A Method for the Verified Solution of Finite Element Models with Uncertain Node Locations

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ABSTRACT: Many problems in structural mechanics are solved using the finite element method (FEM), wherein a model for a mechanical system is constructed by discretising the structure into a finite set of structural elements, connected at nodes, leading to a system of equations to be solved. In the case of linearised geometric displacement equations and linear elastic material behaviour, a system of linear equations is obtained. There may be uncertainty in some or all of the physical model parameters, caused for example by measurement and fabrication imprecision, round-off errors, and various other kinds of inexact knowledge. Intervals can be used to model such parameters when their values are known to lie within certain bounds. In this case, we obtain a system of equations involving interval parameters. However, a naive solution of this system, using interval arithmetic, will typically lead to a solution with result intervals that are hopelessly wide. Previous work has dealt with models where the parameters for uncertain material values (e.g. the Young's modulus and elements' cross-sectional areas) are intervals. However a mechanical frame or truss structure will typically be constructed so that the node positions, before loading, are only known to a tolerance of several millimetres. In this work, we therefore consider not only uncertain material parameters but also uncertain node locations and correspondingly uncertain element lengths, as well as uncertain loading forces. In our approach, firstly guaranteed starting interval enclosures for the node displacements which are relatively wide are computed. These solution intervals are then iteratively tightenend by performing a monotonicity analysis of all the parameters coupled with a solver for interval systems of linear equations. In this way it is possible to provide tight guaranteed enclosures for the node displacements of the model. A simple truss model is presented by way of illustration.

1 INTRODUCTION

Many sources of uncertainty exist in models for the analysis of structural mechanics problems, including measurement imprecision, manufacturing imperfections, and round-off errors. An uncertain quantity is often assumed to be unknown but bounded, i.e. lower and upper bounds for the parameter can be provided (without assigning any probability distribution). These quantities can therefore be represented by intervals, and interval arithmetic, e.g. [1, 10], can be used to track uncertainties throughout the whole computation, yielding an interval result which is guaranteed to contain the exact result.

The finite element method (FEM) is a frequently used numerical method in structural mechanics. However, its accuracy is affected by discretisation and rounding errors and model and data uncertainty. The source of parametric uncertainty (sometimes also called data uncertainty) is the lack of precise data needed for the analysis. In the FEM, parameters describing the geometry, material, and loads may be uncertain. Parametric uncertainty may result from a lack of knowledge (*epistemic uncertainty* or *reducible uncertainty*), e.g. loads are not exactly known, or an inherent variability (*aleatory uncertainty* or *irreducible uncertainty*) in the parameters, e.g. material parameters are only known to vary within known bounds, cf. [7].

In the case of a problem where some of the physical model parameters are uncertain, the application of the FEM results in a system of linear equations with numerous interval parameters which cannot be solved conventionally – a naive implementation in interval arithmetic typically delivers result intervals that are excessively large. The interval arithmetic approach has been variously adapted to handle parameter uncertainty in the application of the FEM to problems in structural mechanics, e.g. [3, 8, 9, 11, 14]. Most of these papers consider the case of affine parametric dependency. Typically, more advanced models involve polynomial or rational parameter dependencies, in which case the coefficients of the systems of linear equations to be solved are polynomial or rational functions of the parameters. In [5, 16] we present approaches to solve such systems, employing a general-purpose fixed-point iteration using interval arithmetic [12], an efficient method for bounding the range of a multivariate polynomial over a given box based on the expansion of this polynomial into Bernstein polynomials [4, 15], and interval tightening methods. Most of the problems treated in the cited works only exhibit uncertainty in either the material values or the loading forces. The problem that the lengths of the bars of a truss system are uncertain, due to fabrication errors or thermal changes, is considered in [8]. However, in real-life problems, not only the lengths are uncertain but also the positions of the nodes are not exactly known. A statically-determinate problem with uncertain node locations was successfully solved in [16].

The approach presented here permits a structural truss problem where *all* of the physical model parameters are uncertain to be treated. Not only the material values and applied loads, but also the positions of the nodes are assumed to be inexact but bounded and are represented by intervals. The interval solution enclosures obtained for the node displacements can be tightened by considering their monotonicity (where it holds) with respect to each of the parameters.

This paper is organised as follows. The following subsection consists of a brief introduction to interval arithmetic. The main methodology is presented in Section 2, detailing the construction of the interval FEM, the algorithm for monotonicity analysis, and a brief description of the solution of interval systems of equations. A simple example model is presented in detail in Section 3, with the results of the application of the new method. We conclude with some suggestions for continuation of this work. [1]

1.1 *Interval Arithmetic*

Let $\mathbb{IR}$ denote the set of the compact, nonempty real intervals. The arithmetic operation $\circ \in \{+, -, \cdot, /\}$ on $\mathbb{IR}$ is defined in the following way. If $a = [\underline{a}, \overline{a}], b = [\underline{b}, \overline{b}] \in \mathbb{IR}$, then

$$\begin{aligned}
a + b &= [\underline{a} + \underline{b}, \overline{a} + \overline{b}], \\
a - b &= [\underline{a} - \overline{b}, \overline{a} - \underline{b}], \\
a \cdot b &= [\min\{\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}\}, \max\{\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}\}], \\
a \,/\, b &= [\min\{\underline{a}/\underline{b}, \underline{a}/\overline{b}, \overline{a}/\underline{b}, \overline{a}/\overline{b}\}, \\
&\qquad \max\{\underline{a}/\underline{b}, \underline{a}/\overline{b}, \overline{a}/\underline{b}, \overline{a}/\overline{b}\}], \ \text{ if } 0 \notin b.
\end{aligned}$$

As a consequence of these definitions we obtain the inclusion isotonicity of the interval arithmetic operations: If $a_1, b_1 \in \mathbb{IR}$ with $a_1 \subseteq a$ and $b_1 \subseteq b$ then it holds that

$$a_1 \circ b_1 \subseteq a \circ b.$$

Note that some relations known to be true in the set $\mathbb{R}$, e.g. the distributive law, are not valid in $\mathbb{IR}$. Here we have the weaker subdistributive law

$$a \cdot (b + c) \subseteq ab + ac \text{ for } a, b, c \in \mathbb{IR}.$$

By $\mathbb{IR}^n$ and $\mathbb{IR}^{n \times n}$ we denote the set of n-vectors and n-by-n matrices with entries in $\mathbb{IR}$, respectively.

Further details on arithmetic with intervals may be found in [1, 10].

2 VERIFIED SOLUTION METHOD

2.1 *Finite Element Assembly*

The usual FEM [2, 17] proceeds by the assemblage of a single large system of linear equations. For each structural element in the problem (see Figure 1), an *element stiffness matrix* is created, expressed in terms of $\cos\theta$, $\sin\theta$, EA, and L, where

- θ is the angle between the two connected nodes;
- E is the Young's modulus;
- A is the cross-sectional area of the element;
- L is the element's length (the distance between the two nodes).

Since the node locations are uncertain, the angles of the various component elements of the structure are also interval quantities. However, the angles and the element lengths are only implicit interval parameters. Therefore, by means of the following substitutions, we can rearrange each element matrix so that it is expressed only in terms of the explicit interval parameters, viz. EA and the node coordinates, (x_l, y_l)

[1] A preliminary version of this paper was presented at the 81st Annual Meeting of the International Association of Applied Mathematics and Mechanics (GAMM) in Karlsruhe, Germany, March 22–26, 2010.

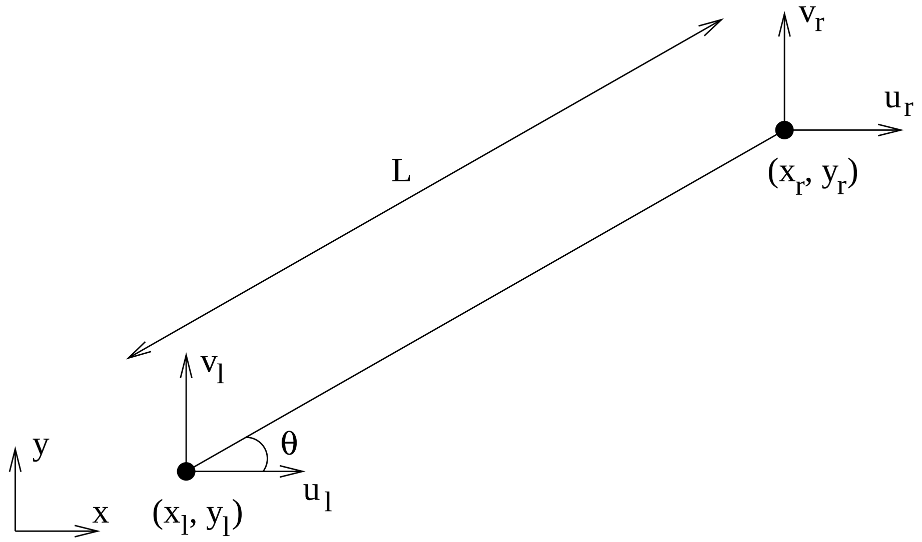

Figure 1: A single element connecting two nodes in a FEM model

and (x_r, y_r):

$$\cos\theta = \frac{x_r - x_l}{L}$$

$$\sin\theta = \frac{y_r - y_l}{L}$$

$$L = \sqrt{(x_r - x_l)^2 + (y_r - y_l)^2}.$$

This yields the element stiffness matrix k as given in Figure 2.

The global system stiffness matrix K is assembled in the usual way, as the aggregation of all component element stiffness matrices, yielding the system of equations

$$Ku = F, \quad (1)$$

where u is the vector of node displacements to be determined and F is the corresponding vector of loading forces. An attempt to solve this system of equations in the conventional fashion, i.e. by substituting each of the variables and parameters by their literal values (in this case, intervals instead of floating-point numbers) can be made using a linear system solver (e.g. interval Gaussian elimination [1, 10]), using interval arithmetic where required. However, this typically results in a solution u whose component intervals are hopelessly wide. It is desired that these result intervals be as tight as possible, whilst still being guaranteed to contain the range of all possible solutions.

2.2 *Solution of interval systems of equations*

There are two general approaches to the solution of the system of equations (1) which might be adopted:

- Computation of the *parametric solution set*: Here, K and F are stored symbolically; their entries are functions of the parameters. The parametric solution set, if computed without overestimation (which is typically not the case), is the "true" solution set to the problem. An iterative method of solution is detailed in [12] and is employed in [5, 16]. As an alternative, global optimisation may be performed, aimed at the minimisation and maximisation of the components of u [6]. The objective functions are the components and the model parameters are the variables. The intervals in which the model parameters vary define the constraints for the variables. The optimisation is performed independently on each component. Therefore, one obtains an interval vector containing the parametric solution set; ideally, it is the tightest interval vector which encloses the parametric solution set. However, the optimisation problem is in general nonconvex. Therefore, this approach requires significant effort and is only suited for models with a small number of parameters.

- Computation of the *interval solution set*: Here, K and F are stored with literal interval entries. In this case, parameter dependencies are lost and the interval solution set is thus a superset of the parametric solution set, and is often considerably wider. However, it is generally easier to compute. Although the classical Gaussian elimination may fail if it is applied to interval matrices, other methods exist which may yield the interval solution set without overestimation, at least for relatively small-sized systems. Here, we employ a solver which proceeds orthant-by-orthant with respect to the solution space and successively computes the interval hull of the intersection of the solution set with each orthant, by the solution of a number of systems of equations corresponding to the vertex values of K and F. It is known that the intersection of the interval solution set with any given orthant is a convex polytope.

In this paper the latter approach is adopted, due to its simplicity. Although the resultant solution for the vector of node displacements u is not sufficiently tight

$$\frac{EA}{((x_r-x_l)^2+(y_r-y_l)^2)^{\frac{3}{2}}} \cdot \begin{pmatrix} (x_r-x_l)^2 & (x_r-x_l)(y_r-y_l) & -(x_r-x_l)^2 & -(x_r-x_l)(y_r-y_l) \\ (x_r-x_l)(y_r-y_l) & (y_r-y_l)^2 & -(x_r-x_l)(y_r-y_l) & -(y_r-y_l)^2 \\ -(x_r-x_l)^2 & -(x_r-x_l)(y_r-y_l) & (x_r-x_l)^2 & (x_r-x_l)(y_r-y_l) \\ -(x_r-x_l)(y_r-y_l) & -(y_r-y_l)^2 & (x_r-x_l)(y_r-y_l) & (y_r-y_l)^2 \end{pmatrix}$$

Figure 2: Element stiffness matrix k

to be an acceptable solution in its own right, it is of sufficient quality to serve as a starting point for the main part of the method which follows.

2.3 *A monotonicity analysis*

Consider the system of equations (1) to be solved, where the entries of K and F may depend upon one or more of the parameters of the problem. Taking the partial derivatives with respect to a single chosen problem parameter p (which may be a material value, loading force, or node co-ordinate) gives

$$K\frac{\partial u}{\partial p} + \frac{\partial K}{\partial p}u = \frac{\partial F}{\partial p}.$$

Rearranging this equation yields

$$K\frac{\partial u}{\partial p} = \left(\frac{\partial F}{\partial p} - \frac{\partial K}{\partial p}u\right). \quad (2)$$

This can be viewed as a system of parametric linear equations where $\frac{\partial u}{\partial p}$ is the unknown quantity. The partial derivatives appearing in the right-hand side can be computed explicitly. Given a suitably tight enclosure for the node displacements u, these systems can be solved for the partial derivatives of the node displacements with respect to each parameter, in turn. It should be noted that since only an outer estimation for u is employed, only an outer estimation for $\frac{\partial u}{\partial p}$ may result. However we are not interested in the exact enclosures for these partial derivatives; it suffices if we can exclude zero and thereby prove monotonicity. In most (but not quite all) cases it seems that the node displacements of a structure do indeed behave monotonically with respect to most or all of the model parameters.

2.4 *Local monotonicity*

In the previous subsection only the monotonicity of u with respect to each parameter over the *entire* parameter domain is considered. Now suppose, e.g., that the interval computed for a certain partial derivative $\frac{\partial u_i}{\partial p_j}$ contains zero, i.e. u_i is not proven to be monotone with respect to p_j over the whole parameter domain. However, it might be that, e.g., $\frac{\partial u_i}{\partial p_k} > 0$ and $\frac{\partial u_i}{\partial p_l} < 0$, where $j \neq k \neq l$. If we wish to find the maximum value that u_i may attain, we can therefore restrict p_k to its maximum value and p_l to its minimum value, and consider the local monotoncity of u_i with respect to p_j over this restricted parameter domain.

This process can be iterated until either local monotonicity of all node displacements is proven with respect to all parameters or there is no further improvement.

2.5 *Algorithm*

The solution procedure consists of the steps detailed below; a similar scheme was used in [13]. The aforementioned orthant-by-orthant interval system solver is used throughout to compute the interval hull of the solution set of a system of linear interval equations.

1. Construct the system of interval equations $Ku = F$. The widths of the intervals appearing in the element stiffness matrices (see Figure 2) can be minimised by optimisation of the interval arithmetic calculations for these particular formulae.

2. Using the interval solver, compute an initial enclosure $u^{\{0\}}$ for the node displacements.

3. Construct systems of interval equations for the partial derivatives of the node displacements with respect to each interval parameter (2), using the current enclosure $u^{\{i\}}$ in place of u in the right-hand side. Using the interval solver, compute outer enclosures for the partial derivatives; where zero is excluded, monotonicity is proven.

4. Attempt to minimise/maximise each solution component in turn by restricting the parameter domain for monotone parameters and thusly reconstructing and solving the original system for the restricted domains.

5. Iterate 3–4, using both successively tighter solution enclosures and monotonicity information obtained so far, until as many of the solution components as possible are found to be monotone over the restricted parameter domains.

3 A SIMPLE EXAMPLE

We consider the simple mechanical truss structure comprising five nodes connected by seven elements as depicted in Figure 3, where the elements are numbered in circles and the coordinates of the nodes are also given. Two of the nodes, 1 and 2, are fixed;

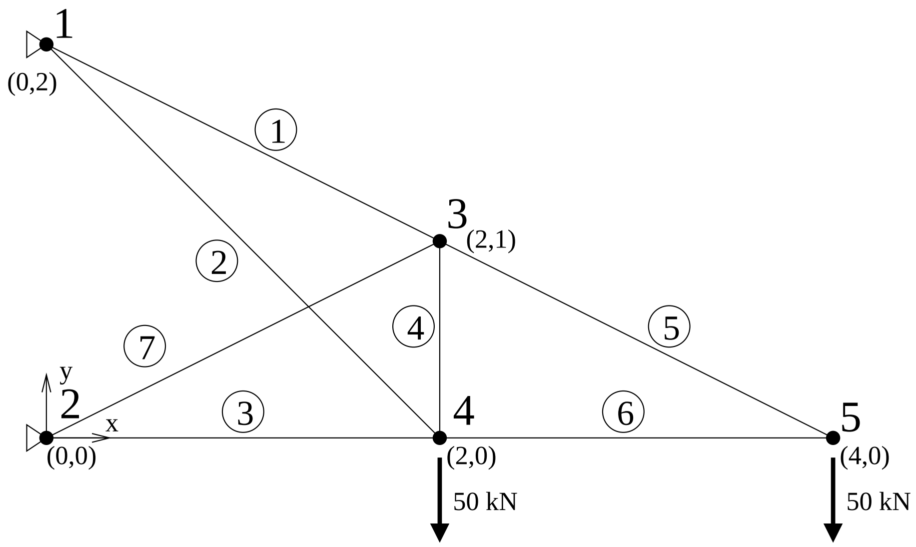

Figure 3: A simple mechanical truss model comprising five nodes and seven elements

Table 1: Interval parameters for the seven-element truss model.

Parameter		**Nominal Value**	**Uncertainty**
Young's modulus $*$ area	EA	422100 kN	± 21105 kN ($\pm 5\%$)
Node coordinates	(x_1, y_1)	$(0, 2)$	± 0.005 m
	(x_2, y_2)	$(0, 0)$	± 0.005 m
	(x_3, y_3)	$(2, 1)$	± 0.005 m
	(x_4, y_4)	$(2, 0)$	± 0.005 m
	(x_5, y_5)	$(4, 0)$	± 0.005 m
Loading forces	F_{x_3}, F_{y_3}	0 kN, 0 kN	± 0 kN
	F_{x_4}, F_{x_5}	0 kN, 0 kN	± 0 kN
	F_{y_4}, F_{y_5}	-50 kN, -50 kN	± 1 kN

the other three are free-moving. A downward loading force of 50kN is separately applied to both nodes 4 and 5. This is adapted from the model in [16] by the addition of an extra element, making it no longer statically-determinate.

Upon loading, we wish to compute the displacements of nodes 3–5, viz. $u_3, v_3, u_4, v_4, u_5, v_5$, and the resultant normal forces in all six elements, $S_1, \ldots, S_6$, (see Conclusion). Each of these is an interval quantity, since the uncertainty in the input data causes uncertainty in the solution. We wish to compute intervals which tightly contain the true ranges of values for each of these variables.

3.1 *Model parameters*

The uncertain parameters of the model are as follows, and are also given in Table 1:

- The positions of the five nodes of the truss (before loading) are subject to an uncertainty of ± 0.005m in both the x- and y-directions. With metres as the coordinate units, this corresponds to a variation of ± 5mm. Correspondingly, the elements are of uncertain length (depending upon configuration, they may vary upto $\pm 10\sqrt{2}$mm).

- The product of the elements' cross-sectional area with the Young's modulus is subject to an uncertainty of $\pm 5\%$. The nominal value is taken as an IPE 160 steel element ($A = 20.1$cm^2, $E = 2.1 * 10^8$kN/m^2). This results in $EA := [400995, 443205]$. Note that there is a single, global EA parameter.

- The two non-zero loading force components are subject to an uncertainty of ± 1kN in the y-direction.

3.2 *Results*

The monotonicity information which is computed as the algorithm proceeds is displayed in Table 2. In somewhat more than half of all cases, a solution component is found to be monotone (over the whole parameter domain) with respect to a particular parameter. Several of the solution components are also locally monotone near the minimum or maximum.

The original system of equations (1) is then solved again 12 times, each over a different restricted parameter domain, so as to compute the minimum and maximum of each solution component in turn. Table 3 shows the results obtained from the interval solver, before and after exploiting the monotonicity tests, and a Monte Carlo simulation with 10^6 runs, for comparison. Point problems chosen randomly from within the parameter domain are used for the Monte Carlo simulation and the interval hull of their solutions is taken. It should be noted that the Monte Carlo result only

Table 2: Monotonicity information; $+$ or $-$ indicates strictly increasing or decreasing over the whole parameter domain; $(+)$ or $(-)$ indicates strictly increasing or decreasing over a subset of the parameter domain where the maximum or minimum occurs.

Parameter:	x_1	y_1	x_2	y_2	x_3	y_3	x_4	y_4	x_5	y_5	F_{y_4}	F_{y_5}	EA
u_3	$-$		$(+)$	$(-)$	$-$	$(-)$		$+$		$+$		$-$	$-$
v_3	$+$	$+$	$+$	$-$							$+$	$+$	$+$
u_4			$+$	$-$		$(+)$			$-$		$+$	$+$	$+$
v_4	$+$	$+$	$+$	$-$		$+$		$-$	$-$	$-$	$+$	$+$	$+$
u_5			$+$	$-$	$+$		$-$				$+$	$+$	$+$
v_5	$+$	$(+)$		$-$	$+$	$+$	$-$		$-$	$+$	$+$	$+$	$+$

Table 3: Enclosures for the node displacements.

	Outer Estimation (Interval Solver)	**Outer Estimation (Monotonicity)**	**Inner Estimation (Monte Carlo)**
u_3	$[0.00018943, 0.00055964]$	$[0.00028110, 0.00038895]$	$[0.00030308, 0.00036239]$
v_3	$[-0.00183366, -0.00058841]$	$[-0.00181882, -0.00059111]$	$[-0.00103362, -0.00086732]$
u_4	$[-0.00119331, -0.00037243]$	$[-0.00097503, -0.00039357]$	$[-0.00066723, -0.00055739]$
v_4	$[-0.00190384, -0.00063106]$	$[-0.00111894, -0.00082202]$	$[-0.00108971, -0.00091329]$
u_5	$[-0.00196654, -0.00071342]$	$[-0.00150969, -0.00082110]$	$[-0.00119584, -0.00098396]$
v_5	$[-0.00869767, -0.00360912]$	$[-0.00577767, -0.00454714]$	$[-0.00556608, -0.00466507]$

serves as an *inner* estimation of the true solution enclosure.

Exploitation of the monotonicity information can thus be seen to yield a significant tightening of the displacement intervals.

4 CONCLUSION

A method for the verified solution of a finite element model which may have uncertain (interval) parameters for node locations, loading forces, and material values has been proposed. The technique relies upon the computation of the interval hulls of the solution sets of systems of linear interval equations, coupled with an analysis of the monotonicity of the solution with respect to its parameters.

A similar procedure can be used for the computation of the normal forces of each structural element, which will be detailed in a forthcoming publication. In the future, we also wish to explore how effectively the method may be applied to truss structures with a greater number of elements and nodes.

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