# Invariance of Hurwitz-Stability of Polynomials of Degree Five under Positive Hadamard Powers

Rola Alseidi* Raoufah Alsaidi† Jürgen Garloff‡

**Abstract**

A complete characterization of the stability-preserving exponent set for fractional Hadamard powers of a monic Hurwitz-stable polynomial $f$ of degree five is presented. The stability problem is reduced to the analysis of a single scalar function depending only on three parameters formed from the coefficients of the polynomial $f$. This reduction leads to a unique stability threshold $p_*(f) \in (0,1)$ such that the $p$-th Hadamard power of $f$ is Hurwitz stable if and only if $p > p_*(f)$. Consequently, the stability-preserving exponent set is precisely $(p_*(f), \infty)$. This threshold depends smoothly on the parameters and provides a global coordinate on the admissible parameter region. Finally, smooth dependence is illustrated by a one-parameter family of Hurwitz-stable polynomials whose complex-conjugate zeros approach the imaginary axis.

## 1 Introduction and Preliminaries

A real polynomial is called (Hurwitz-) stable if all its zeros lie in the open left half of the complex plane. Throughout this paper, we consider monic stable polynomials having only positive coefficients. Their stability is characterized by the Routh–Hurwitz criterion or, equivalently, by the Liénard–Chipart criterion, which requires the positivity of an appropriate subset of the Hurwitz determinants [5, 7, 9].

Let

$$f(x) = \sum_{k=0}^{n} a_k x^{n-k}, \qquad g(x) = \sum_{k=0}^{n} b_k x^{n-k}$$

be two real polynomials. Their *Hadamard product* is defined by coefficientwise multiplication as

$$(f \circ g)(x) = \sum_{k=0}^{n} a_k b_k x^{n-k}.$$

If all coefficients of $f$ are positive, then, for $p > 0$, its $p$-th *Hadamard power* is defined by

$$f^{[p]}(x) = \sum_{k=0}^{n} a_k^p x^{n-k}.$$

Garloff and Wagner proved that the Hadamard product of two stable polynomials is again stable [6]. Consequently, $f^{[p]}$ is stable for every positive integer $p$ whenever $f$ is stable. This preservation property, however, does not extend to fractional exponents.

*Department of Mathematics, Philadelphia University, Amman, Jordan. Email: ralseidi@philadelphia.edu.jo

†Department of Mathematics, Jordan University of Science and Technology, Irbid, Jordan. Email: raofe@just.edu.jo

‡Department of Mathematics and Statistics, University of Konstanz, 78464 Konstanz, Germany; HTWG Konstanz – University of Applied Sciences, Faculty of Computer Science, 78405 Konstanz, Germany. Email: garloff@htwg-konstanz.de

The first negative results for fractional Hadamard powers were obtained by Białas and Białas-Cież, who showed that stability is not preserved for all real exponents $p > 1$ and established sufficient conditions guaranteeing stability for sufficiently large exponents [2]. Later, Białas et al. proved that every stable polynomial of degree at most five remains stable for every exponent $p > 1$, and constructed a degree-six polynomial whose stability-preserving exponent set is not an interval [3].

Related questions have also been studied for Hadamard roots. Białas and Góra investigated the relation between Hadamard factorizability and the stability of Hadamard square roots of polynomials of degree four [4]. More recently, Alsaafin et al. obtained a complete characterization of the Hadamard square root for stable polynomials of degree five [1].

For degree five, so it is known that $f^{[p]}$ is stable for every exponent $p > 1$. But the behavior of fractional Hadamard powers in the interval $0 < p < 1$ remained unknown. The present paper aims at filling this gap.

The analysis developed in this paper is based on a parameterization of the degree-five Hurwitz region, by which the stability problem reduces to the study of a single scalar function depending only on three parameters formed from the polynomial coefficients. This reduction yields a complete characterization of the stability-preserving exponent set for every stable polynomial of degree five.

Throughout this paper, let $H_5$ denote the family of all monic stable polynomials of degree five with positive coefficients. For $f \in H_5$, define

$$\mathcal{P}(f) = \{\, p > 0 : \, f^{[p]} \in H_5 \,\}.$$

Our main result is a complete characterization of the stability-preserving exponent set $\mathcal{P}(f)$ for every polynomial $f \in H_5$.

The characterization is complemented by several structural properties of the stability threshold. In particular, we prove that the threshold depends smoothly on the parameters and provides a global coordinate on the admissible parameter region. We illustrate its behavior by a one-parameter family in $H_5$.

The paper is organized as follows. Section 2 establishes the complete characterization of the stability-preserving exponent set. Section 3 studies the smooth and geometric structure of the stability threshold and illustrates its behavior along a one-parameter family of stable polynomials. Section 4 summarizes the main results and discusses directions for future research.

# 2 Main Results

This section establishes the complete characterization of the stability-preserving exponent set for degree-five stable polynomials. The key step is to reduce the stability conditions to the analysis of a single scalar function.

## 2.1 Reduction to a Scalar Stability Function

Let

$$f(x) = x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5 \in H_5,$$

and, for $p > 0$, let

$$f^{[p]}(x) = x^5 + a_1^p x^4 + a_2^p x^3 + a_3^p x^2 + a_4^p x + a_5^p$$

denote its $p$-th Hadamard power.

We assume that $f$ is stable. By the Liénard–Chipart criterion [9], this is equivalent to the positivity of the second and fourth Hurwitz determinants of $f$, i.e.,

$$\Delta_2(f) = \det \begin{pmatrix} a_1 & a_3 \\ 1 & a_2 \end{pmatrix} > 0,$$

and

$$\Delta_4(f) = \det \begin{pmatrix} a_1 & a_3 & a_5 & 0 \\ 1 & a_2 & a_4 & 0 \\ 0 & a_1 & a_3 & a_5 \\ 0 & 1 & a_2 & a_4 \end{pmatrix} > 0.$$

The first inequality is preserved under Hadamard powers. Introduce the parameters

$$u = \frac{a_3}{a_1 a_2}, \qquad v = \frac{a_2 a_5}{a_3 a_4}, \qquad w = \frac{a_1 a_4}{a_2 a_3}. \tag{1}$$

The positivity of the principal minors of the Hurwitz matrix [8, Theorem 2] implies

$$0 < u, v, w < 1, \tag{2}$$

see also [8, formula (1.10)]. In terms of these parameters, the fourth Hurwitz determinant of $f^{[p]}$ admits the factorization

$$\Delta_4(f^{[p]}) = (a_1 a_2 a_3 a_4)^p \Big[(1-u^p)(1-v^p) - w^p \big(1-(uv)^p\big)^2\Big]. \tag{3}$$

Since $(a_1 a_2 a_3 a_4)^p > 0$, the inequality $\Delta_4(f^{[p]}) > 0$ is equivalent to

$$(1-u^p)(1-v^p) > w^p \big(1-(uv)^p\big)^2.$$

Define

$$h_f(p) = \ln(1-u^p) + \ln(1-v^p) - 2\ln\big(1-(uv)^p\big) - p \ln w. \tag{4}$$

By (2), the logarithms are well defined for every $p > 0$. Therefore, $\Delta_4(f^{[p]}) > 0$ if and only if $h_f(p) > 0$, and the study of stability for the Hadamard powers of $f$ reduces to the analysis of the scalar function $h_f$.

**Lemma 2.1.** *Let $f \in H_5$, and let $h_f$ be defined by* (4)*. Put $u = e^{-\alpha}$ and $v = e^{-\beta}$, where $\alpha, \beta > 0$. Then:*

*(i) The function $h_f$ is strictly concave on $(0, \infty)$,*

*(ii) $h_f(1) > 0$,*

*(iii)*

$$\lim_{p \to 0^+} h_f(p) = \ln\left(\frac{\alpha\beta}{(\alpha+\beta)^2}\right) < 0,$$

*(iv)*

$$\lim_{p \to \infty} h_f(p) = +\infty.$$

*Proof.* *(i)* Differentiation with respect to $p$ gives $p^2 h_f''(p) = -q(\alpha p) - q(\beta p) + 2q((\alpha+\beta)p)$, where $q(x) = x^2 e^x/(e^x - 1)^2$. Moreover, $q'(x)/q(x) = 2/x - \coth(x/2)$. Let $r(x) = x\cosh(x/2) - 2\sinh(x/2)$. Since $r(0) = 0$ and $r'(x) = \frac{x}{2}\sinh(x/2) > 0$ for $x > 0$, it follows that $r(x) > 0$, and hence $2/x < \coth(x/2)$. Therefore $q'(x) < 0$, and so $q$ is strictly decreasing. Since $(\alpha+\beta)p > \alpha p$ and $(\alpha+\beta)p > \beta p$, we obtain $2q((\alpha+\beta)p) < q(\alpha p) + q(\beta p)$, and consequently $h_f''(p) < 0$ for every $p > 0$.

*(ii)* Since $f \in H_5$, one has $\Delta_4(f) > 0$. By the factorization of $\Delta_4$ obtained above, this is equivalent to

$$(1-u)(1-v) - w(1-uv)^2 > 0,$$

and hence $h_f(1) > 0$.

*(iii)* By l'Hospital's rule, we obtain

$$\lim_{p\to 0^+} \ln\left(\frac{1-u^p}{1-(uv)^p}\frac{1-v^p}{1-(uv)^p}\right) = \lim_{p\to 0^+} \ln\left(\frac{\ln u\,\ln v}{u^p v^p \ln^2(uv)}\right)$$
$$= \ln\left(\frac{\ln u\,\ln v}{\ln^2(uv)}\right) = \ln\left(\frac{\alpha\beta}{(\alpha+\beta)^2}\right) < 0.$$

*(iv)* As $p \to \infty$, the first three logarithmic terms in $h_f(p)$ tend to 0, whereas $-p\ln w \to +\infty$ because $0 < w < 1$. Therefore $\lim_{p\to\infty} h_f(p) = +\infty$.

□

**Lemma 2.2.** *For every $f \in H_5$, the equation $h_f(p) = 0$ has a unique solution $p_*(f) \in (0, 1)$.*

*Proof.* Existence follows from Lemma 2.1(ii) and (iii) and the continuity of $h_f$.

Suppose that $h_f$ has two distinct positive zeros $0 < p_1 < p_2$. By the Mean Value Theorem, there exists $\xi \in (p_1, p_2)$ such that $h_f'(\xi) = 0$. Since $h_f$ is strictly concave, $h_f''(\xi) < 0$, and therefore $\xi$ is a local maximum which is also a global maximum. But this contradicts Lemma 2.1(iv).

□

## 2.2 The Stability Threshold

The properties of the stability function $h_f$ derived in Lemmata 2.1 and 2.2 are summarized in the next theorem.

**Theorem 2.3.** *Let $f \in H_5$. Then*

$$\mathcal{P}(f) = \{\, p > 0 : f^{[p]} \in H_5 \,\} = (p_*(f), \infty),$$

*where $p_*(f) \in (0, 1)$ is the unique zero of $h_f$. Moreover, $p_*(f)$ is the unique solution in $(0, 1)$ of the equation*

$$(1-u^p)(1-v^p) = w^p\big(1-(uv)^p\big)^2. \tag{5}$$

**Corollary 2.4.** *For $f \in H_5$, the following are equivalent.*

*(i)* $f^{[1/2]} \in H_5$.

*(ii)* $p_*(f) < \frac{1}{2}$.

*(iii)*
$$(1-\sqrt{u})(1-\sqrt{v}) > \sqrt{w}\,(1-\sqrt{uv})^2.$$

*Proof.* By Theorem 2.3, $f^{[1/2]} \in H_5$ if and only if $p_*(f) < \frac{1}{2}$. On the other hand,

$$h_f\left(\frac{1}{2}\right) = \ln\left(\frac{(1-\sqrt{u})(1-\sqrt{v})}{\sqrt{w}\,(1-\sqrt{uv})^2}\right),$$

so $h_f(1/2) > 0$ is equivalent to

$$(1-\sqrt{u})(1-\sqrt{v}) > \sqrt{w}\,(1-\sqrt{uv})^2.$$

□

*Remark* 2.5. The threshold $p_*(f)$ can be computed by using the strict concavity of $h_f$. We first observe that $h_f'(p) > 0$ for every $p > 0$. Indeed, Lemma 2.1(i) shows that $h_f'$ is strictly decreasing, while

$$\lim_{p\to\infty} h_f'(p) = -\ln w > 0.$$

Thus

$$h_f'(p) > -\ln w > 0, \qquad p > 0.$$

Starting with $p_0 = 1$, define

$$q = p_0 - \frac{h_f(p_0)}{h_f'(p_0)}.$$

Since

$$\frac{d}{dp}\big(ph_f'(p) - h_f(p)\big) = ph_f''(p) < 0$$

and

$$\lim_{p\to\infty}\big(ph_f'(p) - h_f(p)\big) = 0,$$

one has $q > 0$ (note that the linear terms $p \ln w$ cancel). Let $p_1$ be the zero of the secant line through $(q, h_f(q))$ and $(p_0, h_f(p_0))$, that is,

$$p_1 = q - \frac{h_f(q)(p_0 - q)}{h_f(p_0) - h_f(q)}.$$

Strict concavity and the sign properties of $h_f$ imply

$$0 < q < p_*(f) < p_1 < p_0.$$

This step can be iterated to generate a monotonically decreasing sequence $(p_n)_{n\in\mathbb{N}}$ converging to $p_*(f)$, providing at each step an upper bound for it.

We conclude this section with two examples illustrating the threshold characterization.

**Example 2.6.** *For $\varepsilon > 0$, consider the family $f_\varepsilon(x) = (x+\varepsilon)^5$. Its parameters are $u = v = \frac{1}{5}$ and $w = \frac{1}{4}$, independently of $\varepsilon$. The corresponding stability threshold is $p_*(f_\varepsilon) \approx 0.5195$, and therefore $\mathcal{P}(f_\varepsilon) = (0.5195\ldots, \infty)$. Thus, the threshold is independent of $\varepsilon$, even though all zeros converge to the origin as $\varepsilon \to 0^+$.*

**Example 2.7.** *Consider the polynomial from Example 3.9 of [1],*

$$f(x) = x^5 + 15x^4 + 10x^3 + 30x^2 + 10x + 10.$$

*Its zeros are approximately*

$$-14.4485, \qquad -0.0890 \pm 1.2238i, \qquad -0.1868 \pm 0.6518i.$$

*Its parameters are $u = \frac{1}{5}$, $v = \frac{1}{3}$, and $w = \frac{1}{2}$. The corresponding stability threshold is $p_*(f) \approx 0.8449$, and therefore $\mathcal{P}(f) = (0.8449\ldots, \infty)$. Since $p_*(f) > \frac{1}{2}$, the Hadamard square root is not stable. Hence, Corollary 2.4 recovers the conclusion of Example 3.9 in [1].*

## 3 Smooth and Geometric Structure of the Stability Threshold

Theorem 2.3 shows that every polynomial $f \in H_5$ possesses a unique stability threshold $p_*(f) \in (0,1)$, which depends only on the parameters $u, v, w$. It is therefore natural to regard the threshold as a function on the parameter space. For $(u,v,w) \in (0,1)^3$ and $p > 0$, define

$$h(u,v,w,p) = \ln(1-u^p) + \ln(1-v^p) - 2\ln\bigl(1-(uv)^p\bigr) - p\ln w,$$

and let

$$\Omega = \Bigl\{(u,v,w) \in (0,1)^3 : h(u,v,w,1) > 0\Bigr\}.$$

Then $h_f(p) = h(u,v,w,p)$ for the triple associated with $f$. Since $h(u,v,w,1)$ is continuous as a function of $(u,v,w)$, the set $\Omega$ is an open subset of $(0,1)^3$. The arguments of Lemmata 2.1 and 2.2 therefore apply to every $(u,v,w) \in \Omega$. Hence the equation $h(u,v,w,p) = 0$ has a unique solution $p_*(u,v,w) \in (0,1)$, satisfying $h(u,v,w,p) < 0$ for $0 < p < p_*(u,v,w)$ and $h(u,v,w,p) > 0$ for $p > p_*(u,v,w)$. The following results show that the stability threshold depends smoothly on the parameters and provides a global smooth coordinate on the admissible parameter region.

**Theorem 3.1.** *The map*

$$p_* : \Omega \longrightarrow (0,1), \qquad (u,v,w) \longmapsto p_*(u,v,w),$$

*is of class $C^\infty$.*

*Proof.* The function $h : \Omega \times (0,\infty) \to \mathbb{R}$ is of class $C^\infty$. Fix $(u,v,w) \in \Omega$, and let $p_* = p_*(u,v,w)$. Since $p \mapsto h(u,v,w,p)$ is strictly concave, $h(u,v,w,p_*) = 0$, and $h(u,v,w,1) > 0$, it follows that

$$h(u,v,w,1) < h(u,v,w,p_*) + \partial_p h(u,v,w,p_*)(1-p_*),$$

whence $\partial_p h(u,v,w,p_*) > 0$.

The Implicit Function Theorem therefore determines $p$ locally as a $C^\infty$-function of $(u,v,w)$. Since the defining zero is unique by the preceding discussion, these local representations agree wherever their domains intersect and determine a global $C^\infty$-map

$$p_* : \Omega \longrightarrow (0,1).$$

□

Theorem 3.1 shows that the stability threshold depends smoothly on the parameters. Since the defining equation can be solved explicitly for the third parameter, the threshold itself may be used in place of $w$. The next result shows that this yields a global smooth parametrization of the admissible parameter region.

**Theorem 3.2.** *Let*

$$\Phi : (0,1)^2 \times (0,1) \longrightarrow \Omega$$

*be defined by*

$$\Phi(u,v,p) = \left(u, v, \left(\frac{(1-u^p)(1-v^p)}{(1-(uv)^p)^2}\right)^{1/p}\right).$$

*Then $\Phi$ is a $C^\infty$-diffeomorphism between $(0,1)^2 \times (0,1)$ and $\Omega$, with inverse*

$$\Phi^{-1}(u,v,w) = \bigl(u, v, p_*(u,v,w)\bigr).$$

*In particular, the stability threshold provides a global smooth coordinate on the admissible parameter region $\Omega$.*

*Proof.* Let $(u, v, p) \in (0,1)^2 \times (0,1)$, and let $w$ be the third component of $\Phi(u, v, p)$. Since $0 < u, v < 1$, we have

$$1 - (uv)^p > 1 - u^p \qquad \text{and} \qquad 1 - (uv)^p > 1 - v^p.$$

Hence

$$(1 - u^p)(1 - v^p) < \big(1 - (uv)^p\big)^2.$$

and therefore $0 < w < 1$. By the definition of $w$,

$$h(u, v, w, p) = 0.$$

For fixed $u, v, w \in (0,1)$, the function $t \mapsto h(u, v, w, t)$ is strictly concave. Hence its derivative is strictly decreasing, and

$$\lim_{t \to \infty} \partial_t h(u, v, w, t) = -\ln w > 0.$$

Thus $\partial_t h(u, v, w, t) > 0$ for $t > 0$, so $t \mapsto h(u, v, w, t)$ is strictly increasing. Since $p < 1$,

$$h(u, v, w, 1) > h(u, v, w, p) = 0.$$

Hence $(u, v, w) \in \Omega$. The uniqueness of the zero gives $p = p_*(u, v, w)$, and $\Phi$ is well defined.

Let $(u, v, w) \in \Omega$ and put $p = p_*(u, v, w)$. Solving $h(u, v, w, p) = 0$ for $w$ gives

$$w = \left( \frac{(1 - u^p)(1 - v^p)}{\big(1 - (uv)^p\big)^2} \right)^{1/p},$$

hence

$$\Phi\big(u, v, p_*(u, v, w)\big) = (u, v, w).$$

Thus $\Phi$ is surjective. If

$$\Phi(u_1, v_1, p_1) = \Phi(u_2, v_2, p_2),$$

then $u_1 = u_2$ and $v_1 = v_2$. Writing $w$ for the common third component, the first part of the proof gives

$$p_1 = p_*(u_1, v_1, w) = p_*(u_2, v_2, w) = p_2.$$

Hence $\Phi$ is injective, and

$$\Phi^{-1}(u, v, w) = \big(u, v, p_*(u, v, w)\big).$$

The map $\Phi$ is of class $C^\infty$, and by Theorem 3.1, so is its inverse. Hence $\Phi$ is a $C^\infty$-diffeomorphism. □

As an application of the preceding results, consider the one-parameter family

$$f_\varepsilon(x) = (x + a)^3(x^2 + 2\varepsilon x + 1), \qquad a > 0,\ 0 < \varepsilon < 1.$$

Its zeros are $-a, -a, -a, -\varepsilon \pm i\sqrt{1 - \varepsilon^2}$, so the complex-conjugate pair approaches the imaginary axis as $\varepsilon \to 0^+$, while $f_\varepsilon \in H_5$ for every $\varepsilon \in (0, 1)$. Expanding $f_\varepsilon$ gives

$$f_\varepsilon(x) = x^5 + (3a + 2\varepsilon)x^4 + (3a^2 + 6a\varepsilon + 1)x^3 + (a^3 + 6a^2\varepsilon + 3a)x^2 + (2a^3\varepsilon + 3a^2)x + a^3,$$

and hence determines the smooth curve

$$\Gamma_a(\varepsilon) = (u_\varepsilon, v_\varepsilon, w_\varepsilon), \qquad 0 < \varepsilon < 1,$$

where

$$u_\varepsilon = \frac{a^3 + 6a^2\varepsilon + 3a}{(3a + 2\varepsilon)(3a^2 + 6a\varepsilon + 1)}, \qquad v_\varepsilon = \frac{(3a^2 + 6a\varepsilon + 1)a^3}{(a^3 + 6a^2\varepsilon + 3a)(2a^3\varepsilon + 3a^2)},$$

and

$$w_\varepsilon = \frac{(3a+2\varepsilon)(2a^3\varepsilon+3a^2)}{(3a^2+6a\varepsilon+1)(a^3+6a^2\varepsilon+3a)}.$$

Since $\Gamma_a : (0,1) \to \Omega$ is smooth, Theorem 3.1 shows that the function

$$\varepsilon \longmapsto p_*\big(\Gamma_a(\varepsilon)\big) = p_*(f_\varepsilon)$$

is of class $C^\infty$. Moreover, $p_*(f_\varepsilon)$ is characterized implicitly by

$$H_a(p_*(f_\varepsilon), \varepsilon) = 0,$$

where

$$H_a(p,\varepsilon) = \ln(1-u_\varepsilon^p) + \ln(1-v_\varepsilon^p) - 2\ln\big(1-(u_\varepsilon v_\varepsilon)^p\big) - p\ln w_\varepsilon.$$

Differentiation with respect to $\varepsilon$ gives

$$\frac{d}{d\varepsilon}p_*(f_\varepsilon) = -\frac{\partial_\varepsilon H_a(p_*(f_\varepsilon),\varepsilon)}{\partial_p H_a(p_*(f_\varepsilon),\varepsilon)}.$$

Figure 1 illustrates the variation of the stability threshold along the trajectory $\Gamma_a(\varepsilon)$ for representative values of $a$.

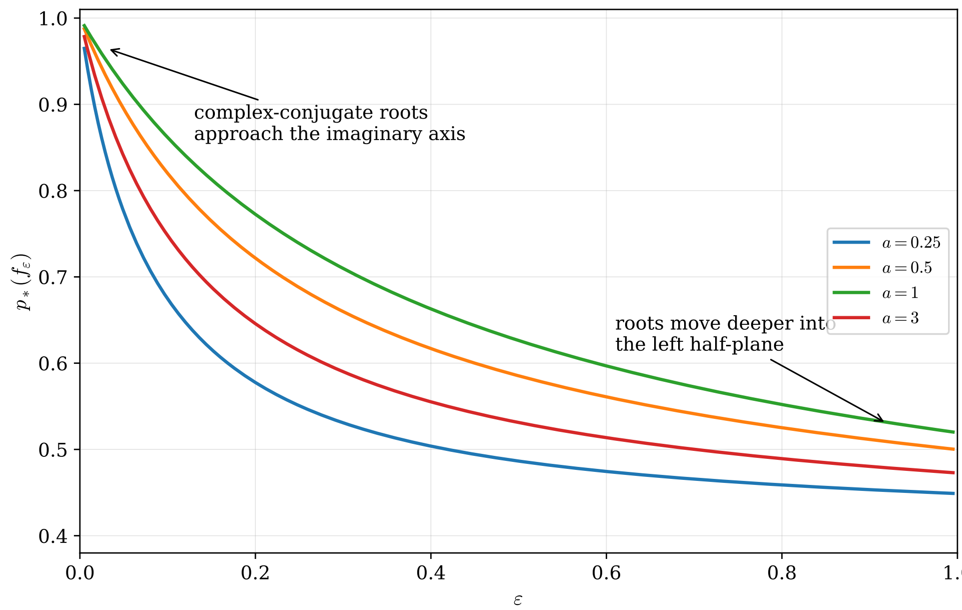


Figure 1: Variation of the stability threshold along the trajectory $\Gamma_a(\varepsilon)$ associated with $f_\varepsilon(x) = (x+a)^3(x^2+2\varepsilon x+1)$ for representative values of $a$.

# 4 Conclusion

This paper provides a complete characterization of the stability-preserving exponent set for fractional Hadamard powers of degree-five stable polynomials with positive coefficients. Using a parameterization of the Hurwitz region, the stability problem is reduced to the analysis of a single logarithmic scalar function. This reduction leads to the existence of a unique stability threshold $p_*(f) \in (0,1)$, and consequently to the complete description $\mathcal{P}(f) = \big(p_*(f), \infty\big)$ for every polynomial $f \in H_5$.

The formulation also reveals additional geometric structure. The stability threshold depends smoothly on the parameters, and the threshold itself provides a global smooth coordinate on the admissible parameter region. The one-parameter family considered in Section 3 illustrates this geometric interpretation by showing how the threshold evolves continuously as the pair of complex-conjugate zeros approaches the imaginary axis.

The characterization obtained in this paper relies on a feature that is specific to degree five, namely that the stability problem is governed by a single Hurwitz determinant. Beginning with degree six, several Hurwitz determinants interact simultaneously, making an analogous reduction substantially more difficult. Nevertheless, the smooth geometric viewpoint developed in this paper provides a starting point for extending these ideas to higher degrees.

## Data Availability Statement

The data on which Figure 1 is based are available on request from the first author.


## Declaration of Funding

The work of the third author was supported by the SRP Program of the HTWG Konstanz - University of Applied Sciences.


## References


[1] Fatimah Alsaafin, Doaa Al-Saafin, and Jürgen Garloff. A study of the validity of Oppenheim's inequality for Hurwitz matrices associated with Hurwitz polynomials. *Electronic Journal of Linear Algebra*, 40:574–584, 2024.

[2] Stanisław Białas and Leokadia Białas-Cież. Comments on 'On Hadamard Powers of Polynomials'. *Mathematics of Control, Signals, and Systems*, 29(3):16, 2017.

[3] Stanisław Białas, Leokadia Białas-Cież, and Michał Kudra. On the Hurwitz stability of noninteger Hadamard powers of stable polynomials. *Linear Algebra and its Applications*, 683:111–124, 2024.

[4] Stanisław Białas and Michał Góra. Simple necessary conditions for Hadamard factorizability of Hurwitz polynomials. *Electronic Journal of Linear Algebra*, 37:671–679, 2021.

[5] Felix R. Gantmacher. *The Theory of Matrices*, volume 2. Chelsea Publishing Company, New York, 1959.

[6] Jürgen Garloff and David G. Wagner. Hadamard products of stable polynomials are stable. *Journal of Mathematical Analysis and Applications*, 202(3):797–809, 1996.

[7] Adolf Hurwitz. Ueber die Bedingungen, unter welchen eine Gleichung nur Wurzeln mit negativen reellen Theilen besitzt. *Mathematische Annalen*, 46(2):273–284, 1895.

[8] J. H. B. Kemperman. A Hurwitz matrix is totally positive. *SIAM Journal on Mathematical Analysis*, 13(2):331–341, 1982.

[9] A. Liénard and M. H. Chipart. Sur le signe de la partie réelle des racines d'une équation algébrique. *Journal de Mathématiques Pures et Appliquées*, 10:291–346, 1914.