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Certification of the Sign Regularity of Matrix Intervals

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Abstract. A sign regular matrix is a matrix having the property that its non-zero minors of all orders have, for each order, an identical sign. The most important instances are the totally nonnegative matrices; these are matrices having only nonnegative minors. In this paper, intervals of real matrices with respect to the usual entry-wise partial ordering are considered. The objective is to determine the minimum number of sign regular vertex matrices with an identical signature of their minors from which it can be inferred that all element matrices in the interval are sign regular with the same signature of their minors. Results obtained so far are surveyed and approaches toward resolving a long-standing open problem are discussed.

Keywords. Totally nonnegative matrix, Sign regular matrix, Matrix interval.

1. Introduction

A long-standing conjecture [16][1] was concerned with totally nonnegative (abbreviated TN) matrices, i.e., matrices having all their minors nonnegative, and matrix intervals with respect to the checkerboard ordering. This ordering is obtained from the usual entry-wise ordering by reversing the inequalities in a checkerboard fashion. It was conjectured that if two matrices of the same order are in this relation and nonsingular TN then all matrices between them are nonsingular TN, too. This conjecture was positively answered in [2]. Later on, the nonsingularity assumption was weakened in [1].

This raises the more general question of whether this interval property holds also for matrices with a general signature of their minors. A real $m \times n$ matrix is called *sign regular* (abbreviated *SR*) if all its non-vanishing minors of the same order are all positive or all negative. It is termed *strictly sign regular* (*SSR*) if it is *SR* and all its minors are non-zero. Such matrices arise, e.g., in

[1] See also [14, Section 3.2] and [24, Section 3.2].

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mechanics [15], computer aided geometric design [23], computer vision [21], and dynamic systems [6].
The motivation for considering the interval property of SR matrices stems, e.g., from the investigation of the linear complementarity problem [13]. Often properties of this problem like solvability, uniqueness, convexity, and finite number of solutions are reflected by properties of the constraint matrix (for a large collection of respective matrix classes, see [12]). In passing, we note that also the SR matrices have to be added to these classes [8]. In the case that one considers the linear complementarity problem with uncertain but bounded data modeled by intervals [5,22], it is important to know whether the matrices obtained by choosing all possible values in the intervals are in the same matrix class. Then it is an enormous advantage if one could ascertain this containment by checking a finite set of matrices [7,20] - in the ideal case, from two matrices. Collections of matrix classes which possess such properties can be found in the survey articles [17,18].
In the next section, we introduce the notation and definitions used in the sequel. In Section 3, we present subclasses of the SR matrices which possess this interval property, possibly with a larger finite set of element matrices from which the sign regularity of the entire matrix interval can be inferred. We discuss the question whether two specified element matrices suffice.

2. Notation and definitions

The *signature* of an $[S]SR$ matrix $A \in \mathbb{R}^{m\times n}$ is the ordered tuple $\epsilon = (\epsilon_1, \ldots, \epsilon_{\min\{m,n\}})$, where ϵ_i is the sign of the non-zero determinants of all $i \times i$ submatrices of A, $i = 1, \ldots, \min\{m, n\}$; we set $\epsilon_i := 0$, if there is no non-zero determinant of order i. We simply say that A is $[S]SR(\epsilon)$.
We consider $\mathbb{R}^{m\times n}$ endowed with the usual entry-wise partial ordering, i.e., for $A, B \in \mathbb{R}^{m\times n}$,

$$A \leq B \quad \text{if and only if} \quad a_{ij} \leq b_{ij}, \quad \text{for } i = 1, \ldots, m, j = 1, \ldots, n.$$

Intervals of matrices in $\mathbb{R}^{m\times n}$ with respect to this ordering are denoted by $[A] = [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m\times n} \mid \underline{A} \leq A \leq \overline{A}\}$, with the *corner matrices* $\underline{A} = (\underline{a}_{ij})$ and $\overline{A} = (\overline{a}_{ij})$. The set of all matrix intervals in $\mathbb{R}^{m\times n}$ is denoted by $\mathbb{I}(\mathbb{R}^{m\times n})$. A matrix $A = (a_{ij}) \in [A] = [\underline{A}, \overline{A}]$ with $a_{ij} \in \{\underline{a}_{ij}, \overline{a}_{ij}\}$, for $i = 1, \ldots, m$, $j = 1, \ldots, n$, is called a *vertex matrix*. Of special interest are the following vertex matrices: Each matrix interval $[A] = [\underline{A}, \overline{A}]$ can be represented as $\{A \in \mathbb{R}^{m\times n} \mid |A - A_c| \leq \Delta\}$, where $A_c = \frac{1}{2}(\overline{A} + \underline{A})$ is the *midpoint matrix* and $\Delta = \frac{1}{2}(\overline{A} - \underline{A})$ is the *radius matrix*, in particular, $\underline{A} = A_c - \Delta$ and $\overline{A} = A_c + \Delta$.
With $Y_p = \{y \in \mathbb{R}^p \mid |y_i| = 1, \ i = 1, \ldots, p\}$ and $T_y = \mathrm{diag}(y_1, y_2, \ldots, y_p)$, for integer p, we define the matrices $A_{yz} = A_c - T_y \Delta T_z$ for all $y \in Y_m$, $z \in Y_n$. The definition implies that for all $i = 1, \ldots, m$, $j = 1, \ldots, n$,

$$(A_{yz})_{ij} = (A_c)_{ij} - y_i(\Delta)_{ij} z_j = \begin{cases} \overline{a}_{ij} & \text{if } y_i z_j = -1, \\ \underline{a}_{ij} & \text{if } y_i z_j = 1, \end{cases}$$

so that all matrices A_{yz}, in particular, $\underline{A}$ and $\overline{A}$, are vertex matrices. We denote by $V([A])$ the set of matrices A_{yz} for all $y \in Y_m$, $z \in Y_n$. Since $A_{-y,-z} = A_{yz}$, the cardinality of $V([A])$ is at most 2^{m+n-1}. The vertex matrices which are obtained for $y = (1, -1, 1, \ldots, (-1)^{m+1})$ and $z = (1, -1, 1, ..., (-1)^{n+1})$, $z = (-1, 1, -1, ..., (-1)^n)$, are the matrices

$$\downarrow A = \begin{bmatrix} \underline{a}_{11} & \overline{a}_{12} & \underline{a}_{13} & \cdots \\ \overline{a}_{21} & \underline{a}_{22} & \overline{a}_{23} & \cdots \\ \underline{a}_{31} & \overline{a}_{32} & \underline{a}_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$$\uparrow A = \begin{bmatrix} \overline{a}_{11} & \underline{a}_{12} & \overline{a}_{13} & \cdots \\ \underline{a}_{21} & \overline{a}_{22} & \underline{a}_{23} & \cdots \\ \overline{a}_{31} & \underline{a}_{32} & \overline{a}_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The matrix interval $[A] \in \mathbb{I}(\mathbb{R}^{m\times n})$ can also be represented as a matrix interval with respect to the *checkerboard partial ordering* on $\mathbb{R}^{m\times n}$ given by

$$\begin{aligned} &A \leq^* B \;\text{ if }\; D_m A D_n \leq D_m B D_n, \text{ where} \\ &D_p := \operatorname{diag}(1, -1, 1, \ldots, (-1)^{p+1}), p \in \{m, n\}, \end{aligned}$$

with the matrices $\downarrow A$ and $\uparrow A$ as corner matrices.

3. Sets of vertex matrices from which the sign regularity of a matrix interval can be inferred

Sign regularity is intimately connected with variation diminution [11]. A matrix is variation diminishing if for any n-vector x, the vector Ax has no more sign changes than x. This connection is exploited in [4, Theorem 3.1] to show the following statement.

Theorem 3.1. *Let $[A] \in \mathbb{I}(\mathbb{R}^{m\times n})$ and let $\epsilon = (\epsilon_1, \ldots, \epsilon_{\min\{m,n\}})$ be a fixed signature. Then all matrices in $[A]$ are $SR(\epsilon)$ if and only if all matrices in $V([A])$ are $SR(\epsilon)$. In addition, for $m = n$, all matrices in $[A]$ are nonsingular if and only if the vertex matrices $\downarrow A$ and $\uparrow A$ are nonsingular.*

Since for a nonsingular SR matrix A the inequality $\epsilon_{n-1}\epsilon_n D_n A^{-1} D_n \geq 0$ holds, one can employ a property of inverse-nonnegative matrices to obtain the following statement [16, Theorem 1], see also [10, Theorem B].

Theorem 3.2. *Let $[A] \in \mathbb{I}(\mathbb{R}^{m\times n})$ and $\epsilon = (\epsilon_1, ..., \epsilon_{min\{m,n\}})$ be a fixed signature. Then all matrices in $[A]$ are $SSR(\epsilon)$ if and only if the vertex matrices in $\downarrow A$ and $\uparrow A$ are $SSR(\epsilon)$.*

We note a large gap in the number of vertex matrices required to conclude that all matrices in the interval are SR with a certain signature and possibly additional properties. On one side, we have classes of nonsingular

SR matrices for which the two vertex matrices $\downarrow A$ and $\uparrow A$ suffice. These include the matrices (of order n) which are:

(i) SSR, see Theorem 3.2,

(ii) almost SSR [3, Theorem 5.5] (a class between the nonsingular SR and the SSR matrices),

(iii) tridiagonal and SR [3, Theorem 5.11],

(iv) $SR(\epsilon)$, where $\epsilon = (\epsilon_1, ..., \epsilon_n)$[2] is one of the following eight periodic (of length 4) signatures

$$\begin{array}{ll} (1,1,1,1,\ldots), & (-1,-1,-1,-1,\ldots), \\ (1,-1,1,-1,\ldots), & (-1,1,-1,1,\ldots), \\ (1,1,-1,-1,\ldots), & (-1,-1,1,1,\ldots), \\ (1,-1,-1,1,\ldots), & (-1,1,1,-1,\ldots), \end{array}$$

or of one of the eight signatures obtained from these sequences by reversing the sign of ϵ_n[3] [19].

On the other hand, Theorem 3.1 requires the set $V([A])$ of vertex matrices. A related result is that the classes of the matrices having all their minors up to a certain order either positive [10, Theorem D] or negative [9, Theorem 5.5] require the same set of vertex matrices.
An open question, see [17, Conjecture 3.1], asks, whether for nonsingular SR matrices, the set V of vertex matrices can be further reduced - in the ideal case to the vertex matrices $\downarrow A$ and $\uparrow A$.
For general, not necessarily nonsingular SR matrices, $\downarrow A$ and $\uparrow A$ do not suffice, as the counterexamples for $n = 3$ in [14, Section 3.2] and for $n = 4$ in [16] show.
A reasonable approach to settle the open question is to approximate the SR matrices by the SSR matrices and apply Theorem 3.2. The well-known approximation process [15, Chapter V, Section 5] specializes to the construction of a sequence of $SSR(\epsilon)$ matrices A^r which tend for $r \to \infty$ to a given nonsingular $SR(\epsilon)$ matrix A. If one uses such sequences which tend to $\downarrow A$ and $\uparrow A$, one is faced with the problem to form a sequence of intervals with respect to the checkerboard ordering having members of both sequences as corner matrices. The following theorem shows that this is possible in a special case.

Theorem 3.3. *[16, Theorem 2] Let ϵ be a fixed signature, and let $[A] = [\underline{A}, \overline{A}] \in \mathbb{I}(\mathbb{R}^{m\times n})$ satisfy $\underline{A} < \overline{A}$ or the condition*

either $\forall i,j \in \{1,...,n\} : \underline{a}_{i,j} = \overline{a}_{i,j} \Rightarrow i+j$ *is even,*

[2]possibly with the further requirement that the sign of one coefficient of $\downarrow A$ or $\uparrow A$ is strict, see [19], which may be weakened by perturbation.
[3]As a consequence, all signatures are covered for $n = 4$.

or $\forall i, j \in \{1, ..., n\} : \underline{a}_{i,j} = \overline{a}_{i,j} \Rightarrow i + j$ *is odd.*

Then all $A \in [A]$ *are (nonsingular and)* $SR(\epsilon)$ *if and only if* $\downarrow A$ *and* $\uparrow A$ *are (nonsingular and)* $SR(\epsilon)$.

Other ways to approximate a nonsingular TN matrix by totally positive matrices, i.e., matrices having all their minors positive, include the use of bidiagonal factorization [14, Section 2.5] and the use of the Cauchon algorithm and the restoration algorithm [25, 3.2]. So far, however, both approaches have not yet been employed for the approximation of general nonsingular SR matrices.

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