Woven Codes with Outer Warp: Variations, Design, and Distance Properties

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Abstract—***In this paper we consider convolutional and block encoding schemes which are variations of woven codes with outer warp. We propose methods to evaluate the distance characteristics of the considered codes on the basis of the active distances of the component codes. With this analytical bounding technique, we derived lower bounds on the minimum (or free) distance of woven convolutional codes, woven block codes, serially concatenated codes, and woven turbo codes. Next, we show that the lower bound on the minimum distance can be improved if we use designed interleaving with unique permutation functions in each row of the warp of the woven encoder. Finally, with the help of simulations, we get upper bounds on the minimum distance for some particular codes and then investigate their performance in the Gaussian channel. Throughout this paper we compare all considered encoding schemes by means of examples, which illustrate their distance properties.***

Keywords— ***woven codes, turbo codes, serially concatenated codes, interleaver design, active distances, bounds on distances.***

I. Introduction

WOVEN convolutional codes (WCC) were first introduced by Höst, Johannesson, and Zyablov [1] in 1997. A series of papers on the asymptotic behavior of WCC show their distance properties [2] and error-correcting capabilities [3], [4]. The characteristics of woven codes were further investigated in [5], [6], [7], [8], [9], [10]. In the original proposal [1], two types of WCC are distinguished: Those with outer warp and those with inner warp. The active distance family recently introduced in [11] plays a key role in the structural analysis of WCC.

In this paper we consider variations of woven codes with outer warp, give design rules and analyze their distance properties. In Section II we give some basic notations and definitions. In Section III we present an overview of some new and old code constructions where we utilize block or convolutional interleavers. We consider all these constructions from the view of the woven code construction. For instance, serially concatenated convolutional codes [12] and parallel concatenated convolutional (turbo) codes [13] are regarded as special cases of woven codes.

In Section IV we give a brief introduction to active distances and based on these, we get some preliminary results which serve for further analysis of the distance properties of woven codes. Section V is concerned with the distance characteristics of woven codes. Lower bounds on the overall free distances or minimum distances are given for all considered code constructions. We derive conditions that allow us to get these lower bounds as the product of the minimum (or free) distances of the component codes.

In Section VI we extend these results. We present designed interleavers, which lead to improved lower bounds on the overall minimum distance, i.e. the overall minimum distance is about twice the product of the minimum distances of the component codes. We give some simulation results in Section VII and discuss them with respect to the distance characteristics of the codes.

II. Basic Notations and Definitions

LET $\mathcal{C}$ be a rate $R = b/c$ binary convolutional code with the rational generator matrix $G(D) = [g_{ij}(D)]$, $i = 1,\ldots,b$, and $j = 1,\ldots,c$. Then the information sequence

$$\mathbf{u(D)} = \ldots \mathbf{u_{-1}D^{-1}} + \mathbf{u_0} + \mathbf{u_1D} + \ldots \quad , \tag{1}$$

where $\mathbf{u_t} = (\mathbf{u_t^{(1)}}, \mathbf{u_t^{(2)}}, \ldots, \mathbf{u_t^{(b)}})$, is encoded as the code sequence

$$\begin{aligned} \mathbf{v(D)} &= \mathbf{u(D)G(D)} \\ &= \ldots \mathbf{v_{-1}D^{-1}} + \mathbf{v_0} + \mathbf{v_1D} + \ldots \quad , \end{aligned} \tag{2}$$

where $\mathbf{v_t} = (\mathbf{v_t^{(1)}}, \mathbf{v_t^{(2)}}, \ldots, \mathbf{v_t^{(c)}})$. For simplicity we sometimes use the notation $\mathbf{u} = \ldots \mathbf{u_0u_1u_2} \ldots$ and $\mathbf{v} = \ldots \mathbf{v_0v_1v_2} \ldots$ instead of $\mathbf{u(D)}$ and $\mathbf{v(D)}$, respectively.

The constraint length [14], [15] for the ith input sequence is defined as

$$\nu_i = \max\{\deg \mathrm{f_{i1}(D)}, \deg \mathrm{f_{i2}(D)}, \ldots, \deg \mathrm{f_{ic}(D)}, \deg \mathrm{q_i(D)}\}, \tag{3}$$

where $g_{ij}(D) = f_{ij}(D)/q_i(D)$, $j = 1,2,\ldots,c$, and $f_{i1}(D), f_{i2}(D), \ldots, f_{ic}(D), q_i(D)$ are polynomials in D and $\gcd(\mathrm{f_{i1}(D)}, \mathrm{f_{i2}(D)}, \ldots, \mathrm{f_{ic}(D)}, \mathrm{q_i(D)}) = 1$.
The overall constraint length ν is defined as the sum of all constraint lengths $\nu = \sum_{i=1}^{b} \nu_i$. The memory m is defined as the maximum of the constraint lengths $m = \max_{i=1,\ldots,b} \nu_i$, and the minimal constraint length is $\nu_{min} = \min_{i=1,\ldots,b} \nu_i$. A rate $R = b/c$ convolutional encoder of a convolutional code with generator matrix $\mathbf{G(D)}$ is a realization of $\mathbf{G(D)}$ as linear sequential circuit.

III. Encoding Schemes

WITH binary woven convolutional codes, several convolutional encoders are combined in such a way that the overall code is a convolutional code. In [1], two possible constructions were introduced, namely woven convolutional codes with outer and inner warp. In this paper we only consider woven encoders with outer warp. We present some new encoder constructions. Contrary to the original proposal, we investigate encoder constructions which lead to overall convolutional codes and to overall block codes.

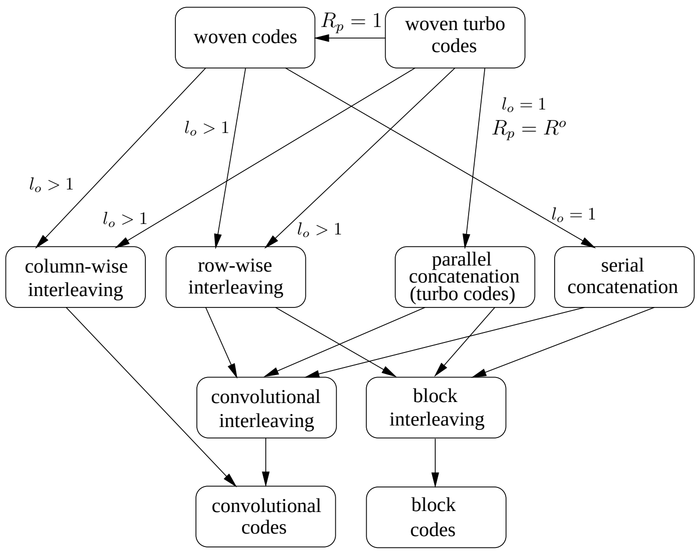

Fig. 1. Overview of woven code constructions.

Serially concatenated convolutional codes with interleavers introduced in [12], as well as parallel concatenated convolutional codes (turbo codes [13]) can be considered as special cases of this new encoder constructions. An overview of possible variations of woven code constructions is illustrated in Figure 1, where l_o denotes the number of outer encoders employed in the woven scheme and the rates R_p and R^o will be defined later on. Below, we present different encoding schemes of woven codes and point out the associations among them.

Woven Convolutional Codes (WCC) [1]: A woven convolutional encoder with outer warp as depicted in Figure 2 consists of l_o outer convolutional encoders which have the same rate $R^o = b^o/c^o$. The information sequence $\mathbf{u}$ is divided into l_o sub-sequences $\mathbf{u}_\mathbf{l}^\mathbf{o}$ with $l = 1, \ldots, l_o$. These sequences are fed into the parallel outer encoders. The outer code sequences $\mathbf{v}_1^o, \ldots, \mathbf{v}_{l_o}^o$ are written row-wise into a buffer of l_o rows. The binary code bits are read column-wise and the resulting sequence constitutes the input sequence $\mathbf{u}^i$ of the single inner rate $R^i = b^i/c^i$ convolutional encoder. The resulting woven convolutional code has overall rate

$$R^{WCC} = R^i R^o \ . \tag{4}$$

Woven Turbo Codes (WTC) [16]: A woven turbo encoder consists of l_o outer convolutional encoders and one inner convolutional encoder (see Figure 3). The information sequence $\mathbf{u}$ is subdivided into l_o sequences which are the input sequences to the l_o rate $R^o = b^o/c^o$ outer encoders. Parts of the symbols of the outer code sequences ($\mathbf{v}_\mathbf{l}^{\mathbf{o},(\mathbf{1})}$) which are located on the same bit positions are multiplexed to the sequence $\mathbf{u}^\mathbf{i}$. The sequence $\mathbf{u}^\mathbf{i}$ is the input sequence of the inner encoder. The other symbols of the outer code sequences ($\mathbf{v}_\mathbf{l}^{\mathbf{o},(\mathbf{2})}$, dashed lines in Figure 3) are not encoded by the inner encoder. These sequences $\mathbf{v}_\mathbf{l}^{\mathbf{o},(\mathbf{2})}$ form the sequence $\mathbf{v}^{\mathbf{o},(\mathbf{2})}$ which together with the inner code sequence $\mathbf{v}^\mathbf{i}$ constitute the overall code sequence $\mathbf{v}$.

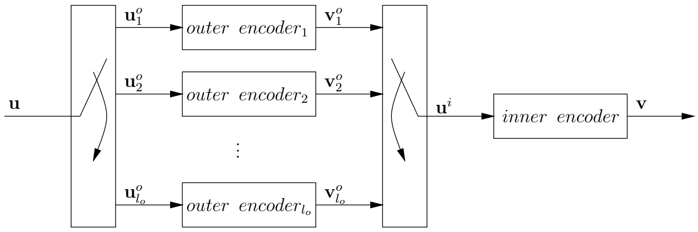

Fig. 2. Woven encoder with outer warp.

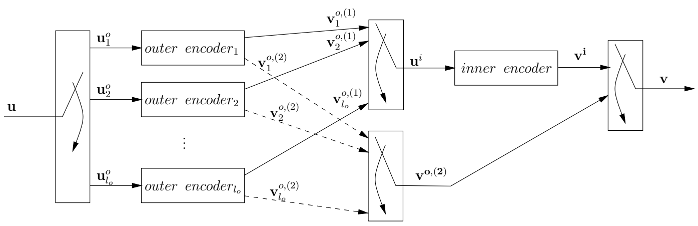

Fig. 3. Woven turbo encoder.

Similar to the puncturing of convolutional codes we describe the partitioning of the outer code sequences into two so-called partial code sequences $\mathbf{v}_\mathbf{l}^{\mathbf{o},(\mathbf{1})}$ and $\mathbf{v}_\mathbf{l}^{\mathbf{o},(\mathbf{2})}$ by means of a partitioning matrix $\mathbf{P}$. Consider a rate $R^o = b^o/c^o$ outer convolutional code. $\mathbf{P}$ is a $c^o \times k_p$ matrix with matrix elements $p_{ij} \in \{0,1\}$, where $k_p \geq 1$ is any integer. A matrix element $p_{ij} = 1$ means that the corresponding code bit will be mapped to the partial code sequence $\mathbf{v}_\mathbf{l}^{\mathbf{o},(\mathbf{1})}$, while a code bit corresponding to $p_{ij} = 0$ will appear in the partial code sequence $\mathbf{v}_\mathbf{l}^{\mathbf{o},(\mathbf{2})}$. With $b_p = \sum_{i,j} p_{ij}$ - the number of ones in the partitioning matrix - and with the partitioning period $c_p = c^o k_p$ we have

$$R_p = \frac{b_p}{c_p} = \frac{\sum_{i,j} p_{ij}}{c^o k_p}. \tag{5}$$

We call R_p the partial rate, that is the fraction of outer code bits which will be encoded by the inner encoder of a woven turbo encoder. With $R_p = 1$ we have a woven encoder with outer warp as given in Figure 2. Finally, we obtain the rate of the overall woven turbo code

$$R^{WTC} = \frac{R^o R^i}{R_p + R^i(1 - R_p)}, \tag{6}$$

where R^i is the inner code rate.

Interleaving [17], [10]: First of all, we note that if no interleaving is used, both considered encoding schemes result in convolutional codes. When interleaving is employed, one can use either column-wise or row-wise interleaving as indicated in Figure 1. The former leads to overall convolutional codes. We mainly focus our attention to the design of row-wise block interleavers (see Figure 4). Here, each outer code sequence $\mathbf{u_l^o}$ is interleaved by arbitrary and independent interleavers. The interleaved sequences $\tilde{\mathbf{v}}_l^o$ are fed into the inner encoder. Block interleaving can be described by means of a permutation function $\pi(\cdot) : \{1, \ldots, N\} \rightarrow \{1, \ldots, N\}$, which is one-to-one and onto. N defines the interleaver size. We note, that there is also the possibility of using row-wise convolutional interleaving [18]. As a result, no termination is required and the overall woven code remains a convolutional code.

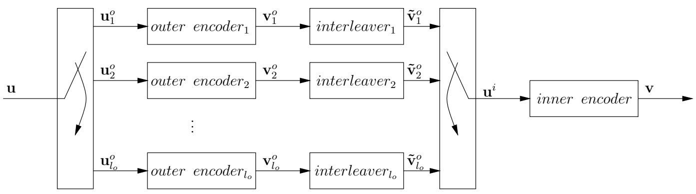

Fig. 4. Woven encoder with row-wise interleaving.

Woven Block Codes (WBC) [17], [19]: Here, we use block interleavers in the woven encoder. We assume that all convolutional encoders generate terminated code sequences. The resulting woven code is a woven block code. The l_o outer information sequences $\mathbf{u_l^o} = \mathbf{u_0 u_1} \ldots \mathbf{u_{K^o-1}}$ are of finite length and consist of K^o information bits. The overall information sequence has length $K^{WBC} = K^o l_o$. The sequences $\mathbf{u_l^o}$ are encoded by the outer encoders and we obtain outer code sequences $\mathbf{v}_l^o$ of length

$$N^o = \frac{K^o + \nu^o}{R^o}, \tag{7}$$

where ν^o denotes the overall constraint length of the outer encoders. The sequences $\tilde{\mathbf{v}}_l^o$ consist of the interleaved code bits of the outer code sequence $\mathbf{v_l^o}$. Let ν^i be the overall constraint length of the inner encoder. Then, we have

$$N^{WBC} = \frac{l_o N^o + \nu^i}{R^i}, \tag{8}$$

for the length of the overall code sequence. Thus, we obtain an overall code rate

$$\begin{aligned} R^{WBC} &= \frac{K^{WBC}}{N^{WBC}} \\ &= R^o R^i \frac{K^{WBC}}{K^{WBC} + l_o \nu^o + R^o \nu^i} \\ &= R^o R^i R_f, \end{aligned} \tag{9}$$

where R_f represents the fractional rate loss due to termination.

In place of outer convolutional encoders we may use encoders for binary block codes. Then each information sequence $\mathbf{u_l^o}$ is sub-divided into M blocks of length k^o. Each block is independently encoded with help of the same generator matrix $\mathbf{G^o}$. We call a codeword encoded by $\mathbf{G^o}$ a *basic codeword*. The sequence $\mathbf{v_l^o}$ consists of M basic codewords of length n^o, i.e. $N^o = Mn^o$ code bits. Then we use interleaving and obtain the output code sequence $\tilde{\mathbf{v}}_l^o$ of the lth row after interleaving the code bits of $\mathbf{v_l^o}$. Using an $N^o \times N^o$ permutation matrix $\mathbf{Z_l}$ to describe the row-wise interleaving we may express the encoding of the lth output sequence as :

$$\tilde{\mathbf{v}}_\mathbf{l}^\mathbf{o} = \mathbf{u_l^o} \left(\mathbf{I_M} \otimes \mathbf{G^o} \right) \cdot \mathbf{Z_l}, \tag{10}$$

where $\mathbf{I_M}$ is an $M \times M$ identity matrix and $\otimes$ denotes the Kronecker product.

Serially Concatenated Codes (SCC) [12]: Serially concatenated encoders consist of a cascade of an outer encoder, an interleaver, and an inner encoder (see Figure 5). Note, that we may consider this construction as a special case of a woven encoder with row-wise interleaving where we choose $l_o = 1$. We may distinguish between serially concatenated convolutional codes (SCCC), if only non-terminated convolutional encoders and convolutional interleavers are used, and serially concatenated block codes (SCBC).

Fig. 5. Serial concatenation with interleaving.

Parallel Concatenated Codes (PCC) [13]: Consider the encoder of a woven turbo code depicted in Figure 3. As mentioned above, additional interleaving can be used. Then if we choose $R_p = R^o$, $l_o = 1$ and use systematic outer and inner encoding we obtain a parallel (turbo) encoder (cf. Figure 1).

IV. Active Distances and Generating Tuples

Definition 1 (Encoder state, encoder state space [15]) The *encoder state* σ of a realization of a rational generator matrix $G(D)$ is the contents of its memory elements. The set S of encoder states is called the *encoder state space*.

If the encoder is realized in controller canonical form, then the dimension μ, with $|S| = 2^\mu$, of the encoder state space is equal to the overall constraint length ν. We consider only encoder realizations in controller canonical form [15]. The denominator polynomials of a generator matrix with degree greater than zero are realized with feedbacks in the encoder. We call convolutional encoders with feedback *recursive* encoders. Encoders without feedback are called *polynomial*, as all elements $g_{ij}(D)$ of the corresponding generator matrix are polynomials in D, i.e. $q_i(D) = 1, \; i = 1, \ldots, b$.

The active distance measures [11] are defined as the minimal weight of a set of code sequence segments $\mathbf{v_{[t_1, t_2-1]}} = \mathbf{v_{t_1} v_{t_1+1} \ldots v_{t_2-1}}$ which is given by a set of encoder state sequences. Let $S_{[t_1,t_2]}^{\sigma_s,\sigma_e}$ denote the set of encoder state sequences $\sigma_{[t_1,t_2]} = \sigma_{t_1}\sigma_{t_1+1} \ldots \sigma_{t_2}$ that start at depth t_1 in some state $\sigma_{t_1} \in \sigma_s$ and terminate at depth t_2 in some state $\sigma_{t_2} \in \sigma_e$ and do not have all-zero state transitions along with all-zero information block weight in between:

$$S^{\sigma_s,\sigma_e}_{[t_1,t_2]} = \{\sigma_{[t_1,t_2]} \mid \sigma_{t_1} \in \sigma_s, \sigma_{t_2} \in \sigma_e \text{ and not}$$
$$\sigma_i = \mathbf{0}, \sigma_{\mathbf{i+1}} = \mathbf{0} \text{ with } \mathbf{u_i} = \mathbf{0}, \, \mathbf{t_1} \leq \mathbf{i} \leq \mathbf{t_2} - \mathbf{1}\}, \quad (11)$$

where $\sigma_\mathbf{s}$ and $\sigma_\mathbf{e}$ denote the sets of possible starting and ending states. This definition of state sequences was presented in [17]. It differs slightly from the original definition presented in [11]. Here, we included all-zero to all-zero state transitions that are not generated by all-zero information blocks in order to consider partial (unit) memory codes.

Definition 2 (Active distances [11]) Let $\mathcal{C}$ be a convolutional code encoded by a rational generator matrix $\mathbf{G}(\mathbf{D})$ with memory m which is realized in controller canonical form.

The *jth order active burst distance* is

$$a^b_j \overset{\text{def}}{=} \min_{S^{0,0}_{[0,j+1]}} \{wt(\mathbf{v}_{[\mathbf{0},\mathbf{j}]})\} \quad , \quad (12)$$

where $j \geq \nu_{min}$ and $wt(\cdot)$ denotes the (Hamming) weight of the sequence.
The *jth order active column distance* is

$$a^c_j \overset{\text{def}}{=} \min_{S^{0,\sigma}_{[0,j+1]}} \{wt(\mathbf{v}_{[\mathbf{0},\mathbf{j}]})\} \quad , \quad (13)$$

where σ denotes any encoder state.
The *jth order active reverse column distance* is

$$a^{rc}_j \overset{\text{def}}{=} \min_{S^{\sigma,\mathbf{0}}_{[0,j+1]}} \{wt(\mathbf{v}_{[\mathbf{0},\mathbf{j}]})\} \quad , \quad (14)$$

where σ denotes any encoder state.
The *jth order active segment distance* is

$$a^s_j \overset{\text{def}}{=} \min_{S^{\sigma_\mathbf{1},\sigma_\mathbf{2}}_{[m,m+j+1]}} \{wt(\mathbf{v}_{[\mathbf{0},\mathbf{j}]})\} \quad , \quad (15)$$

where $\sigma_\mathbf{1}$ and $\sigma_\mathbf{2}$ denote any encoder state.

The active distances are encoder properties, not code properties. For the free distance of a non-catastrophic convolutional encoder we obtain:

$$d_f = \min_j (a^b_j) \quad . \quad (16)$$

In general, all active distances can be lower bounded by linear functions with the same slope α. Therefore, we can write [11]:

$$\begin{aligned} a^b_j &\geq \tilde{a}^b(j) = \alpha \cdot j + \beta^b & (17)\\ a^c_j &\geq \tilde{a}^c(j) = \alpha \cdot j + \beta^c & (18)\\ a^{rc}_j &\geq \tilde{a}^{rc}(j) = \alpha \cdot j + \beta^{rc} & (19)\\ a^s_j &\geq \tilde{a}^s(j) = \alpha \cdot j + \beta^s \quad , & (20)\end{aligned}$$

where $\tilde{a}^*(j)$ denote the lower bounds on the active distances and $\beta^b, \beta^c, \beta^{rc}, \beta^s$ are rational constants.

Below, we consider the generating tuples of an input sequence $\mathbf{u}(\mathbf{D})$ of a convolutional encoder. Moreover, we prove that each generating tuple generates at least d_g many non-zero bits in the encoded sequence $\mathbf{v}(\mathbf{D})$, where d_g is some weight in the region $\beta^b \leq d_g \leq d_f$. In order to prove this result, we have to restrict our considerations to the class of encoders for which $deg(\mathbf{u}(\mathbf{D})) \leq \mathbf{deg}(\mathbf{v}(\mathbf{D}))$ [15].

Definition 3 (Burst) Consider a convolutional encoder and its active burst distance. We call a segment of a convolutional code sequence *burst* if it corresponds to an encoder state sequence starting in the all–zero state and ending in the all–zero state, and having no consecutive all–zero states in between which correspond to an all–zero input tuple of the encoder. A burst of length $j+1$ has at least weight $\tilde{a}^b(j)$.

Let d_g denote some weight and $\beta^b \leq d_g \leq d_f$. We define the generating length for d_g as

$$j_g = \left\lceil \frac{2d_g - \beta^b}{\alpha} \right\rceil, \quad (21)$$

i.e. j_g is the minimum j for which $\tilde{a}^b(j) \geq 2d_g$ holds.

Definition 4 (Generating and neighboring tuples) Let t_1 be the time index of the first non-zero tuple $\mathbf{u_t} = (\mathbf{u}^{(\mathbf{1})}_\mathbf{t}, \mathbf{u}^{(\mathbf{2})}_\mathbf{t}, \ldots, \mathbf{u}^{(\mathbf{b})}_\mathbf{t})$ of a sequence $\mathbf{u}(\mathbf{D})$. Let t_2 be the time index of the first non-zero tuple with $t_2 \geq t_1 + j_g$, and so on. We call the tuples $\mathbf{u}_{\mathbf{t_1}}, \mathbf{u}_{\mathbf{t_2}}, \ldots$ *generating tuples*. All other non-zero tuples are called *neighboring tuples*. Non-zero bits which belong to a generating or neighboring tuple are named *generating bits* or *neighboring bits*, respectively.

We note, that the generating length given by (21) is defined in b-tuples and the corresponding number of bits is equal to bj_g.

Lemma 1: Let $\mathbf{u}(\mathbf{D})$ be the input sequence of a convolutional encoder with N_g generating tuples with generating length j_g. Then the weight of the corresponding code sequence $\mathbf{v}(\mathbf{D})$ satisfies

$$wt(\mathbf{v}(\mathbf{D})) \geq \mathbf{N_g d_g}. \quad (22)$$

Proof: Consider an encoder with j_g according to equation (21). The weight of a burst that is started by a generating tuple of the encoder will be at least d_f if the next generating tuple enters the encoder outside the burst and at least $2d_g$ if the next generating tuple enters the encoder inside the burst. This approach can be generalized. Let N_i denote the number of generating tuples corresponding to the ith burst. For $N_i = 1$ the weight of the ith burst is greater or equal to d_f, which is greater or equal to d_g . The length of the ith burst is at least $(N_i-1)j_g+1$ for $N_i > 1$ and we obtain

$$\begin{aligned} wt(burst_i) &\geq \tilde{a}^b((N_i-1)j_g) \\ &\geq \alpha(N_i-1)j_g + \beta^b. \quad (23)\end{aligned}$$

With equation (21) we have $\alpha j_g \geq 2d_g - \beta^b$ and it follows:

$$\begin{aligned} wt(burst_i) &\geq (N_i-1)(2d_g-\beta^b)+\beta^b \\ &\geq N_i d_g + (N_i-2)(d_g-\beta^b). \end{aligned} \quad (24)$$

Taking into account that $d_g \geq \beta^b$, i.e. $d_g - \beta^b \geq 0$, we obtain

$$wt(burst_i) \geq N_i d_g \;\; \forall \;\; N_i \geq 1. \quad (25)$$

Finally, with $N_g = \sum_i N_i$ we obtain

$$wt(\mathbf{v}(\mathbf{D})) \geq \sum_{\mathbf{i}} \mathbf{wt}(\mathbf{burst_i}) \geq \sum_{\mathbf{i}} \mathbf{N_i d_g} = \mathbf{N_g d_g}. \quad (26)$$

■

Definition 5 (Effective length) Consider a convolutional encoder and its active burst distance. We define the *effective length*:

$$l_{eff} = b\left\lceil \frac{2d_f-\beta^b}{\alpha} \right\rceil \;. \quad (27)$$

Then, it follows from Lemma 1:

Lemma 2: Let $\mathbf{u}(\mathbf{D})$ be the input sequence of a convolutional encoder with N_g generating tuples with generating length $j_g = l_{eff}/b$. Then the weight of the corresponding code sequence $\mathbf{v}(\mathbf{D})$ satisfies

$$wt(\mathbf{v}(\mathbf{D})) \geq \mathbf{N_g d_f}. \quad (28)$$

V. Lower Bounds on the Distances

WOVEN **Convolutional Codes:** Consider the encoder of a woven convolutional code with outer warp depicted in Figure 4. Let d_f^o, d_f^i, d_f^{WCC} denote the free distances of the outer codes, the inner code and the overall woven convolutional code, respectively. Let l_{eff}^i denote the effective length of the inner encoder.

Theorem 1: [17], [8] The free distance of the WCC with $l_o \geq l_{eff}^i$ outer convolutional encoders satisfies the following inequality:

$$d_f^{WCC} \geq d_f^o d_f^i. \quad (29)$$

Proof: Due to the linearity of the considered codes, the free distance of the woven convolutional code is given by the minimal weight of all possible inner code sequence, except the all-zero sequence. If $l_o \geq l_{eff}^i$ holds and one of the outer code sequences $\mathbf{v_l^o}$ is non-zero, then there exist at least d_f^o generating tuples in the inner information sequence which generate a weight greater or equal to d_f^i. Thus, inequality (29) follows from Lemma 2. ■

Note, that similarly to [20] we can also estimate the free distance of a WCC for some values of $l_o < l_{eff}^i$ with the help of Lemma 1.

Consider a burst of the inner encoder. We need at least one non-zero bit to start a burst. Because we assume rational generator matrices, the encoding may be recursive and thus we may require several non-zero input bits to terminate the burst. In order to prove the following theorem, we introduce a new time base and write $\mathbf{u^i} = \mathbf{u_0}, \ldots, \mathbf{u_t}, \ldots$, where t denotes the time index of bits. Let u_1 be non-zero and starting bit of a burst with weight d_f^i. We consider all non-zero input bits $u_1, u_{t_1}, u_{t_2}, \ldots$ corresponding to this burst and define a set $\mathcal{T}=\{t_0 = 1, t_1, t_2, \ldots\}$ of their time indices.

Theorem 2: The free distance of the WCC with $l_o \geq l_{eff}^i$ identical outer convolutional encoders and identical interleavers satisfies:

$$d_f^{WCC} = d_f^o d_f^i. \quad (30)$$

Proof: Consider the warp of the woven encoder. Without loss of generality we assume that the first outer codeword $\mathbf{v_1^o}$ has weight d_f^o. The corresponding information sequence is $\mathbf{u_1^o}$. For each outer information sequence $\mathbf{u_l^o}, \mathbf{l} \in \mathcal{T}$ we assume that $\mathbf{u_l^o} = \mathbf{u_1^o}$ and all other outer information sequences are zero.

Due to the fact that all outer encoders and all interleavers are identical, all non-zero codewords contain zeros and ones in the same positions. Therefore, there exist d_f^o generating tuples in the inner input sequence, each generating tuple corresponds to a burst of weight d_f^i and the weight of the output sequence is $d_f^o d_f^i$. Thus, the free distance of the woven code cannot exceed $d_f^o d_f^i$. ■

Woven Block Codes: Consider the encoder of a woven block code (see Figure 4), where we employ terminated convolutional codes or sequences of binary block codes as outer codes. Let d^o, d^{WBC}, d_f^i denote the minimum distances of the outer codes, the overall woven block code, and the free distance of the inner convolutional code, respectively.

Theorem 3: The minimum distance of the WBC with $l_o \geq l_{eff}^i$ rows of outer block code sequences satisfies the following inequality:

$$d^{WBC} \geq d^o d_f^i, \quad (31)$$

with equality if all outer block codes and all interleavers are identical.

Proof: Theorem 3 immediately follows from Theorem 1 and Theorem 2. ■

Example 1: Let us construct a woven encoder employing l_o outer block codes with generator matrix

$$\mathbf{G_1^o} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

and minimum distance $d^o = 2$. The generator matrix of the inner convolutional code is $\mathbf{G^i}(\mathbf{D}) = (\mathbf{1}, \frac{\mathbf{1+D^2}}{\mathbf{1+D+D^2}})$, i.e. $d_f^i = 5$. The lower bound on the active burst distance of the inner encoder is given by $\tilde{a}^{b,i}(j) = 0.5j + 4$

and we have $l_{eff}^i = 12$. Then, with $l_o = l_{eff}^i = 12$ we obtain the minimum distance $d^{WBC} = 10$ which corresponds to the statement of Theorem 3, as all outer codes are identical. Now, let the warp consist of 12 encoders made up of alternating matrices $\mathbf{G_1^o}$ and $\mathbf{G_2^o}$, where

$$\mathbf{G_2^o} = \mathbf{G_1^o} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

We have verified that this woven code has the minimum distance $d^{WBC} = 12$ which exceeds the bound presented in Theorem 2. With this construction we have used a permutation matrix in every second row of the warp which leads to a slightly improved overall minimum distance. In Section VI we will extend this concept to designed row-wise interleaving in order to improve the lower bound on the minimum distance.

Serially Concatenated Codes: A serially concatenated encoder as presented in [12] consists of a cascade of an outer encoder, an interleaver, and an inner encoder (see Figure 5). We have already mentioned in Section III that this encoder construction can be considered as a special case of a woven encoder if we choose $l_o = 1$ (cf. Figure 4). We will now consider the distance properties of serially concatenated block and convolutional codes. First, we deal with SCBC with randomly chosen interleavers, as given in [12]. Next, we show that the product distance can be achieved if the employed interleaver fulfills some design criteria. Finally, we give the corresponding criterion for convolutional interleavers.

Let d^{SCBC} denote the minimum distances of the SCBC.

Theorem 4: The minimum distance of the SCBC with randomly chosen interleaver satisfies the following inequality:

$$d^{SCBC} \geq \max\left\{ d^i, \tilde{a}^{b,i}\left(\left\lceil \frac{d^o}{b^i} \right\rceil - 1 \right) \right\} \quad , \tag{32}$$

where $\tilde{a}^{b,i}(j)$ is the lower bound on the active burst distance of the inner encoder.

Proof: The minimum distance of the concatenated code is given by the minimal weight of all possible inner code sequences, except the all-zero sequence. A non-zero outer code sequence has at least weight d^o. The d^o non-zero bits may occur in direct sequence after interleaving and therefore may be encoded as a burst of length $\lceil d^o/b^i \rceil$. Such a burst has at least weight $\tilde{a}^{b,i}(\lceil d^o/b^i \rceil - 1)$. However, the weight of the inner code sequence can not be less then d^i and consequently we get (32). ■

Now we focus on the interleaver structure.

Definition 6 ((l_1, l_2)-interleaver) Let t denote the time index of a bit before interleaving. We consider all possible pairs of indices $(t, t'), t \neq t'$ with $|t - t'| < l_1$. Let $\pi(t)$ and $\pi(t')$ denote the corresponding indices after interleaving. We call an interleaver (l_1, l_2)*-interleaver* if for all such pairs (t, t') it satisfies:

$$|\pi(t) - \pi(t')| \geq l_2. \tag{33}$$

Theorem 5: For a given pair (l_1, l_2) there exist (l_1, l_2)-block interleavers with size

$$N \geq l_1 l_2, \tag{34}$$

and (l_1, l_2)-block interleavers with size

$$N < l_1 l_2 - \left\lfloor \frac{l_2 - 1}{l_1} \right\rfloor \tag{35}$$

do not exist.

Proof: First, consider the function

$$\pi(t) = t \cdot q \bmod (N+1), \quad q \not| \; N+1 \tag{36}$$

where t, q, N are integers, and $t = 1, \ldots, N$. As q does not divide $N + 1$, the function $\pi(t)$ is a permutation which defines the interleaving. Let q be from the region

$$l_2 \leq q \leq \frac{N}{l_1}. \tag{37}$$

Then N obviously satisfies (34). We have $\pi(t) - \pi(t') = q(t - t') \bmod (N + 1)$. With $1 \leq |t - t'| < l_1$ and (37) it follows $l_2 \leq |q(t - t')| \leq N$. Applying the modulo operation given by (36) we obtain $l_2 \leq |\pi(t) - \pi(t')|$. Thus, such an interleaver is an (l_1, l_2)-interleaver according to Definition 6, which proves the first assertion.

Next, assume that $\pi(\cdot)$ defines an (l_1, l_2)-block interleaver. We consider sets $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_l = \{t_1^{(l)}, \ldots, t_{l_1}^{(l)}\}, \ldots$ of l_1 time indices such that no two elements of $\mathcal{T}_l$ have temporal distance greater than $l_1 - 1$. Consider the corresponding set $\pi(\mathcal{T}_l) = \{\pi(t_1^{(l)}), \ldots, \pi(t_{l_1}^{(l)})\}$. As the function $\pi(\cdot)$ is one-to-one and onto, we may assume without loss of generality $1 \leq \pi(t_1^{(l)}) < \ldots < \pi(t_{l_1}^{(l)}) \leq N$.

With (33) we get

$$\begin{array}{ccc} 1 & \leq & \pi(t_1^{(l)}), \\ l_2 + 1 & \leq & \pi(t_2^{(l)}), \\ & \vdots & \\ (l_1 - 1)l_2 + 1 & \leq & \pi(t_{l_1}^{(l)}) \leq N. \end{array}$$

From $N \geq (l_1 - 1)l_2 + 1$ we conclude that there must exist $l_2 - \lfloor (l_2 - 1)/l_1 \rfloor$ disjoint sets $\mathcal{T}_1, \ldots, \mathcal{T}_{l_2 - \lfloor (l_2-1)/l_1 \rfloor}$. One of these sets satisfies

$$\begin{array}{ccc} l_2 - \lfloor (l_2 - 1)/l_1 \rfloor & \leq & \pi(t_1^{(l)}), \\ & \vdots & \\ (l_1 - 1)l_2 + l_2 - \lfloor (l_2 - 1)/l_1 \rfloor & \leq & \pi(t_{l_1}^{(l)}) \leq N. \end{array}$$

Finally, we have $N \geq l_1 l_2 - \lfloor (l_2 - 1)/l_1 \rfloor$ if $\pi(\cdot)$ is a permutation function of an (l_1, l_2)-block interleaver. ■

Definition 7 (Minimum length [21]) Consider a rate $R = b/c$ convolutional encoder and its active distances. Let j_s denote the minimum j for which $j_s = \min_j\{j \mid \tilde{a}^s(j) \geq d_f\}$ holds. Let j_c denote the minimum j for which $j_c = \min_j\{j \mid \tilde{a}^c(j) \geq d_f\}$ holds. Let j_{rc} denote the minimum j for which $j_{rc} = \min_j\{j \mid \tilde{a}^{rc}(j) \geq d_f\}$ holds. We define the *minimum segment length*:

$$l^s_{min} = c(j_s + 1) \tag{38}$$

and the *minimum column length*:

$$l^c_{min} = \min(c(j_c + 1), c(j_{rc} + 1)) \ . \tag{39}$$

Consider a serial concatenation where the employed (l_1, l_2)-block interleaver has size $N^o \geq l_1 l_2$. Let l^i_{eff} be the effective length of the inner encoder and $l^{c,o}_{min}$ be the minimum column length of the outer encoder according to Definition 5 and Definition 7, respectively.

Theorem 6: The minimum distance of the SCBC with an (l_1, l_2)-block interleaver ($l_1 = l^{c,o}_{min}$ and $l_2 = l^i_{eff}$) satisfies the following inequality:

$$d^{SCBC} \geq d^o d^i. \tag{40}$$

Proof: Again, we are looking for the minimum weight sequence among all possible inner code sequences. A terminated non-zero outer code sequence consists of one or more bursts of arbitrary length. The definition of the minimum length ensures that we have to consider at most $l^{c,o}_{min}$ code bits of a burst to obtain a code segment with at least d^o non-zero bits. The definition of the effective length, (33) and Theorem 5 guarantee that $l^{c,o}_{min}$ successive bits in the outer code sequence are sufficiently interleaved to belong to independent generating tuples, but within the interleaver size. Thus, there exist at least d^o generating tuples in each non-zero input sequence to the inner encoder and inequality (40) follows from Lemma 2. ∎

For (l_1, l_2)-convolutional interleavers, results similar to Theorem 5 can be derived [18]. Now, consider SCCC employing convolutional interleavers. Let $l^{s,o}_{min}$ be the minimum segment length of the outer encoder.

Theorem 7: The free distance of the SCCC with an (l_1, l_2)-convolutional interleaver ($l_1 = l^{s,o}_{min}$ and $l_2 = l^i_{eff}$) satisfies the following inequality:

$$d^{SCCC}_f \geq d^o_f d^i_f. \tag{41}$$

Proof is similar to the proof of Theorem 6. We take into consideration that non-terminated non-zero code sequences do not necessarily include a burst.

Example 2: Consider a rate $R^o = 2/3$ outer code with generator matrix

$$\mathbf{G^o(D)} = \begin{pmatrix} 1+D & 1+D & 1 \\ 0 & D & 1+D \end{pmatrix}.$$

The outer code has minimum (free) distance $d^o = 3$ and $\alpha^o = 0.5$, $\beta^{c,o} = 1$, $\beta^{rc,o} = 1.5$, and $\beta^{s,o} = 0$. The inner generator matrix is $\mathbf{G^i(D)} = (\mathbf{1+D^2}, \mathbf{1+D+D^2})$, i.e. $R^i = 1/2$, $d^i = 5$, $\alpha^i = 0.5$, and $\beta^{b,i} = 4$. For an SCBC with randomly chosen interleaver we obtain the minimum distance $d^{SCBC} \geq 5$ according to Theorem 4 regardless of the interleaver size.

Now, on the basis of the same outer and inner generator matrices, we construct an SCBC with a designed permutation satisfying (36). We have: $j^i_b = 12$, $j^o_c = 4$, $j^o_{rc} = 3$, thus we obtain $l^{c,o}_{min} = 12$ and $l^i_{eff} = 12$. The permutation function

$$\pi(t) = 12 \cdot t \mod 145, \forall\ t \in \{1, \ldots, 144\}$$

defines a $(12, 12)$-block interleaver of size $N^o = 144$. Thus, with $R^o = 2/3$ we have 96 corresponding input bits, where we use two bits for termination. We obtain overall code dimension 94. With the inner code we have $N^o = 144$ input bits plus two bits for termination. We have overall length 292 and rate $R^{SCBC} \approx 0.322$. The overall minimum distance satisfies $d^{SCBC} \geq 15$ according to Theorem 6.

We may also construct an SCCC with free distance $d^{SCCC}_f \geq 15$ and rate $R^{SCCC} = 1/3$. However, with $j^o_s = 6$ we obtain the minimum segment length $l^{s,o}_{min} = 21$, and require an $(21, 12)$-convolutional interleaver.

Woven Turbo Codes: Consider a woven turbo encoder depicted in Figure 3. We employ one inner and l_o equal outer codes, where the outer partial code sequences $\mathbf{v^{o,(1)}_1}$ are multiplexed to the inner input sequence $\mathbf{u^i}$. We may also apply interleaving to each outer partial code sequence $\mathbf{v^{o,(1)}_1}$ before multiplexing.

Definition 8 (Partial Distance) Let $\mathbf{v_{[0,j]}}$ denote a burst of length $j+1$. Considering all possible bursts we define the partial distance with respect to a partitioning matrix $\mathbf{P}$ as the minimum weight of the partial code sequence $\mathbf{v^{(2)}_{[0,j]}}$, where we fix the weight of the partial code sequence $\mathbf{v^{(1)}_{[0,j]}}$:

$$d_p(w) \quad = \min_{\mathbf{v_{[0,j]}}, \mathbf{wt}(\mathbf{v^{(1)}_{[0,j]}} = \mathbf{w})} \left\{ wt\left(\mathbf{v^{(2)}_{[0,j]}}\right)\right\}. \tag{42}$$

Example 3: The rate $R = 1/2$ convolutional code with generator matrix $\mathbf{G(D)} = (\mathbf{1+D+D^2}, \mathbf{1+D^2})$ has free distance $d_f = 5$. With partitioning matrix

$$\mathbf{P} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

we obtain the partial distances $d_p(w) = 4, 2, 2, 2, 2, \ldots$ with $w = 2, 3, 4, \ldots$. Note, that $d_p(w = 1)$ does not exist. With partitioning matrix

$$\mathbf{P'} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

we obtain the partial distances $d'_p(w) = 3, 2, 2, 2, 2, \ldots$ with $w = 2, 4, 6, \ldots$. These partial distances exist only for even values of w.

Let $d_p^o(w)$ denote the outer code partial distance. Let d_f^i be the free distance of the inner code and l_{eff}^i denotes the effective length of the inner encoder according to equation (27).

Theorem 8: The free distance of the WTC with $l_o \geq l_{eff}^i$ outer codes satisfies:

$$d_f^{WTC} \geq \min_w \left\{ w \cdot d_f^i + d_p^o(w) \right\} . \tag{43}$$

Proof: Due to the linearity of the considered codes, the free distance of the woven turbo code is given by the minimal weight of all possible overall code sequences $\mathbf{v}$, except the all-zero sequence. If $l_o \geq l_{eff}^i$ holds and one of the outer partial code sequences $\mathbf{v_l^{o,(1)}}$ has weight w then there exist at least w generating tuples in the inner information sequence. Thus, with Lemma 2 we have $wt(\mathbf{v}) \geq \mathbf{w} \cdot \mathbf{d_f^i}$. However, the weight of the corresponding partial code sequence $\mathbf{v_l^{o,(2)}}$ is at least $d_p^o(w)$ and we have $wt(\mathbf{v}) \geq \mathbf{w} \cdot \mathbf{d_f^i} + \mathbf{d_p^o}(\mathbf{w})$. Consequently, we obtain (43). ∎

We note, that when block interleaving and termination is applied one can prove results similar to Theorem 8, where the overall free distance is replaced by the overall minimum distance.

Example 4: We construct a WTC with equal inner and outer rate $R^i = R^o = 1/2$ codes with generator matrices $\mathbf{G^i(D)} = \mathbf{G^o(D)} = (\mathbf{1}, \frac{\mathbf{1+D^2}}{\mathbf{1+D+D^2}})$. After outer encoding we apply the partitioning matrix $\mathbf{P}$ as given in Example 3. Then, the overall rate is $R^{WTC} = 1/3$ according to (6). The lower bound on the active burst distance of the inner encoder is given by $\tilde{a}^{b,i}(j) = 0.5j + 4$ and we have $l_{eff}^i = 12$. Thus, with $l_o \geq 12$ outer codes we obtain an overall code with free distance $d_f^{WTC} \geq 14$ according to Theorem 8. Note, that partitioning of the outer code sequences according to $\mathbf{P}'$ from Example 3 would lead to $d_f^{WTC} \geq 13$.

For this example we have verified that with $l_o \geq 24$ outer codes, we can further improve the free distance of the woven turbo code and we obtain $d_f^{WTC} \geq 20$. However, for a rate $R^{TC} = 1/3$ turbo code (TC) with the same inner and outer generator matrices as given above and with a randomly chosen block interleaver we can only guarantee a minimum distance $d^{TC} \geq 7$.

VI. Designed Interleaving

The use of designed interleavers is motivated by the asymptotic coding gain, which for un-quantized channels is given by [22]:

$$G_a = 10 \log(Rd) \quad , \tag{44}$$

where R is the rate and d is the minimum distance of the code. This formula implies that for fixed rates the codes should be constructed with minimum distances as large as possible, in order to ensure efficient performance for high signal to noise ratios (SNR).

In this section we present a woven encoder construction where we use block interleaving with unique permutation functions in each row of the warp and obtain an improved lower bound on the minimum distance [19]. Note, that the derivation of the new bound given below is not valid for constructions with inner polynomial encoders.

In particular we consider only *rational* generator matrices, where the elements $g_{ij}(D) = f_{ij}(D)/q_i(D)$ are rational functions in D and no feedback polynomial $q_i(D)$ has zero degree, i.e. we use inner recursive convolutional encoding. Therefore, we need at least one non-zero information bit to start a burst in the convolutional encoder and one non-zero information bit to terminate this burst.

In the following we consider a similar interleaver construction as one given in [23]. Let $p = N + 1$ be prime and perform all multiplications in GF(p). For each lth row in the warp we use a different interleaver with an unique permutation function:

$$\pi_l(i) = i \cdot u_l, \;\; i \in \{1, \ldots, N\} \quad , \tag{45}$$

where $l \in \{1, \ldots, l_o\}$ and each u_l is a fixed element of $GF(p)$ which satisfies the following condition.

Condition 1:

$$\begin{aligned} u_l &\geq 2, && (46)\\ u_l &\leq \frac{N}{n^o - 1}, && (47)\\ u_l^{-1} &\geq n^o, && (48)\\ |\delta_1 u_l - \delta_2 u_j| &\geq 3, &&\\ \forall\ \delta_1, \delta_2 &\in \{-n^o+1, \ldots, -1, 1, \ldots, n^o-1\} &&\\ \text{and for any} &\ \text{pair } l \neq j;\ \ l, j \in \{1, \ldots, l_o\}. && (49) \end{aligned}$$

Consider a woven encoder employing an inner recursive convolutional encoder with l_{eff}^i according to Definition 5.

Theorem 9: The minimum distance of the WBC with $l_o \geq l_{eff}^i$ rows of outer block codes and interleavers fulfilling Condition 1 satisfies:

$$d^{WBC} \geq (2d^o - 1)d_f^i. \tag{50}$$

Proof: In order to prove Theorem 9 we consider the following cases:

a) There is only one non-zero basic codeword in the warp. Here, we prove that two non-zero bits of this basic codeword are separated by at least one zero bit after interleaving. The resulting code sequence has at least weight $2d^o d_f^i$.

b) There is only one outer non-zero code sequence $\mathbf{v_l^o}$. We prove that if after interleaving two successive bits of this sequence are non-zero, then there must exist at least two non-zero basic codewords in this row. Thus, there are at least $2d^o$ generating tuples.

c) There are two non-zero outer code sequences $\mathbf{v_l^o}, \mathbf{v_j^o}$ and each non-zero sequence contains only one non-zero

basic codeword. Here, we show that there exist at most two non-zero bits in the inner input sequence which are not separated by at least l^i_{eff} zero bits. All other non-zero bits are separated by at least l^i_{eff} zero bits. Therefore, there are at least $2d^o - 1$ generating tuples.
d) There are at least two non-zero outer code sequences $\mathbf{v_l^o}$ and $\mathbf{v_j^o}$. If there exist more then two non-zero bits in the inner input sequence which are not separated by l^i_{eff} zeros then there must be one outer code sequence in the warp with at least two non-zero basic codewords. We obtain at least $2d^o$ generating tuples.
Considering the cases b,c,d, with Lemma 2 the resulting code sequences have at least weight $(2d^o - 1)d^i_f$ which concludes the proof.

a) First, we consider the case when only one basic codeword in the warp is non-zero, i.e. all non-zero bits of this codeword are generating bits. Assume that $\mathbf{v_l^o}$ is non-zero and let i_1, i_2 denote the positions of two non-zero code bits before interleaving. It follows that $n^o - 1 \geq |i_1 - i_2| \geq 1$. Now with $u_l \geq 2$ we have

$$\begin{aligned} |(i_1 - i_2)|u_l &= \\ |\pi_l(i_1) - \pi_l(i_2)| &\geq 2. \end{aligned}$$

Furthermore, with $\frac{N}{n^o-1} \geq u_l$ we have

$$\begin{aligned} |(i_1 - i_2)|u_l &= \\ |\pi_l(i_1) - \pi_l(i_2)| &\leq N. \end{aligned}$$

Therefore, the conditions (46) and (47) guarantee that two successive non-zero bits are separated by at least one zero bit after interleaving.

Now taking into account that $d^i_f \geq \beta^{b,i}$ and that we need at least one non-zero bit to terminate a burst it follows from above discussions that the inner code sequence has weight $2d^o d^i_f$ or more.

b) Now assume that only one outer code sequence is non-zero and after interleaving two successive bits of this sequence are non-zero. Let $\pi_l(i_1)$, $\pi_l(i_2)$ denote the positions of the two bits after interleaving. Then we have

$$\begin{aligned} |\pi_l(i_1) - \pi_l(i_2)| &= \\ |(i_1 - i_2)|u_l &= 1 \end{aligned}$$

and it follows that $|(i_1 - i_2)| = u_l^{-1}$. With $u_l^{-1} \geq n^o$ (formula (48)) we obtain $|(i_1 - i_2)| \geq n^o$, which means that there exit at least two non-zero basic codewords in this row and therefore we have at least $2d^o$ generating tuples.

c) We consider the case when there are two non-zero outer code sequences $\mathbf{v_l^o}, \mathbf{v_j^o}$ and each non-zero sequence contains only one non-zero basic codeword. Let i_1, i_2 denote the positions before interleaving of any two non-zero code bits of the lth and jth row, respectively. If

$$|\pi_l(i_1) - \pi_j(i_2)| \geq 2$$

holds for all possible pairs i_1, i_2, then there exist at least $2d^o$ generating tuples.

Consider $|\pi_l(i_1) - \pi_j(i_2)| \leq 1$, that means that after interleaving the bit at position $\pi_j(i_2)$ might be a neighboring bit to the bit at position $\pi_l(i_1)$, or vice versa. In the following, we prove that if there exist any neighboring bits, then we still have $2d^o - 1$ or more generating tuples.

Assume that two sequences $\mathbf{v_l^o}, \mathbf{v_j^o}$ are non-zero and the bits corresponding to i_1, i_2 are neighboring after interleaving. We obtain

$$\begin{aligned} \pi_l(i_1) - \pi_j(i_2) &= \\ i_1 u_l - i_2 u_j &= k, \ \ k \in \{-1, 0, 1\}. \end{aligned}$$

Now we consider all other bits which may be non-zero, i.e. all bits which might belong two the same basic codewords as i_1, i_2. All these bits should be separated by at least l^i_{eff} zero bits in the input sequence of the inner encoder. Note, that i_1, i_2 may have any position within the non-zero basic codewords. Therefore, we consider all bits $i'_1 = i_1 + \delta_1$ and $i'_2 = i_2 + \delta_2$ which are separated by at most $n^o - 1$ positions from i_1, i_2. We have

$$\delta_1, \delta_2 \in \{-n^o + 1, \ldots, -1, 1, \ldots, n^o - 1\}$$

and the following inequality should be satisfied:

$$\begin{aligned} |\pi_l(i'_1) - \pi_j(i'_2)| &= \\ |\pi_l(i_1 + \delta_1) - \pi_j(i_2 + \delta_2)| &= \\ |i_1 u_l + \delta_1 u_l - i_2 u_j - \delta_2 u_j| &= \\ |\delta_1 u_l - \delta_2 u_j + k| &\geq 2, \ \ k \in \{-1, 0, 1\} \end{aligned}$$

which is true due to inequality (49).

d) Finally, we consider the case where at least two outer code sequences $\mathbf{v_l^o}$ and $\mathbf{v_j^o}$ are non-zero. If there exist more then two non-zero bits in the inner input sequence which are not separated by l^i_{eff} zeros then there must be one outer code sequence in the warp with at least two non-zero basic codewords. We assume that there exist at least two pairs i_1, i_2 and i'_1, i'_2, both neighboring, such that $\pi_l(i_1) - \pi_j(i_2) = k$ and $\pi_l(i'_1) - \pi_j(i'_2) = k'$ for $k, k' \in \{-1, 0, 1\}$. We obtain for $\delta_1, \delta_2 \in \{-n^o + 1, \ldots, -1, 1, \ldots, n^o - 1\}$ the following condition:

$$\begin{aligned} \pi_l(i'_1) - \pi_j(i'_2) &= \\ \pi_l(i_1 + \delta_1) - \pi_j(i_2 + \delta_2) &= \\ i_1 u_l + \delta_1 u_l - i_2 u_j - \delta_2 u_j &= \\ \delta_1 u_l - \delta_2 u_j + k &= k'. \end{aligned}$$

The last equality contradicts (49). Thus, there must exist one outer code sequence with two non-zero basic codewords and therefore with weight at least $2d^o$. ∎

Example 5: We construct a woven encoder with row-wise interleaving, where we use $\mathbf{G^i(D)} = (\mathbf{1}, \frac{\mathbf{1+D^2}}{\mathbf{1+D+D^2}})$ as inner generator matrix. We employ $l_o = l^i_{eff} = 12$ rows of single parity check codes with

$$\mathbf{G^o} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} .$$

Each row consists of $M = 416$ basic codewords. For the interleavers according to equation (45) we use

$$u_l \in \{7, 10, 17, 23, 26, 29, 37, 40, 43, 49, 55, 61\}$$

which satisfy conditions (46) - (49). The resulting woven code has rate $R = 1/3$ and dimension $K = 9984$. With the minimum distances $d^o = 2$ and $d_f^i = 5$ we obtain $d^{WBC} \geq 15$. For comparison, a serially concatenated block code with the same outer and inner codes, but with a randomly chosen interleaver has minimum distance $d^{SCBC} \geq 5$. The code parameters are summarized in Table I, where K and N are the overall dimension and code length, respectively. The lower bound for the minimum distance d is also presented.

	R^o	R^i	l_o	K	N	d
WBC	2/3	1/2	12	9984	29956	≥ 15
SCBC	2/3	1/2	1	9984	29956	≥ 5

TABLE I
CODE PARAMETERS

Finally, Theorem 9 is also valid if we use outer terminated convolutional codes. We simply replace n^o by $l_{min}^{(c,o)}$, the minimum column length of the outer convolutional codes. The proof is almost identical.

VII. Simulation Results

In the following we investigate the characteristics of the presented constructions using simulations.

Consider SCBC, WBC, TC, and WTC with overall dimension $K = 100$. All employed block interleavers are randomly chosen, except for the WBC case where we use a designed interleaver. For all constructions the inner convolutional component encoders are recursive encoders with $\mathbf{G^i(D)} = (\mathbf{1}, \frac{\mathbf{1+D^2}}{\mathbf{1+D+D^2}})$. For TC and WTC we use $\mathbf{G^o(D)} = (\mathbf{1}, \frac{\mathbf{1+D^2}}{\mathbf{1+D+D^2}})$ and partitioning matrix $\mathbf{P}$ as given in Example 3. With WBC and SCBC we employ $\mathbf{G^o(D)} = \begin{pmatrix} 1+D & 1+D & 1 \\ 0 & D & 1+D \end{pmatrix}$. For WTC and WBC we have $l_o = 12$. The overall codes have rates $R \approx 1/3$, due to the fractional rate loss. Using the lower bounds derived in the previous sections we get the minimum distances: $d^{SCBC} \geq 7$, $d^{WBC} \geq 15$, $d^{TC} \geq 7$, and $d^{WTC} \geq 14$. We can also estimate the actual minimum distance of each construction by searching for low weight codewords. For WTC and TC we encode all possible information sequences of weight two and store the minimum weight of the corresponding code sequences. For WBC and SCBC we randomly generate information sequences of weight one, two, and three. It is obvious that the actual minimum distance can be upper bounded by the lowest weight found. In Figure 6 we depict the simulation results, where we have investigated 1000 different interleavers per construction. For almost 90 percent of the considered turbo codes we get $d^{TC} \leq 10$ and we found one example with lowest weight equal to 7. However, with woven turbo codes the lowest weight found was 15 and, therefore, $d^{WTC} \leq 15$. For SCBC and WBC with worst lowest weight we have $d^{SCBC} \leq 7$ and $d^{WBC} \leq 17$. For $K = 1000$ we have investigated 100 different interleavers per construction and obtained: $d^{WTC} \leq 18$, $d^{TC} \leq 10$, $d^{WBC} \leq 17$, and $d^{SCBC} \leq 9$. Thus, for these examples we observe only small differences between analytical lower bounds and upper bounds obtained by simulation.

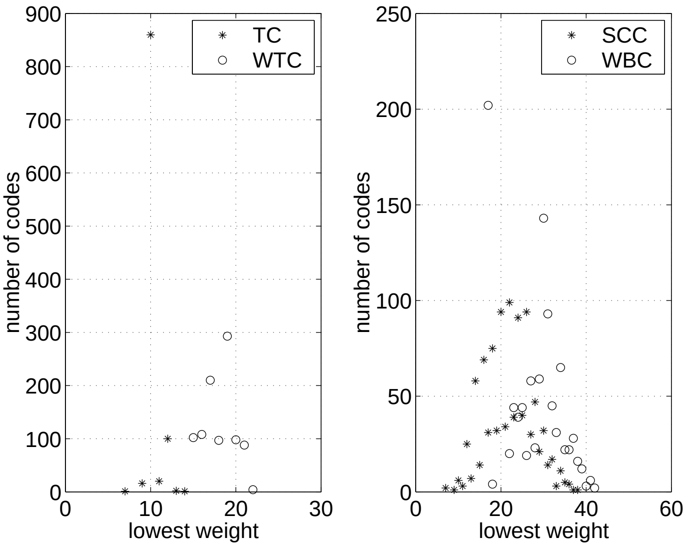

Fig. 6. Distribution of lowest weight codewords.

Now, we give some simulation results for the additive white Gaussian noise (AWGN) channel with binary phase-shift keying. We employ an iterative decoding procedure [24]. For the decoding of the component codes we use the sub-optimum symbol-by-symbol a-posteriori probability algorithm [24], which is a variant of the BCJR algorithm [25]. All results are obtained for 10 iterations of iterative decoding. First, we consider the WBC and SCBC given in Example 5. The bit error rates (BER) (see Figure 7) for the SCBC employing pseudorandom interleavers represent the average performance of 100 different interleavers. Next, we consider TC and WTC constructions as given above in this section, but with overall dimension $K = 1000$. Simulation results are depicted in Figure 8, where the left hand figure presents bit error rates and the right hand figure presents block (or word) error rates (WER) for blocks of 1000 information bits.

One can argue that:

- the minimum (or free) distance determines only the asymptotic performance of the code and plays a key role only for very high signal to noise ratios, i.e. at BER which are lower than the target BER of many practical systems;
- at low and moderate BER at which many systems operate the distance spectrum and corresponding error coefficients are more important than the minimum distance of sufficiently large woven (woven turbo) codes

when iterative decoding is employed;
- the error coefficients are optimized by using a pseudorandom interleaver between component codes and the designed interleaver generally introduces structure to the interleaver and thus destroys the very randomness that results in such excellent performance.

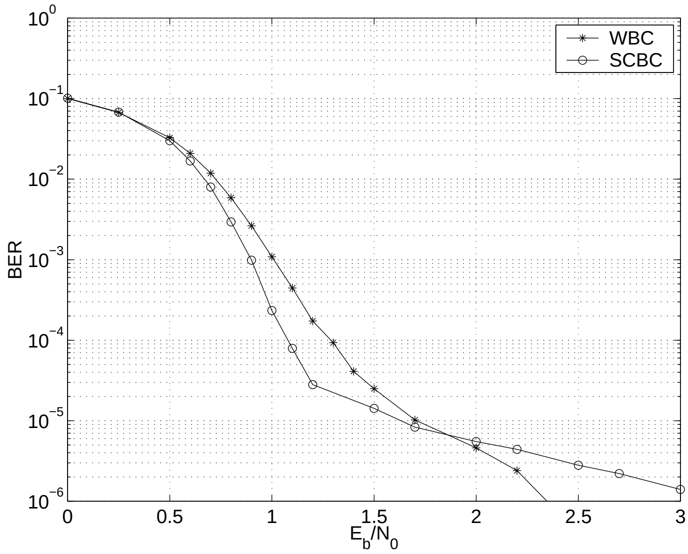

Fig. 7. Simulation results for SCBC and WBC.

The simulation results in Figure 7 show that at least for this particular example the WBC with designed interleaver performs almost as well as the comparable SCBC with pseudorandom interleaver at low and moderate BER, but it is much better suited to provide near-error-free performance. Besides, the WBC starts to outperform the SCBC at a realistic BER. Similarly, WTC possessing less randomness but with higher minimum distance has the same performance at low and moderate BER compared to TC. The performance becomes better at high SNR. Furthermore, the WTC outperforms the TC for the whole considered region of WER and achieves significantly better performance at high SNR.

VIII. Conclusions

We have presented encoding schemes for woven codes with outer warp and in this context we have considered new and old constructions. With the help of active distances we derived lower bounds on the minimum (or free) distances of the considered codes. Crucial was the introduction of designed interleavers which lead to substantially improved lower bounds compared with pseudorandom interleaving. For particular examples, the obtained lower bounds have been compared with the corresponding upper bounds obtained by simulation. The observed differences between theoretical lower bounds and simulation results were small. It was also demonstrated that higher minimum distances are obtainable with the new woven and woven

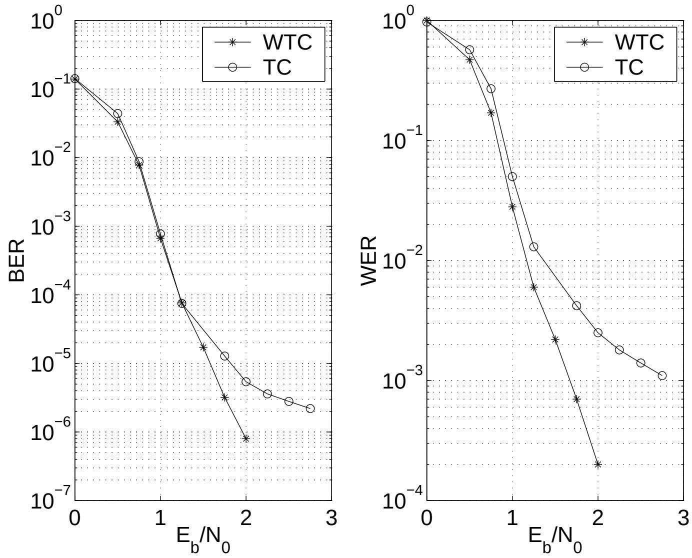

Fig. 8. Simulation results for TC and WTC.

turbo code constructions compared with the traditional parallel and serially concatenated codes.

The comparison by means of simulation in AWGN channels shows that for these particular examples the woven code with designed interleaving performs almost as well as comparable serially concatenated codes with pseudorandom interleaving at low and moderate BER when iterative decoding is employed. The former becomes much better for high signal to noise ratios, but still within BER regions of practical interest. Similarly, a woven turbo code outperforms the comparable turbo code for high SNR with respect to the BER. However, considering the WER we observe better performance throughout the complete simulated SNR region. On the other hand, we are aware that although the minimum distance remains the important criterion for construction of woven codes, it would be desirable to characterize the complete distance spectrum of such codes. This would allow for a prediction of the absolute performance of woven codes. We consider this topic as the subject of future investigations.

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