As database served 570 of the biggest securities of the Tokyo Stock Exchange, which were traded without interruption in the years from 1.12.1980 until 1.11.1999. The securities

[7] see Feinstein C. D. and Thapa M. N. (1993).
[8] see Konno H. and Yamazaki H. (1991).
[9] see Konno H. and Yamazaki H. (1991).

were divided into three sets S0, S1, S2 with different skewness. The skewness was computed for the complete time interval. S0 contained 190 assets with skewness between –0,5 and 0,5. S1 included 248 with skewness between 0,5 and 1,5 and the last set S2 with 132 stocks had a skewness higher than 1,5. The three sets were used separate or together as database for the random selection of 30 securities for the portfolio optimization for both models. The small number of securities offer the possibility to generate many different portfolios. By this process 17987 portfolios were optimized with the M-V- and the M-AD-model. The expected return μ of the two optimization processes of both models was fixed in most cases at “return of the minimal variance portfolio + 5%”.

The annual returns were produced like the 365 momentum in the chart analysis. Every possible time interval with 365 days distance can be used for an annual return. If every day annual returns were computed, the number of periods T would be some thousands. If every year only one annual return is computed, starting with 1.12.1980, the number of periods T would be 18. By that process it was possible to change the number of return periods without changing the time unit (from month to week etc.)

The portfolio selection was optimized by the mixed-integer software CPLEX 7.1 which was embedded in a C++ software program which selected the sample etc.. The statistical package SPSS 8.0 was used for the analysis of the results.

2.2 Results

The quality of the solution of the M-AD–model depends on the **number of periods** for the characterization of the return distribution. If there are only some few returns computed, the results would not be as robust as the results of the M-V–model. The norm of the **difference of the solution-vector x of both models** will be used for the observation of the influence of the number of time intervals used in the optimization process. Usually this norm is unequal zero due to the different models.

In Figure 2 for each set S0, S1 and S2 a set of stocks were random selected for portfolio optimization with different numbers of periods. Low numbers of periods seem to produce portfolios with strong changing solutions. For higher numbers of periods, the difference of the solutions of both models seem to be more stabile. Further more, the results of the different skewed data in Table 1a illustrate that the norm of the difference of the solutions depend on the skewness. The analysis of the variance (ANOVA) of the three sets in Table 1a show a significant level of 0,00[10].

Skewness	Sample size	Mean of X-norm difference
S0	4049	0,1298
S1	4448	0,1615
S2	4558	0,1803
total	13055	0,1582

Table 1a: Mean of X-norm difference and skewness

[10] The significant level of 0,00 means, that the exact value is lower than 0,005.

The norm of the difference of the solutions of the M-AD– and the M-V-model can easy be interpreted, using portfolios with equal weighted assets. The weightings for stock i in the solution of the two models are represented by x_i^{AD} resp. x_i^{V}. In the case of a portfolio out of the set S0 (resp. S1 or S2), the mean of 0,1298 (resp. 0,1615 or 0,1803) results if one of 11 (resp. 9 or 8) stocks is different. Extreme cases showed a difference of about 0,5. This would stand for the case that one of two equal weighted stocks is different. In Table 1b the difference of the solutions with equal weighted assets is demonstrated. The different selected stocks in the portfolios of the two models are marked with grey color. The scatter-plot of Figure 1 also illustrates, that the difference is not dependent on the number of assets, which are selected in the M-AD-portfolio.

Skewness	1. stock	2. stock	3. stock	4. stock	5. stock	6. stock	7. stock	8. stock	9. stock	10. stock	11. stock
≈ 0	$x_1^{AD}=x_1^V$	$x_2^{AD}=x_2^V$	$x_3^{AD}=x_3^V$	$x_4^{AD}=x_4^V$	$x_5^{AD}=x_5^V$	$x_6^{AD}=x_6^V$	$x_7^{AD}=x_7^V$	$x_8^{AD}=x_8^V$	$x_9^{AD}=x_9^V$	$x_{10}^{AD}=x_{10}^V$	
≈ 1	$x_1^{AD}=x_1^V$	$x_2^{AD}=x_2^V$	$x_3^{AD}=x_3^V$	$x_4^{AD}=x_4^V$	$x_5^{AD}=x_5^V$	$x_6^{AD}=x_6^V$	$x_7^{AD}=x_7^V$	$x_8^{AD}=x_8^V$			
≈ 2	$x_1^{AD}=x_1^V$	$x_2^{AD}=x_2^V$	$x_3^{AD}=x_3^V$	$x_4^{AD}=x_4^V$	$x_5^{AD}=x_5^V$	$x_6^{AD}=x_6^V$	$x_7^{AD}=x_7^V$				

Table 1b: Mean of X-norm difference and skewness in portfolios with equal weighted assets.

To separate that influence of different skewed returns, in Fig. 3 only the portfolios were regarded with at least 7 securities and in Fig. 4 only the set S0 was included. In Fig. 3 and 4 the change of the norm of the difference was used to neutralize the absolute difference between the models[11]. Obviously the mean of the change of the difference is approximate zero for high numbers of periods, but not for small numbers. The result recommends to use at least 200 periods in the optimization to avoid too strong estimation errors[12].

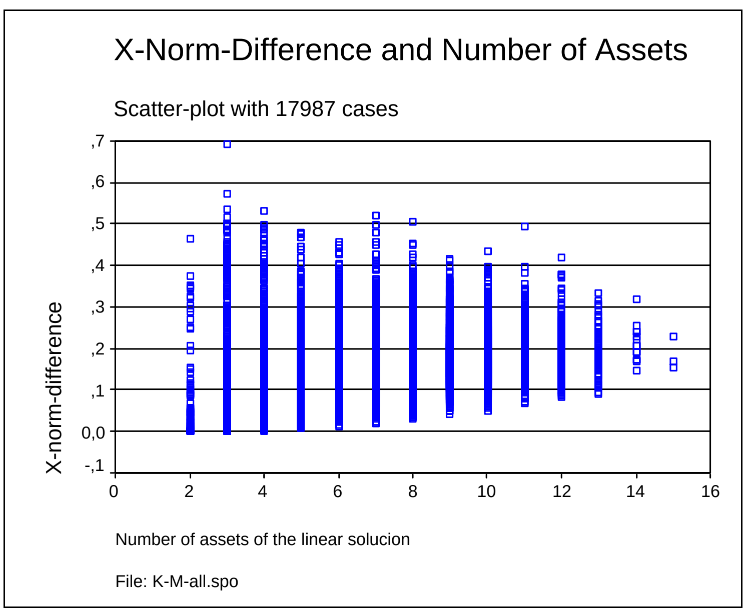

Fig. 1: X-norm-difference and number of assets in the M-AD-portfolio.

[11] A change can occur between the number of periods T1 and the next number T2 with (T1<T2). The change must be positioned at T1 in the Fig. 3 and Fig. 4 because there exists different T2 for every T1.
[12] Referred to the total database of about 18 years, at least every month a return should be computed..

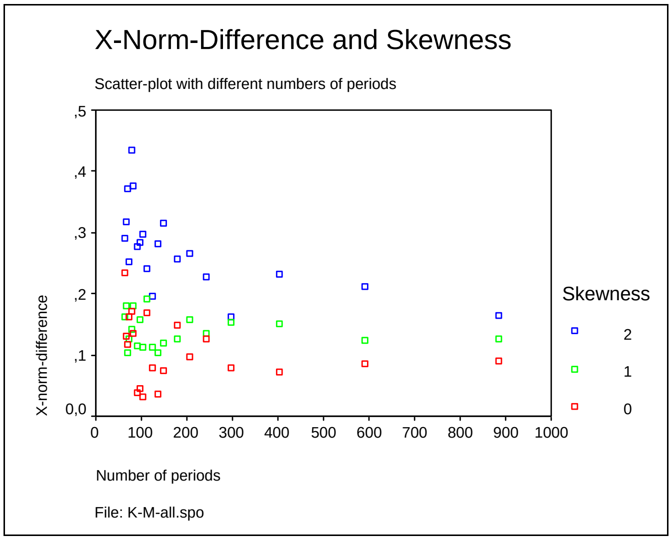

Fig. 2: X-norm-difference and skewness.

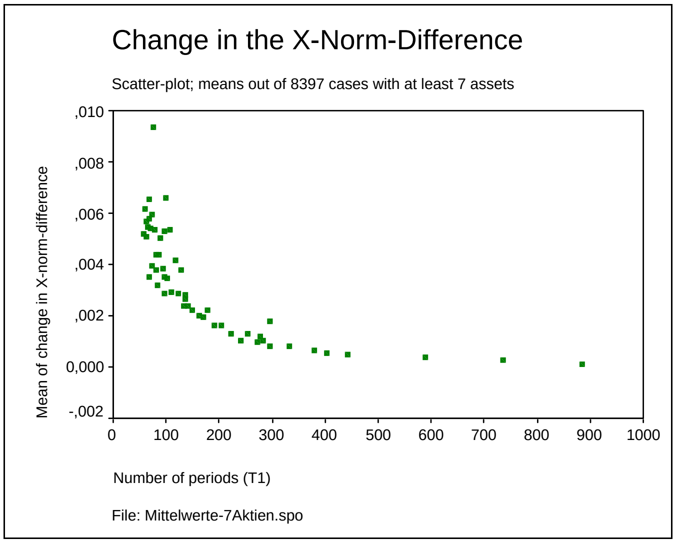

Fig. 3: Mean of X-norm-difference by at least 7 assets in the portfolios.

The **ex ante standard deviation** difference (in % of the standard deviation of the M-V-portfolio) for all 17987 cases is on average 2,76%. The mean of **the ex post standard deviation** difference is 6,82%. The ex post standard deviation can be computed out of the realized returns after one time period. In the following section the ex ante standard deviation is discussed.

If the returns are *skewed distributed* the ex ante standard deviation in % seem to be bigger (see Figure 5). In the scatter-plot of Figure 5 most of the cases have a difference in standard deviation which is smaller than 10%[13]. Only high skewed data with small numbers of periods produce higher differences. The standard deviation difference is not zero for portfolios of the set S0 because the returns are still weak skewed distributed.

[13] Konno H. and Yamazaki H. (1991) observed, that the difference in standard deviation is at most 10%.

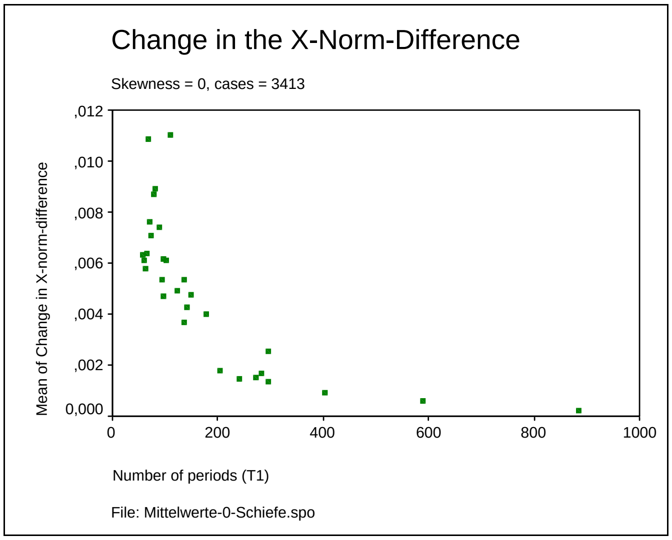

Fig. 4: Mean of X-norm-difference and skewness = 0.

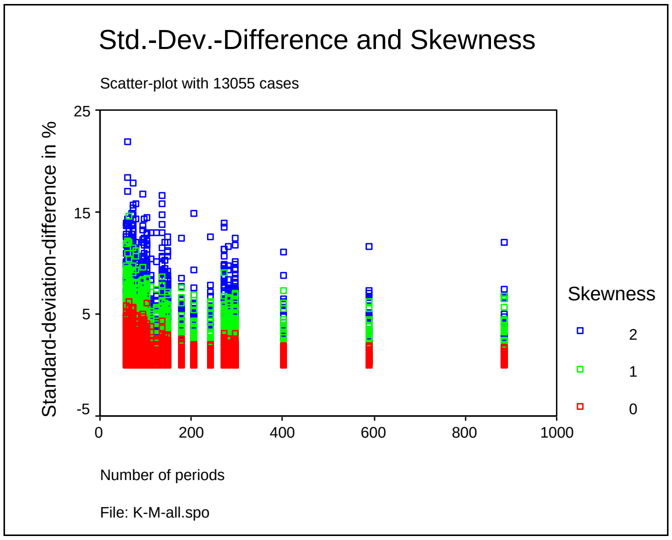

Fig. 5: Scatter-plot of the standard deviation difference and skewness.

The influence of the skewness on the standard deviation difference shows Figure 5 and also Table 2. Besides the mean of the standard deviation difference (in %), the standard deviation of this mean rises also with the skewness. The differences of the sets S0 to S2 were confirmed by ANOVA with a significant level of 0,01.

The *economic business cycles* of the analyzed market were divided in clusters “Hausse”, “Equal” and “Baisse” dependent of the return movement of the market-index in the year of the realized return. The market index was constructed out of all 570 securities with equal weightings. In the different periods of the economic business cycles, the difference of the ex ante standard deviation do not vary strong compared with the ex post standard deviation (in %). The reason is founded in the date, when the different features are registered. The ex ante standard deviation is computed when the portfolios are optimized and the ex post standard deviation is related to the period after this date. In bullish markets, the difference of the standard deviation (in %) between the two models seem to be stronger.

Skewness	Mean of difference of standard deviation (%) ex ante	Sample size	Standard deviation of ex ante mean	
0	1,2041	4049	1,0472	
1	2,5747	4448	1,9880	
2	4,6243	4558	2,7773	
Total	**2,8652**	**13055**	**2,5188**	
Business cycle				**Mean of difference of standard deviation (%) ex post**
Baisse	3,1948	3384	2,6203	3,05
Equal	2,8988	3377	2,5437	7,50
Hausse	2,7454	3363	2,5038	13,80
total	**2,9468**	**10124**	**2,5631**	**5,48**

Table 2: Mean of difference of standard deviation (%) and skewness resp. economic business cycles

The point of most interest is the **realized return** one period after the optimization of the portfolios. The absolute mean of the return difference computed for all 17987 cases is 1,3%. This small advantage of the M-AD-model was confirmed by a t-test for means by a significant level of 0,00. The result may be caused by the separation of the assets in sets S0, S1 and S2 which are normally not visible clusters in the exchange board. The cases (exactly: 13055) which were selected out of one of the separated clusters S0, S1 and S2 produced a higher return advantage of 1,69%. The return advantage of the M-AD-model depends obviously on the *skewness* (see Table 3a). While the differences in the cluster S0 is very small, it rises up in S1 and S2. The results of the different clusters were tested by ANOVA and confirmed by a significant level of 0,00. Therefore the 4932 cases which were selected out of all sets (S0+S1+S2) were analyzed (see Table 3b) and the 7108 cases in which the portfolios contained at least 7 securities (see Table 3c). The result in Table 3b was 0,34% return difference with a significant level of 0,01. It is only a small, but important difference which up values the M-AD–model. Surprising is the mean of the return difference if only portfolios with at least 7 securities in the portfolios are regarded like in Table 3c. The return advantage of the M-AD-portfolios of 2,43% is higher than in Table 3a, where the number of securities in the portfolios were not restricted. The reason is the dominance of cluster S2 in the sample in Table 3c.

Skewness	Mean	Sample size	Std. dev. of mean
0	0,2620	4049	0,0707
1	1,3067	4448	0,0926
2	3,3438	4558	0,1650
Total	**1,6939**	**13055**	**0,0701**

Table 3a: Mean of the difference of realized returns.

Skewness	Mean	Sample size	Std. dev. of mean
Total	**0,3391**	**4932**	**0,1311**

Table 3b: Mean of the difference of realized returns without separation S0, S1, S2.

Skewness	Mean	Sample size	Std. dev. of mean
0	0,3604	1163	0,1554
1	1,5485	1885	0,1462
2	3,4357	4060	0,1765
Total	**2,4320**	**7108**	**0,1119**

Table 3c: Return differences and skewness with at least 7 assets in the portfolios.

The consequences of this results are, if there are sectors of an economy with high skewed return distribution, the M-AD-model would produce better results on average, even if the portfolio contains at least 7 securities.

The influence of the *economic business cycle* on the return difference of the both models was also analyzed. The return differences in the different business cycle clusters of Table 4a were significant (ANOVA: sign. level: 0,00). Especially in bullish markets, the M-AD-portfolio produced an average of 1,9% higher returns. In the analysis only the 10124 cases with clear identifiable economic business cycle were used. The scatter-plot of Figure 6 shows symmetric distributed return differences. Extreme differences are higher than 20% resp. lower than –20%. Obviously the differences are not so strong in bearish markets than in bullish markets.

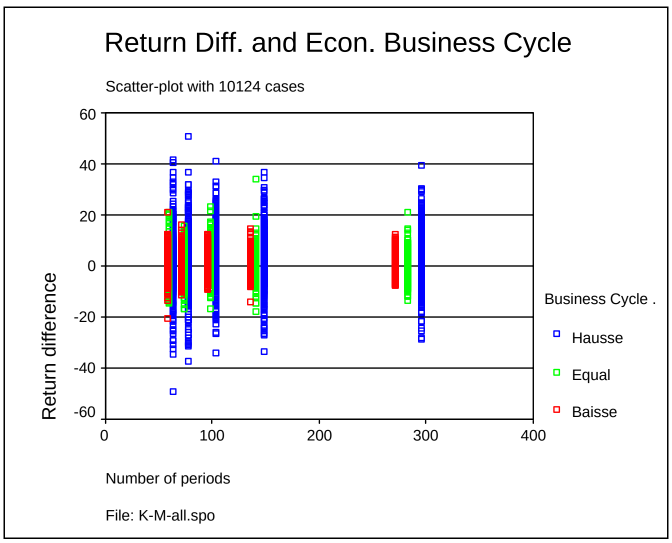

Figure 6: Scatter-plot of return differences and economic business cycle.

Economic business cycle	Mean	Sample size	Std. dev. of mean
Baisse	0,8716	3384	0,0621
Equal	0,3682	3377	0,0797
Hausse	1,9133	3363	0,1617
Total	**1,0497**	**10124**	**0,0637**

Table 4a: Mean of the return difference and the economic business cycle.

For the Table 4b without the effect of the skewness clusters could not enough cases in the sample be found. If only the portfolios with at least 7 securities are analyzed, the better results of the M-AD model in bullish markets are confirmed.

Economic business cycle	**Mean**	**Sample size**	**Std. dev. of mean**
Baisse	1,0937	2227	0,0805
Equal	0,2313	1903	0,1126
Hausse	3,5586	1556	0,2949
Total	1,4796	5686	0,0961

Table 4c: Return differences and economic business cycle with at least 7 assets in the portfolios[14].

Splitting the clusters of economic business cycle by the skewness of the distribution of the returns like in Table 5 the above mentioned effects can be seen. Securities with positive skewed return distributions offer higher returns in bullish markets. The M-AD-model seems to manage better skewed data than the M-V-model. The standard deviation of the mean demonstrates, that most of the results are significant. Low significant results are marked with gray color. In markets which are moving sideways (Equal), or falling down (Baisse), the effect of skewness is weaker than in bullish markets, where the advantage of the M-AD-model is in the average 4,4%.

			Baisse			**Equal**			**Hausse**		
		Return difference	**mean**	cases	std. dev. of mean	**mean**	cases	std. dev. of mean	**mean**	cases	std. dev. of mean
Schiefe	**0**		**0,4593**	1097	0,1093	**0,3737**	1053	0,1442	**-,0234**	1047	0,1435
	1		**1,1262**	1172	0,1104	**0,6065**	1128	0,1373	**1,0324**	1116	0,2427
	2		**1,0096**	1115	0,1015	**0,1387**	1196	0,1332	**4,4222**	1200	0,3608

Table 5: Return difference and skewness resp. economic business cycle.

The **number of securities in the portfolio** was relative small, due to the random selected databases of 30 securities. The Bravais-Pearson correlation coefficient of the number of assets in the portfolios of the both models computed with 17987 cases was 0,84 on a significant level of 0,00. The following Table 6 illustrates the relationship in a cross-table[15].

[14] The differentiation in the Tables 4b and 4c were not planed in the sample of the empirical simulation with the consequence, that sometimes the number of cases is zero like in the missing Table 4b in which the results without the clusters S0, S1 and S2 should be analyzed.
[15] Weightings x_i smaller than 0,00001 were not counted.

Number of assets		**M- V**															
		2	***3***	***4***	***5***	***6***	***7***	***8***	***9***	***10***	***11***	***12***	***13***	***14***	***15***	***16***	***17***
M-AD	**2**	413	82	9													
	3	111	531	191	51	13	2	2									
	4	17	209	605	344	148	42	13	2								
	5	3	45	318	801	579	272	85	38	8	2						
	6		5	99	432	868	670	317	150	58	17	6					
	7			25	147	516	890	739	438	158	54	11	3	1		1	
	8			4	32	169	471	830	714	357	124	48	13	1			
	9		1		7	40	186	419	652	454	246	108	27	2	1		
	10				2	7	52	171	323	360	300	134	52	10			
	11					1	11	39	108	167	191	118	55	15	3		
	12					1	3	13	28	60	93	68	20	11	2		
	13								7	20	27	28	12	7	2		
	14										2	3	3	5	1	1	1
	15												3				

Table 6: Cross-table for the number of assets in the selected portfolios of the models M-V and M-AD.

3. M-TSP-portfolios versus M-V-portfolios

In recent years, the Target-Shortfall-Probability (TSP) was discussed as an alternative measure of risk[16]. If the return r_i of the securities i (i=1, ..., n) are multivariate normally distributed, then minimizing the TSP is equivalent to minimizing the variance[17]. The only difference is, that the efficient frontier of the M-TSP-portfolios will not include the portfolios near the minimal variance point of the M-V-portfolio. From the utility-theoretical point of view, the TSP is not perfect. Furthermore it is criticized due to the insufficient description of the risk. The advantages of the TSP are the usage independent of the return distribution and the intuitive understanding of this risk measure by the investor.

The use of a TSP-vector[18] reduces the utility-theoretical disadvantage of a single TSP and offers an approximate sufficient description of risk[19]. The Mean-TSP-vector model is a mixed-integer linear program. The CPU-time to solve the model demonstrated that the model is suitable for practical applications with some hundred assets[20].

The objective function of the M-TSP–model maximizes the expected return

[16] Roy A. D. (1952) proposed first to use the TSP for the selection of portfolios. Many other researchers modified his suggestion.
[17] see Baumol W. J. (1963). A more general proof can be found by Schubert L. (1996).
[18] see Engesser K., Schubert L., Woog M. (1997).
[19] see Schubert L. (2002). The risk measure “TSP-vector” is not restricted to the utility-function of risk-averse investors. The investor itself determines the utility function by selecting different targets and probabilities.
[20] Due to the mixed-integer problem, the optimization needs much more time than the M-V-model. In most cases it was possible to find the optimal solution, although near 700 stocks were sometimes included (see Schubert L. (2002)).

$$\sum_{i=1}^{n} x_i \mu_i \tag{6}$$

under the condition (2) and m TSP-constraints

$$\sum_{i=1}^{n} x_i r_{it} \le (1-\delta_{tk})M + \tau_k, \quad (t=1,\dots,T, \quad k=1,\dots,m) \tag{7a}$$

$$\sum_{i=1}^{n} x_i r_{it} \ge \tau_k + \varepsilon - \delta_{tk} M, \quad (t=1,\dots,T, \quad k=1,\dots,m) \tag{7b}$$

$$\frac{1}{T}\sum_{t=1}^{T} \delta_{tk} \le \alpha_k, \quad (k=1,\dots,m) \tag{7c}$$

In the restrictions (7a) and (7b) are binary variables δ_{tk} (t = 1, ..., T, k = 1, ..., m) included counting the cases, when the target-restriction is violated. The restriction (7c) determines for the target τ_k that the shortfall probability must be at most α_k. The rank-order of the targets $\tau_1 < \tau_2 < ... < \tau_m$ must be equal to the rank-order of the probabilities $\alpha_1 < \alpha_2 < ... < \alpha_m$. The parameter ε (resp. M) is a very small (resp. big) number.

To compare the M-TSP and the M-V-model, answers to the following questions should be found: How strong is influence of the number of targets and the skewness on the realized return in the post period? Is the result of the portfolio depending on the economic business cycle? What will be the difference in the risk-measures like standard deviation and shortfalls? How many assets will be in the optimized portfolios of the two models?

3.1 Data and Software

The database contains the same 570 securities resp. returns and the same clusters S0, S1 and S2 as in the empirical simulation above[21]. The software CPLEX 7.1 and SPSS 8.0 were used in the same way.

Every simulated case was generated for a given target with its shortfall-probability (t, α)[22]. For this TSP, the optimal portfolio resp. expected return (μ) was computed (see equation (6)). This expected return of the M-TSP-portfolio was a restriction when in the second part the solution for the M-V–portfolio was searched (see equation (3)). By this process 11895 portfolios were optimized with each model.

3.2 Results

The whole sample of 11895 cases showed a small disadvantage of -0,22% on average (sign. level: 0.01) of less **realized return** produced by M-TSP-portfolios. If we exclude the cases with only one *target*, because of the insufficient description of risk using only one target-shortfall probability, return difference disappear (see Table 7a). The remaining return

[21] Per random selection 30 to 50 securities were given as database for every case of the sample.

[22] For the M-TSP–model, all cases (11895) are divided in cases with one (5395), two (3151) and three (3349) targets.

difference means of –0,059% resp. –0,068% do not have a significant levels (with 0,71 resp. 0,67) in a t-test of the means. Disregarding the different risk measures, a M-TSP-vector portfolio with at least 2 targets seems to produce return results like the M-V-portfolio. For only one target, the negative result of –0,41% on average has a high significant level.

Number of targets	Mean	Sample size	Std. dev. of mean
1	-0,4047	5395	0,1322
2	-0,0591	3151	0,1621
3	-0,0681	3349	0,1574
Total	**-0,2184**	**11895**	**0,0860**

Table 7a: Return differences and number of targets.

The return differences of cases with one target and the cases with more than one target are confirmed as significant by an ANOVA test (sign. level: 0,05).

Due to the separate sampling of the cases according to their skewness (either clusters S0 or S1 or S2), the same tests for return differences were done for the 1636 cases, which were sampled out of all 570 assets without skewness cluster. Under this circumstances, the portfolios with one target produced a negative return difference of -11.81% (sign. level: 0,05). It must be mentioned, that in the sample size existed only of 8 cases (see Table 7b). Two (resp. three) targets effect a result of positive return differences with sign. level 0,50 (resp. 0,01). In Table 7c the cases with at least 7 securities in the portfolios were analyzed. Under this condition, the negative result on average for M-TSP-portfolios with only one target remains, but without significant level like in the cases with more than one target which produced positive results.

Number of targets	Mean	Sample size	Std. dev. of mean
1	-11,8049	8	4,9383
2	0,2204	822	0,3060
3	0,7501	806	0,3027
Total	**0,4226**	**1636**	**0,2165**

Table 7b: Return differences and number of targets (without S0, S1, S2).

Number of targets	Mean	Sample size	Std. dev. of mean
1	-0,6409	635	0,4786
2	0,2128	39	1,3887
3	0,1618	270	0,5421
Total	**-0,3761**	**944**	**0,3619**

Table 7c: Return differences and number of targets with at least 7 assets in the portfolios.

The results of the three tables (7a-7c) confirm a return disadvantage of the M-TSP-model, if only one target is used. In the case of more targets the return differences of the models on average are small and not significant; with short words: there seem to be no return-difference.

The *skewness* of the returns and the influence on the realized return were analyzed too. The skewness of an asset was computed with all time periods which existed in the database. This is the reason why the analysis was restricted to 4087 cases, which were computed at the 2. June 1997 or later[23] (see Table 8a). While the cases in cluster S0 offered a negative return difference of –1,77% (sign. level 0,00), higher skewness seem to be an advantage of the M-TSP-portfolio (see Table 8a). The M-TSP-portfolios, whose asset-returns had a skewness of 2, offered 2,10% additional return on average (sign. level: 0,00) than the M-V-portfolios. Only in the case of the skewness of 1, the results had a low significant level of 0,54. The variance analytical test (ANOVA) shows, that the three clusters in Table 8a effect different results (sign.-level: 0,00). In the Table 8b, the cases are included which were sampled without the differentiation in clusters S0 to S2. Under this condition, the M-TSP-portfolios produced on average a return advantage of 1,39% with a significant level of 0,01. In Table 8c only cases with at least 7 securities are included. Under this condition the return difference in the different skewness cluster is like in the Table 8a but with lower level of significance for cluster S0 and S1: negative difference in cluster S0 and positive in cluster S1 and S2.

Skewness	Mean	Sample size	Std. dev. of mean
0	-1,7665	1036	0,3052
1	0,1912	1133	0,3096
2	2,1030	1378	0,3806
Total	**0,3622**	**3547**	**0,2007**

Table 8a: Return differences and skewness.

Skewness	Mean	Sample size	Std. dev. of mean
Total	**1,3869**	**540**	**0,4936**

Table 8b: Return differences and skewness (without S0, S1, S2).

Skewness	Mean	Sample size	Std. dev. of mean
0	-2,2073	99	1,1504
1	1,8753	97	1,3666
2	4,3520	203	0,9985
Total	**2,1224**	**399**	**0,6826**

Table 8c: Return differences and skewness with at least 7 assets in the portfolios.

The result of the Tables 8a and 8c show, that the M-TSP-model does not offer an return advantage, if the returns are not skewed distributed. But if strong skewness exists, the M-TSP-portfolio produces higher returns on average.

The analysis of the effect of the *economic business cycle* includes only the 11780 cases, were the economic business cycle could be clearly identified. Table 9a demonstrates, that the M-TSP-model is concerned to the risk. Therefore, in bearish markets the mean of

[23] The simulation of the last chapter (M-TSP-model versus M-V-model) used only cases which were produced after the 1. June 1997. In the simulation of this chapter cases which were computed at an earlier date are included to get more cases of different economic business cycles.

the return difference of 0,74% is significant better (level: 0,00) compared with the M-V-model (see Table 9a). In bullish markets the opposite result can be regarded. In this times, the mean of the return difference is –1,60% (sign. level: 0,00). In markets which are moving sideways (Equal), the effect is with 0,35% weaker than in bearish markets (significant level: 0,00). The better results of the M-TSP-Portfolio are also confirmed by the results of the Tables 9b resp. 9c in which only the cases were analyzed which were computed without the separation in clusters S0, S1 and S2 resp. which contain at least 7 assets in the portfolios.

Economic business cycle	Mean	Sample size	Std. dev. of mean
Baisse	0,7354	3782	0,1061
Equal	0,3460	3996	0,1175
Hausse	-1,6040	4002	0,1950
Total	**-0,1915**	**11780**	**0,0850**

Table 9a: Return differences and economic business cycle.

Economic business cycle	Mean	Sample size	Std. dev. of mean
Baisse	1,0332	533	0,2603
Equal	0,8841	575	0,3136
Hausse	-0,5256	520	0,5105
Total	**0,4826**	**1628**	**0,2153**

Table 9b: Return differences and economic business cycle (without S0, S1, S2).

Economic business cycle	Mean	Sample size	Std. dev. of mean
Baisse	1,2208	295	0,4633
Equal	0,4022	359	0,4398
Hausse	-2,9921	286	0,9169
Total	**-0,3736**	**940**	**0,3608**

Table 9c: Return differences and economic business cycle with at least 7 assets in the portfolios.

Under the circumstances of Table 9b, the portfolios with TSP as risk measure earned 1,03% more (sign. level: 0,00) in times of bearish markets. In markets with are moving sideways, the return difference was 0,88% (sign. level: 0,01). In bullish markets, the return result of the M-TSP-portfolios was also negative compared with the M-V-model, but with a low significant level of 0,30. In the Table 9c, the advantage in bearish markets and the disadvantage in bullish markets have also a high significant level. In bearish markets it can be summarized, the return result of the M-TSP-portfolio is better on average and in bullish markets worse.

The Table 10 demonstrates the effect of both variables, the skewness and the economic business cycle. While the influence of the skewness is like above (high skewness means better returns for the M-TSP-model) the return difference is better, if only bearish markets were compared with bullish markets. When the markets are moving sideways, the results of the M-TSP-model are always better than in bullish markets. Due to the standard deviation of the mean of the differences, in this economic business cycle the results have low significant level (see gray marked means).

	return difference	**Baisse** mean	cases	std. dev. of mean	**Equal** mean	cases	std. dev. of mean	**Hausse** mean	cases	std. dev. of mean
Schiefe	**0**	**-1,7732**	365	0,4607	**0,5966**	337	0,4411	**-4,1435**	334	0,6423
	1	**2,1481**	404	0,3236	**0,7349**	368	0,4209	**-2,5530**	361	0,7728
	2	**2,6903**	404	0,4132	**4,3740**	405	0,3842	**0,6223**	462	0,9085

Table 10: Return Difference and skewness resp. economic business cycle.

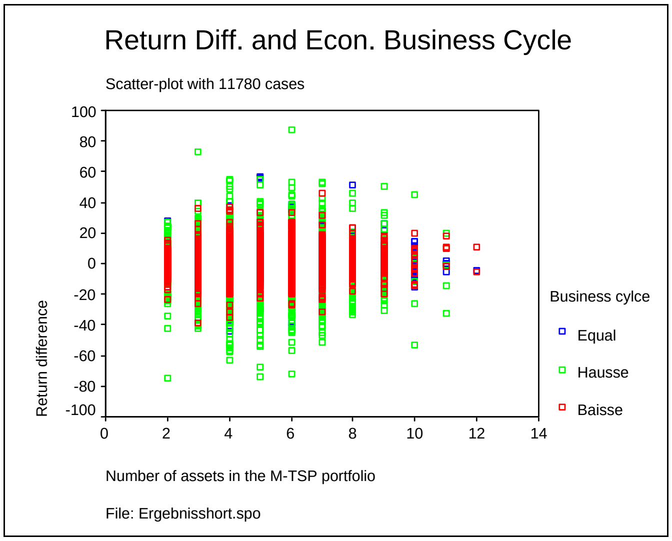

Figure 7: Scatter–plot of the return difference and economic business cycles.

In the scatter-plot of Figure 7 can be seen, that the return differences vary stronger in bullish markets than in bearish markets.

The average of the **ex ante standard deviation** difference (in % of the standard deviation of the M-V-solution) between the M-TSP- and the M-V-model is for all 11895 cases 12,05%. The average of the **ex post standard deviation** difference is only 0,50%. The ex post standard deviation can be computed out of the standard deviation of the realized returns of the models.

The influence of the *skewness* on the ex ante standard deviation difference in % shows Table 11. Like above, for the analysis of skewness, only the cases which were computed after the 1. June 1997 were included. Table 11 shows, that high skewness results high standard deviation difference. The analysis of the mean of the ex ante standard deviation always must show results with high significant level. Therefore, the significance will not be mentioned for the following tables concerning the standard deviation.

The ex ante standard deviation of the return of the M-TSP-portfolios is sometimes very high compared with the ex ante standard deviation of M-V-portfolios. The scatter-plot of Figure 8 illustrates that high differences occur if the skewness is high.

Skewness	Mean of std. dev. diff. %	Sample size	Std. dev. of mean
0	6,5963	1036	0,1697
1	7,9213	1133	0,2101
2	14,2151	1378	0,3606
Total	**9,9795**	**3547**	**0,1728**

Table 11: Ex ante standard deviation difference % and skewness.

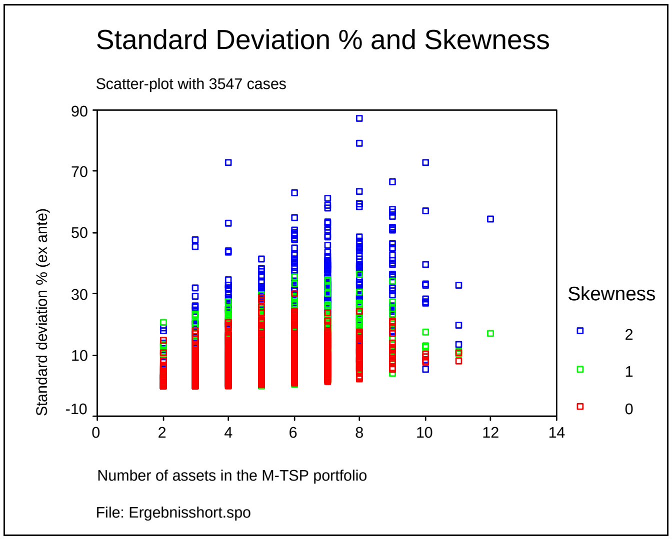

Figure 8: Scatter–plot of the std. deviation difference % and skewness.

The following Tables 12a – 12c demonstrate, that the standard deviation difference in % obviously depends a little bit on the *number of targets*. Table 12a contains all cases[24], Table 12b only the cases which were computed without splitting in skewness cluster S0, S1 and S2 and Table 12c only the cases with at least 7 assets in the portfolios.

Number of targets	Mean of std. dev. diff. %	Sample size	Std. dev. of mean
1	14,1475	5395	0,2044
2	8,4849	3151	0,1807
3	12,0340	3349	0,2237
Total	**12,0524**	**11895**	**0,1237**

Table 12a: Ex ante standard deviation difference (%) and number of targets.

Number of targets	Mean of std. dev. diff. %	Sample size	Std. dev. of mean
1	11,5576	8	2,9602
2	10,0104	822	0,4004
3	11,3915	806	0,4205
Total	**10,6984**	**1636**	**0,2895**

Table 12b: Ex ante standard deviation difference (%) and number of targets (without S0, S1, S2).

[24] For some cases, no definition of the business cycle exists (compare sample size of Table 9a and Table 12a).

Number of targets	**Mean of std. dev. diff. %**	**Sample size**	**Std. dev. of mean**
1	25,6844	635	0,7088
2	14,7920	39	1,6916
3	21,7695	270	0,8906
Total	**24,1147**	**944**	**0,5513**

Table 12c: Ex ante standard deviation difference (%) and number of targets of portfolios with at least 7 assets.

If only one target constraints the portfolio optimization, the difference of the standard deviation (in %) has the highest level under the condition of Tables 12a – 12c. The lowest standard deviation difference on average had portfolios with two targets and not the portfolios with three targets.

Economic business cycle	**Mean of std. dev. diff. % (ex ante)**	**Sample size**	**Std. dev. of mean (ex ante)**	**Mean of std. dev. diff. % (ex post)**
Baisse	12,4171	3782	0,2273	15,72
Equal	11,4467	4002	0,2173	10,88
Hausse	12,4387	3996	0,2031	4,84
total	**12,0948**	**11780**	**0,1246**	**0,50**

Table 13a: Standard deviation difference (%) and economic business cycle.

Economic business cycle	**Mean of std. dev. diff. % (ex ante)**	**Sample size**	**Std. dev. of mean (ex ante)**	**Mean of std. dev. diff. % (ex post)**
Baisse	10,7456	533	0,5145	7,43
Equal	9,5201	520	0,5080	7,16
Hausse	11,7082	575	0,4852	1,86
total	**10,6941**	**1628**	**0,2906**	**0,09**

Table 13b: Standard deviation difference (%) and economic business cycle (without S0, S1, S2).

In the different periods of the *economic business cycles*, the difference of the ex ante standard deviation do not vary as strong as the ex post standard deviation (in %) (see Table 13a and 13b). The reason is founded in the date, when the different features are registered. The ex ante standard deviation is computed when the portfolios are optimized and the ex post standard deviation as the economic business cycle is related to the period after this date. In bullish markets the difference of the ex post standard deviation seem to be smaller than in bearish markets. If the total number of cases in the Tables 13a and 13b are regarded, differences of the ex post standard deviation between the models disappear.

The Tables 14a-c show the **target shortfall** of the M-TSP and the M-V-model. The target shortfall was only registered for portfolios with one target, this means in 5395 cases. The advantage of the M-TSP-model is very weak. The model produced in about 0,2% (= 34,4%-34,6%) of all 5395 cases less target shortfalls. The Table 14b refers to cases which were computed without the splitting into cluster S0, S1 and S2. The results were for both models equal (see Table 14b). If only the cases were analyzed, which had portfolio solutions with at least 7 assets, the M-TSP-portfolios failed in about 0,8% (= 27,7%-26,9) more often than the M-V-portfolios (see Table 14c).

M-TSP	**Return**	**Number of cases**	**Percent**	**Cumulated Percent**
target	ok	3548	65,8	65,8
target	below	1847	34,2	100,0
M-V				
target	ok	3540	65,6	65,6
target	below	1855	34,4	100,0
Total		**5395**	**100,0**	

Table 14a: Target shortfall.

M-TSP	**Return**	**Number of cases**	**Percent**	**Cumulated Percent**
target	ok	8	100,0	100,0
target	below	0	0,0	100,0
M-V				
target	ok	8	100,0	100,0
target	below	0	0,0	100,0
total		**8**	**100,0**	

Table 14b: Target shortfall (without S0, S1, S2) .

M-TSP	**Return**	**Number of cases**	**Percent**	**Cumulated Percent**
target	ok	459	72,3	72,3
target	below	176	27,7	100,0
M-V				
target	ok	464	73,1	73,1
target	below	171	26,9	100,0
total		**635**	**100,0**	

Table 14c: Target shortfall with at least 7 assets in the portfolios.

If the M-TSP-model uses only one target, there cannot be insisted any advantage concerning the risk measure of target shortfalls.

The target shortfall and *skewness* was analyzed for both models. In Table 15a only the 1798 cases were included, which were computed after the 1. June 1997. In 500 (resp. 579) cases, the M-TSP (resp. M-V) portfolios missed the target. The target missing difference shows, in how much cases, the M-TSP-model was better. The Table 15a shows, that the target missing difference depends on the skewness.

In 2,86% of the cases of the cluster S2 in Figure 9a, only the M-TSP-portfolios missed the target while the M-V-portfolio did not. But in 8,90% the M-V portfolios failed the target, while the M-TSP did not[25]. The difference between the skewness clusters remain under the condition if only cases with at least 7 assets in the M-TSP- and M-V-portfolios were

[25] The target shortfall difference of Table 15a can be computed e.g. for cluster S2 by: 629 * (8,90-2,86) / 100 ≈ 38.

regarded (see Table 15c and Figure 9c). High skewness means a target shortfall advantage for the M-TSP model.

Skewness		TSP missed target	V missed target	Target missing (difference)
0	cases	547	547	547
	target-missing	**187**	**188**	**1**
1	cases	622	622	622
	target-missing	**166**	**206**	**40**
2	cases	629	629	629
	target-missing	**147**	**185**	**38**
Total	cases	1798	1798	1798
	target-missing	**500,00**	**579,00**	**79**

Table 15a: Target shortfall difference and skewness.

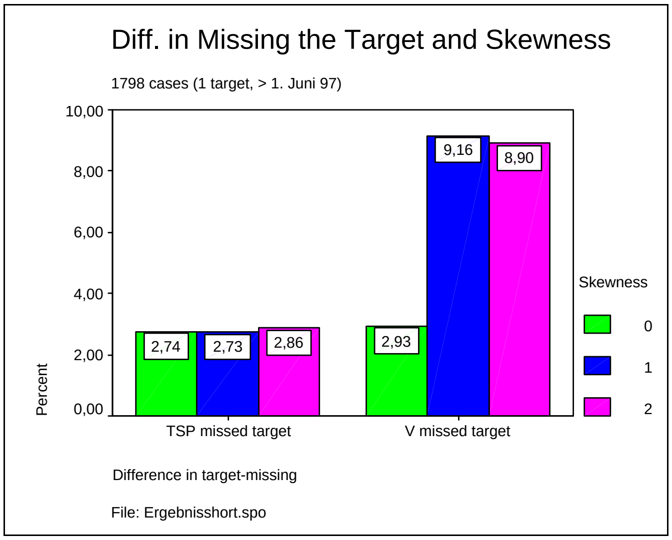

Figure 9a: Target shortfall and skewness.

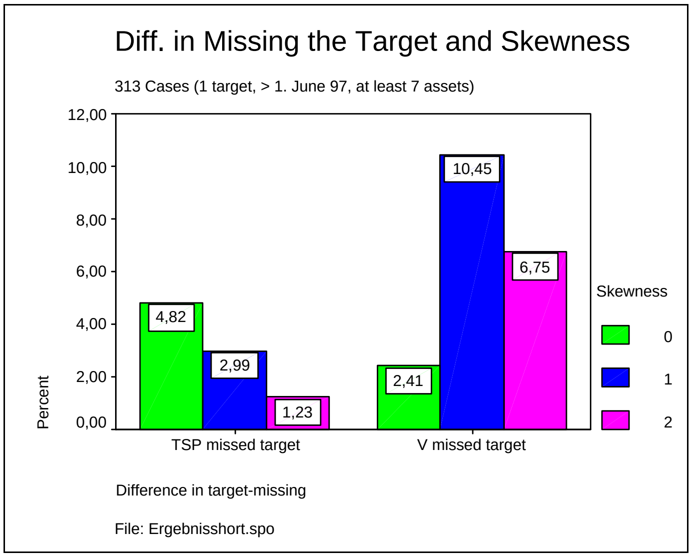

Figure 9c: Target shortfall and skewness with at least 7 assets in the portfolios.

Skewness		TSP missed target	V missed target	Target missing (difference)
0	cases	83	83	83
	target-missing	**21**	**19**	**-2**
1	cases	67	67	67
	target-missing	**28**	**33**	**5**
2	cases	163	163	163
	target-missing	**61**	**70**	**9**
Total	cases	313	313	313
	target-missing	**110**	**122**	**12**

Table 15c: Target shortfall difference and skewness with at least 7 assets in the portfolios.

The *economic business cycle* seem to have an influence on the target shortfalls of the models. In times of bearish markets, the M-V-portfolios failed in 6,69% of the cases the target, while the M-TSP-portfolios produced at least the target return. In bullish markets however, the M-TSP-portfolios failed more often (2,01%) the target (see Figure 10a). With the more realistic condition of at least 7 securities in the portfolio, the results seem to be similar (see Figure 10c)[26].

The 11895 cases had different numbers of securities in the portfolio. Table 16 demonstrates, that the M-TSP-portfolios used nearly as much assets as the M-V-portfolio. The average was 4,5 and 4,7 assets. The Bravais-Pearson correlation coefficient between the numbers of assets in the portfolios of the two models is 0,61.

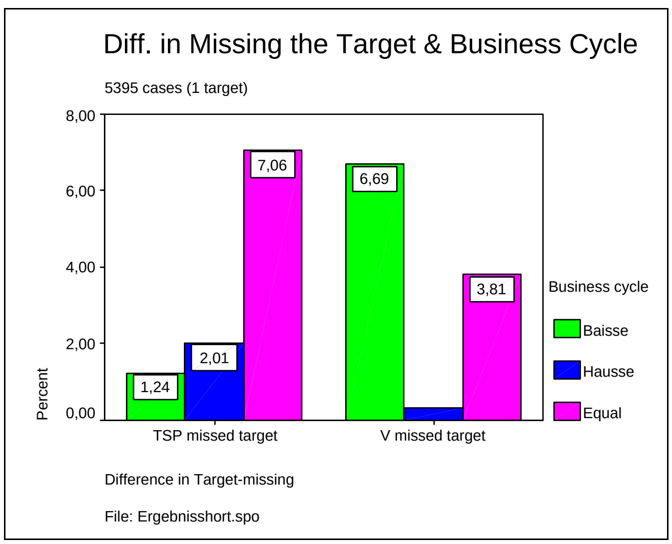

Figure 10a: Target shortfall difference and economic business cycle.

[26] Due to less cases in the sample, it was not possible to design Figure 10b (without splitting in cluster S0, S1 and S2).

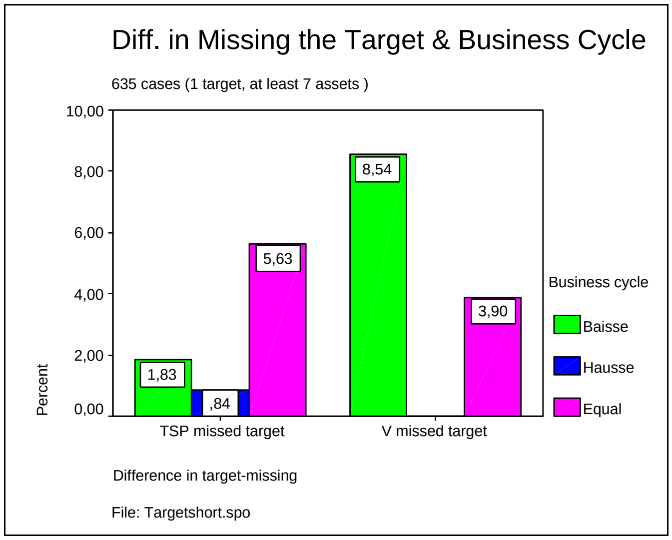

Figure 10c: Target shortfall difference and economic business cycle with at least 7 assets in the portfolios.

Number of assets														
		M- V												
		2	***3***	***4***	***5***	***6***	***7***	***8***	***9***	***10***	***11***	***12***	***13***	***14***
M-TSP	**2**	977	406	138	56	13	6	2						
	3	527	794	527	273	120	59	19	4	3	1			
	4	269	590	661	451	287	142	77	20	16	6	2		
	5	102	311	439	442	341	266	141	73	39	15	5		2
	6	51	143	253	283	266	251	172	108	57	22	4	4	1
	7	9	51	107	139	161	168	142	94	68	27	7	2	1
	8	1	21	26	62	60	72	77	59	28	20	9	1	
	9		7	11	20	21	26	35	23	15	17	3	3	
	10			1	5	10	4	7	11	5	1	1	1	
	11						3	4	3	2	2	1	1	
	12					2	1							

Table 16: Cross-table for the number of assets in the selected portfolios of the models M-V and M-TSP.

4. Constraints in portfolio optimization

In recent years, researchers tried to find efficient portfolios, which are feasible under certain conditions. The efficiency of a portfolio depends also on the transaction- and administration costs. Therefore, constraints of transaction costs but also threshold- and cardinality restrictions must be integrated. Besides this, there exist conditions by law which must be fulfilled e.g. by funds or some individual conditions of investors. Models which are designed to respect such conditions have often the consequence, that mixed integer problems have to be solved. Mixed integer optimization with binary variables means concerning the efficient frontier in the mean-variance space, that the shape of the efficient frontier will be discontinuous.[27]

To find the optimal solution of such mixed integer problems with quadratic objective functions is more complicated than within a linear model. The reason is, that in linear models the optimal solution contains always at least one extreme point of the feasible area. Optimal solutions which are extreme points of the feasible area can be easier found than optimal solution which are within the feasible area like in the case of quadratic mixed integer optimization problems. Therefore the classical portfolio optimization with mixed integer variables has to use heuristic methods[28]. Only in cases with few securities and one restriction a complete enumeration or an algorithm for continuous problems[29]—which is concerned to only a single restriction—is possible for finding the global optima.

For linear mixed integer problems, branch&bound algorithms are successful instruments. If the algorithm is not adjusted to the specific restrictions the CPU time for solving the problem can be too long. But if the time for finding the global optima is restricted, branch&bound algorithms always offer some information about the quality of the best found solution and the potential maximum loss if the algorithm will be interrupted.

Linear optimization seem to be more suitable to find efficient portfolios under consideration of different mixed integer constraints.

5. Conclusions

For the analysis of the results of linear and quadratic optimization models, the database was divided into three skewness cluster. On the one hand the clusters offered the possibility to show the advantages of the linear models, if the return distributions are positive skewed. On the other hand this could be criticized due to the separation of securities with high skewed return distributions. Normally, funds are designed out of stocks of a specific sector of the economy or of a country etc.. But it could happen, that in some sector companies with high skewed return distributions are dominant or that a fund manager like to construct something like a high skewness fund. For this cases linear optimization models can achieve higher returns, if some skewness will stay in the distribution of the portfolio returns.

The skewness seem not to be diversified away by six or seven assets like some researchers found. The reason could be the splitting in different skewness cluster, too. Therefore, some results were checked under the condition, that there is no splitting into skewness cluster or that there are at least 7 stocks in the portfolios. This check could not always be realized due to the content of the database. In most cases the differences between the models continued to exist.

Concerning the economic business cycle, the M-TSP-model seem to realize a better return on average in bearish markets, while the M-AD-model showed good results especially

[27] see Jobst N. J., Hornimann M. D., Lucas C. A., Mitra G. (2001).

[28] see Beasley J. E., Meade N., Chang T.-J. (2003); Chang T. J., Meade N., Beasly J. E., Sharaiha Y.M. (2000); Crama Y., Schyns M. (1999); Derigs U., Nickel N. H. (2003); Jobst N. J., Hornimann M. D., Lucas C. A., Mitra G. (2001).

[29] Jansen R., Dijk van R. (2002) substituted the cardinality constraint: $\lim_{p\to 0} \sum_{i=1\to 100} x_i^p = |\{x_i / x_i > 0\}|$.

in bullish markets compared with the results of the M-V-model. Without differentiation in skewness or periods of the economic business cycle, the M-TSP-model showed a small return disadvantage while the M-AD-model realized on average higher returns than the M-V-model. According to this return results and the capital market theory, the difference of the ex-post standard deviation (in % of the standard deviation of the M-V-portfolios) was in the case of the M-TSP-model only +0,50% and in the case of the M-AD-model +6.82% on average.

The M-TSP-model can be recommended as an alternative model under certain conditions (bearish markets, high skewness) if more than one target are used, while the M-AD-model seem to be an alternative approach in general. Taking into account, that this linear model is more flexible if constraints have to be respected or that the reward of portfolio manager is sometimes based on the AD[30], it is surprising, that portfolio management is still dominated by the M-V-model.

6. References

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