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THE SPECTRA OF MATRICES HAVING SUMS OF PRINCIPAL MINORS WITH ALTERNATING SIGN*

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Abstract. We present an observation on the localization of the spectrum of a matrix having sums of principal minors with alternating sign.

Key words. principal minors, PN-matrix, P-matrix, eigenvalues

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In the past fifteen years, considerable attention has been paid in the economic literature to the class of the so-called PN-matrices and semi-PN-matrices (see [3], [6], [8, Chap. 7], [9]). A matrix is a (semi-) PN-matrix if every principal minor of odd order is positive, and every principal minor of even order is negative (provided that the order of the minors is greater than 1). We call a real matrix a SPN-matrix if all the sums of its principal minors of odd order are nonnegative and all the sums of its principal minors of even order are nonpositive. The class of the SPN-matrices obviously contains the PN-matrices as well as the nonnegative semi-PN matrices.

Example. Let the $n \times n$ matrix $A = (a_{ij})$ be defined by

$$a_{ij} = \begin{cases} 1 & \text{if } j \geqq i \\ a_j & \text{if } j < i \end{cases} \qquad (i,j = 1, \cdots, n),$$

see [7]. Then the principal minors of order $k+1$ are of the form $(1 - a_{i_1})(1 - a_{i_2}) \cdot \cdots \cdot (1 - a_{i_k})$, where $1 \leqq i_1 < i_2 < \cdots < i_k \leqq n-1$. Thus, A is a PN-matrix and a SPN-matrix if and only if for all k, $a_k > 1$ and $a_k \geqq 1$, respectively.

The purpose of this note is to present an observation on the spectra of SPN-matrices. We note that a matrix A with the sign of the sums of its principal minors of order k equal to $(-1)^k$ or 0 can be transformed to a matrix having nonnegative sums of its principal minors by considering $-A$. Theorems concerning the spectra of such matrices may be found in [1].

Let $n \geqq 2$ and A be an $n \times n$ SPN-matrix. The characteristic polynomial of A is given by

$$p(x) = (-x)^n + s_1(-x)^{n-1} + s_2(-x)^{n-2} + \cdots - s_{n-1}x + s_n, \tag{1}$$

where s_k denotes the sum of the principal minors of order k of A. By definition, we have sign $s_k = (-1)^{k+1}$, $k = 1, \cdots, n$. Without loss of generality we may assume that A is nonsingular since otherwise we can divide $p(x)$ by x^μ, where μ is the multiplicity of the eigenvalue 0, to obtain a polynomial of lower degree whose coefficients have the same sign as the corresponding coefficients of $p(x)$. By using the companion matrix of p and the Perron–Frobenius theorem one obtains that A has a simple positive eigenvalue λ_1, say, equal to its spectral radius (see [4]).

Let the eigenvalues of A which are different from λ_1 be denoted by $\lambda_2, \cdots, \lambda_n$. It is easy to see that λ_2 is negative if $n = 2$. We therefore assume without loss of generality that $n \geqq 3$.

A matrix is called a P-matrix if all its principal minors are positive.

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THEOREM. *Let A be a nonsingular* SPN-*matrix with spectral radius* λ_1. *Then* $-\lambda_2, \cdots, -\lambda_n$ *are the eigenvalues of a* P-*matrix.*

Proof. We divide the characteristic polynomial $p(x)$, cf. (1), of A by $x - \lambda_1$ and denote the resulting polynomial by $p_1(x)$. By the Horner scheme we obtain the following recurrence formula for the coefficients a_i of $p_1(x) = a_0x^{n-1} + a_1x^{n-2} + \cdots + a_{n-1}$

$$\begin{aligned} &a_0 = (-1)^n, \\ &a_k = a_{k-1}\lambda_1 + (-1)^{n-k}s_k, \qquad k = 1, \cdots, n-1. \end{aligned} \tag{2}$$

From the equality $a_{n-1}\lambda_1 + s_n = 0$ we conclude that sign $a_{n-1} = (-1)^n$ and by (2) recursively, sign $a_{n-k} = (-1)^n$, $k = 2, \cdots, n-1$. Hence by Vieta's formula, the kth elementary symmetric function σ_k of the eigenvalues $\lambda_2, \cdots, \lambda_n$,

$$\sigma_k(\lambda_2, \cdots, \lambda_n) = \sum_{2 \leqq i_1 < i_2 < \cdots < i_k \leqq n} \lambda_{i_1} \cdot \cdots \cdot \lambda_{i_k}, \qquad k = 1, \cdots, n-1,$$

has the sign $(-1)^k$. Then $\sigma_k(-\lambda_2, \cdots, -\lambda_n)$ is positive for $k = 1, \cdots, n-1$. By [1, Prop. 4] there exists a P-matrix such that $-\lambda_2, -\lambda_3, \cdots, -\lambda_n$ are the eigenvalues of this matrix. $\square$

This theorem enables the use of the results on the localization of the spectra of P-matrices [1], [2], [5] in order to localize the spectra of SPN-matrices. The most important conclusions are given in the following corollary.

COROLLARY. *Let A be a nonsingular* SPN-*matrix with spectral radius* λ_1. *Then*

(i) $$|\arg \lambda_k| > \frac{\pi}{n-1}, \qquad k = 2, \cdots, n.$$

(ii) *There is at least one eigenvalue with negative real part; if there is exactly one such eigenvalue then*

$$|\arg \lambda_k| > \frac{\pi}{3}, \qquad k = 2, \cdots, n;$$

this bound is independent of n and cannot be improved.

(iii) *If $n > 2m + 3$ and there are exactly $m + 1$ eigenvalues with positive real parts or exactly m eigenvalues with negative real parts then there exists α satisfying*

$$|\arg \lambda_k| > \alpha > \frac{\pi}{n-1}, \qquad k = 2, \cdots, n.$$

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