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Bounds on the Range of Multivariate Rational Functions

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By utilising the expansion of a polynomial into Bernstein polynomials, we construct bounds for the range of a multivariate rational function over a box.

1 Introduction

Computing good quality bounds for a given function is of great importance in, e.g., global optimization when a branch and bound approach is used. Tight bounds for multivariate polynomials over a box can be constructed by utilising the coefficients of the expansion of the given polynomial into Bernstein polynomials. This approach is now a well established tool which has been applied to a variety of problems, cf. [2]. In this paper we show how this expansion can be employed to construct bounds for the range of a multivariate rational function over a box. The application to the problem of enclosing the solution set of a parametric system of linear equations, see [1], is delegated to a follow-up paper. Therein applications to the verified solution of some finite element models for truss structures will also be presented.

2 Bernstein Expansion

Comparisons and the arithmetic operations on multiindices $i = (i_1, \ldots, i_n)^T$ are defined componentwise. For $x \in \mathbf{R}^n$ its monomials are $x^i := x_1^{i_1} \ldots x_n^{i_n}$. Using the compact notation $\sum_{i=0}^{l} := \sum_{i_1=0}^{l_1} \ldots \sum_{i_n=0}^{l_n}$, $\binom{l}{i} := \prod_{\mu=1}^{n} \binom{l_\mu}{i_\mu}$, an n-variate polynomial p, $p(x) = \sum_{i=0}^{l} a_i x^i$, $x \in I = [0,1]^n$, can be represented as $p(x) = \sum_{i=0}^{l} b_i B_i(x)$, where $B_i(x) = \binom{l}{i} x^i (1-x)^{l-i}$ is the ith Bernstein polynomial of degree l, and the so-called *Bernstein coefficients* b_i are given by $b_i = \sum_{j=0}^{i} \frac{\binom{i}{j}}{\binom{l}{j}} a_j$, $0 \leq i \leq l$. Without loss of generality we may consider here the unit box I since any bounded box of $\mathbf{R}^n$ can be mapped thereupon by an affine transformation. In particular, we have the *endpoint interpolation property*

$$b_i \;=\; p(\frac{i}{l}), \quad \text{for all } \; i = 0, \ldots, l \; \text{ with } \; i_\mu \in \{0, l_\mu\}\,, \quad \mu = 1, \ldots, n. \tag{1}$$

For an efficient computation of the Bernstein coefficients, see [7].

A fundamental property for our approach is the *interval enclosing property*

$$\min_{i=0,\ldots,l} b_i \;\leq p(x) \leq \max_{i=0,\ldots,l} b_i, \quad \text{for all } \; x \in I. \tag{2}$$

A disadvantage of the direct use of (2) is that the number of the Bernstein coefficients to be computed explicitly grows exponentially with the number of variables n. Therefore, we use a method [6] by which the number of coefficients which are needed for the enclosure only grows approximately linearly with the number of the terms of the polynomial.

3 Interval Arithmetic

Let $\mathbf{IR}$ denote the set of the compact, nonempty real intervals. The arithmetic operation $\circ \in \{+, -, \cdot, /\}$ on $\mathbf{IR}$ is defined in the following way. If $a = [\underline{a}, \overline{a}], b = [\underline{b}, \overline{b}] \in \mathbf{IR}$, then

$$\begin{aligned}
a + b &= [\underline{a} + \underline{b}, \overline{a} + \overline{b}], \\
a - b &= [\underline{a} - \overline{b}, \overline{a} - \underline{b}], \\
a \cdot b &= [\min\{\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}\}, \max\{\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}\}], \\
a \,/\, b &= [\min\{\underline{a}/\underline{b}, \underline{a}/\overline{b}, \overline{a}/\underline{b}, \overline{a}/\overline{b}\}, \max\{\underline{a}/\underline{b}, \underline{a}/\overline{b}, \overline{a}/\underline{b}, \overline{a}/\overline{b}\}], \;\; \text{if } \; 0 \notin b.
\end{aligned}$$

Details on interval arithmetic and methods which rely on it which can be used to bound the range of an arbitrary function may be found in [3].

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4 Bounding the Range of a Rational Function

In this section we utilise the above polynomial bounding technique for the construction of an enclosure of the range over I of a multivariate rational function $f = p/q$, where p and q are polynomials in $x = (x_1, \ldots, x_n)$. A simple way is to divide the Bernstein enclosures P and Q of the ranges of the polynomials p and q, respectively. This method, termed the *naive method* below, neglects the dependency between the variables of both polynomials and may therefore result in gross overestimation in the range of f.

Following a suggestion by Arnold Neumaier [5], we represent f in the following form

$$f(x) \;=\; \frac{p(x)}{q(x)} \;=\; r(x) + \frac{p(x) - r(x)q(x)}{q(x)}. \tag{3}$$

where r is a linear approximation to f. In our approach we use for r the linear least squares approximation of the control points $(i/l, b_i(p)/b_i(q)) = (i/l, f(i/l))$ associated with the vertices of I, see (1). Here $b_i(p)$ and $b_i(q)$ denote the Bernstein coefficients of p and q, respectively. The advantages of the representation (3) is that the range of r over I can be given exactly and that the Bernstein enclosure of the range of $p - rq$ over I is often tighter than the Bernstein enclosure P. As in the naive method, we employ the Bernstein enclosure Q.

4.1 Examples [4]

Let f be given by

$$f \;:=\; \frac{a(w^2 + x^2 - y^2 - z^2) + 2b(xy - wz) + 2c(xz + wy)}{w^2 + x^2 + y^2 + z^2},$$

where

$$a \in [7, 9],\; b \in [-1, 1],\; c \in [-1, 1], w \in [-0.9, -0.6],\; x \in [-0.1, 0.2],\; y \in [0.3, 0.7],\; z \in [-0.2, 0.1].$$

The range is $[-2.9560\ldots, 8.0093\ldots]$. The naive method gives $[-5.9111\ldots, 16.8222\ldots]$, whereas the method based on (3) yields $[-5.4356\ldots, 10.9532\ldots]$. A related problem is to find the range of

$$g \;:=\; \frac{2(xz + wy)}{w^2 + x^2 + y^2 + z^2},$$

where the intervals for w, x, y, z are as given above. The lower endpoint of the range is -1, and the upper endpoint about -0.5263. The naive method gives $[-2.9777\ldots, -0.2370\ldots]$, whereas the new method yields $[-1.3301\ldots, -0.4250\ldots]$. Since the naive method also provides a valid enclosure we may intersect both enclosures, however this gives no improvement in our examples. It is remarkable that even with 100 subdivision steps the enclosures obtained by (3) can be improved only marginally. This corresponds to a related observation reported in [4]. More advanced examples which document the superiority of the new method over the naive method will be given in the follow-up paper (see the Introduction).

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