Total nonnegativity of finite Hurwitz matrices and root location of polynomials

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Abstract

In 1970, B.A. Asner, Jr., proved that for a real quasi-stable polynomial, i.e., a polynomial whose zeros lie in the *closed* left half-plane of the complex plane, its finite Hurwitz matrix is totally nonnegative, i.e., all its minors are nonnegative, and that the converse statement is not true. In this work, we explain this phenomenon in detail, and provide necessary and sufficient conditions for a real polynomial to have a totally nonnegative finite Hurwitz matrix.

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1. Introduction

This paper is devoted to total nonnegativity of Hurwitz matrices. We remind the reader that given a real polynomial of degree n

$$p(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n, \qquad a_0, \ldots, a_n \in \mathbb{R}, \quad a_0, a_n > 0, \tag{1.1}$$

its *finite* Hurwitz matrix has the form

$$\mathcal{H}_n(p) = \begin{pmatrix} a_1 & a_3 & a_5 & a_7 & \dots & 0 & 0 \\ a_0 & a_2 & a_4 & a_6 & \dots & 0 & 0 \\ 0 & a_1 & a_3 & a_5 & \dots & 0 & 0 \\ 0 & a_0 & a_2 & a_4 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & a_{n-1} & 0 \\ 0 & 0 & 0 & 0 & \dots & a_{n-2} & a_n \end{pmatrix}. \tag{1.2}$$

In 1970, B.A. Asner, Jr., established in [6] that if the polynomial p is quasi-stable (that is, all its zeros lie in the *closed* left half-plane of complex plane), then the matrix $\mathcal{H}_n(p)$ is totally nonnegative. This means that all its minors are nonnegative. Asner noted that the converse statement is not true. As an example, he provided the polynomial $p(z) = z^4 + 198z^2 + 10201$ with zeros $\pm 1 \pm i10$ whose finite Hurwitz

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matrix $\mathcal{H}_4(p)$ is totally nonnegative. In fact, Asner implicitly established that if the finite Hurwitz matrix of a real polynomial is nonsingular and totally nonnegative, then this polynomial is (Hurwitz) stable (that is, all its zeros lie in the *open* left half-plane of the complex plane). In 1980, J. H. B. Kemperman [23] considered the *infinite* Hurwitz matrix

$$H_\infty(p) = \begin{pmatrix} a_0 & a_2 & a_4 & a_6 & a_8 & a_{10} & \dots \\ 0 & a_1 & a_3 & a_5 & a_7 & a_9 & \dots \\ 0 & a_0 & a_2 & a_4 & a_6 & a_8 & \dots \\ 0 & 0 & a_1 & a_3 & a_5 & a_7 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{1.3}$$

and proved by a method different from Asner's one that the matrix (1.3) is totally nonnegative if the polynomial p given in (1.1) is quasi-stable. Later on, O. Holtz [17] gave a very simple proof of this fact. However, both Kemperman and Holtz did not discuss the converse statement.

In [19, Theorem 3.44], a general theorem was proved which implies that the total nonnegativity of the infinite Hurwitz matrix of a given real polynomial is equivalent to the quasi-stability of this polynomial. This fact was also mentioned in [11].

To make the present work self-contained, we mention some important properties of stable and quasi-stable polynomials. We use these properties to obtain our main result on the total nonnegativity of finite Hurwitz matrices and prove that the total nonnegativity of the infinite Hurwitz matrix of a polynomial is equivalent to the quasi-stability of this polynomial. Note that Asner and Kemperman initially proved their theorems for stable polynomials and extended the results to quasi-stable polynomials by approximating quasi-stable polynomials by stable polynomials. Holtz dealt only with stable polynomials. Here we consider quasi-stable polynomials directly and obtain all results for stable polynomials as a particular case.

The main results of the paper provide necessary and sufficient conditions on the zeros of a given real polynomial for its *finite* Hurwitz matrix to be totally nonnegative. Note that our (sharp) necessary condition does not coincide with our (sharp) sufficient condition. To obtain these conditions we use results by I. Schoenberg [30, 31] on the polynomials from the class of the Pólya frequency functions, see Section 5 for details. These conditions require that the given polynomial does not have zeros in a specified sector in the right half-plane. Such polynomials appear in the stability analysis of fractional differential equations, e.g., commensurate fractional-order linear time-invariant systems [27] and fractional-order Lotka-Volterra predator-prey models [2].

In passing we note some properties of the Hurwitz matrix which are stronger than its total nonnegativity. It was noted by Kemperman [23] that the infinite Hurwitz matrix associated with a stable polynomial is almost totally positive, i.e., besides its total nonnegativity, each of its square submatrices has a positive determinant if and only if all of the diagonal entries of this submatrix are positive. It was shown in [15] that, in fact, the latter positivity condition suffices to be hold only for all square submatrices formed from consecutive rows and columns. Characterizations of the almost total positivity of the infinite matrices of Hurwitz type, see Definition 3.2, can be found in [1]. We mention also that in [26] the smallest possible constant c_n was determined such that the positivity of the coefficients of the polynomial p given by (1.1) and the satisfaction of the inequalities $a_k a_{k+1} > c_n a_{k+2} a_{k-1}$, $k = 1, \dots, n-2$, imply the stability of p (see also [22] for extensions of this result). Furthermore, it was shown in [24] that if p has positive coefficients and satisfies the inequality

$$\frac{a_0 a_3}{a_1 a_2} + \frac{a_1 a_4}{a_2 a_3} + \cdots + \frac{a_{n-3} a_n}{a_{n-2} a_{n-1}} < 1,$$

then p is stable.

The organization of the paper is as follows. In Section 2, we state our main results. We provide some auxiliary facts on R-functions, the definition of the finite and infinite matrices of Hurwitz type, as well as their factorizations in Section 3. In Section 4, we recall and prove some properties of stable and quasi-stable polynomials and reproduce the results of Asner, Kemperman, and Holtz. In Section 5, we prove the main results of this work, Theorems 2.1 and 2.3. Section 6 is devoted to the eigenstructure of totally nonnegative finite Hurwitz matrices. Here we generalize the results by Asner [6] and Lehnigk [25] on the eigenvalues and Jordan form of totally nonnegative Hurwitz matrices. In Section 7, we draw some conclusions and pose an open problem.

2. Main results

In this section we state our main results which are to be proved in Section 5.

Theorem 2.1. *Let p be a polynomial of degree $n \geqslant 2$ given in (1.1). If its finite Hurwitz matrix $\mathcal{H}_n(p)$ is totally nonnegative, then p has no zeros in the sector*

$$|\arg z| < \begin{cases} \dfrac{\pi}{4} \cdot \dfrac{n+1}{n-1} & \text{for odd } n, \\[2ex] \dfrac{\pi}{4} \cdot \dfrac{n}{n-1} & \text{for even } n. \end{cases} \tag{2.1}$$

The constants in (2.1) are sharp.

For example, for even n the finite Hurwitz matrix of the following polynomial

$$p(z) = \prod_{j=1}^{\frac{n}{2}} \left(z^2 + e^{i\frac{\pi}{2}\frac{n-4j+2}{n-1}} \right)$$

is totally nonnegative, but p has zeros on the border of the sector (2.1). Analogously, for odd n the polynomial

$$p(z) = (z+1) \prod_{j=1}^{\frac{n-1}{2}} \left(z^2 + e^{i\frac{\pi}{2}\frac{n-4j+1}{n-1}} \right)$$

provides the sharp constant in Theorem 2.1.

Since $\frac{n+1}{n-1} > \frac{n}{n-1} > 1$ for any $n \geqslant 2$, we can give a universal estimate for the sector free of zeros of p which is independent on the degree of the polynomial p.

Corollary 2.2. *If the finite Hurwitz matrix $\mathcal{H}_n(p)$ of a real polynomial p of degree n is totally nonnegative, then p has no zeros in the sector*

$$|\arg z| \leqslant \frac{\pi}{4}.$$

The next theorem provides a sharp sufficient condition on a real polynomial to have its finite Hurwitz matrix totally nonnegative.

Theorem 2.3. *Let a polynomial p of degree $n \geqslant 4$ given in (1.1) have no zeros in the sector*

$$|\arg z| < \frac{\pi}{2} \cdot \frac{n-2}{n-1}, \tag{2.2}$$

and satisfy the "reflection property": if $p(\lambda) = 0$ for some $\lambda \in \mathbb{C}$ such that $\operatorname{Re} \lambda > 0$, then $p(-\lambda) = 0$.

Then the finite Hurwitz matrix $\mathcal{H}_n(p)$ of the polynomial p is totally nonnegative. The constant in (2.2) is sharp.

The polynomial p of degree n, $1 \leqslant n \leqslant 3$, is quasi-stable if and only if $\mathcal{H}_n(p)$ is totally nonnegative.

The polynomial

$$p(z) = (z+1)^{n-4}(z^4 + 2z^2 \cos\theta + 1), \quad \theta = \frac{\pi}{2(n-1)} + \varepsilon,$$

with $\varepsilon > 0$ arbitrarily small, has roots inside the sector (2.2), and its finite Hurwitz matrix $\mathcal{H}_n(p)$ is not totally nonnegative. This means that we cannot decrease the angle in (2.2), so the result of Theorem 2.3 is sharp.

3. Auxiliary facts: *R*-functions and finite Hurwitz matrix factorization

Many properties of Hurwitz matrices and stable polynomials are related to properties of the so-called rational R-functions [8, 14, 19].

Consider a rational function
$$R(z) = \frac{q(z)}{p(z)}, \tag{3.1}$$
where p and q are real polynomials

$$p(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n, \qquad a_i \in \mathbb{R},\ i = 0, 1, \ldots, n,\ a_0 > 0, \tag{3.2}$$
$$q(z) = b_0 z^n + b_1 z^{n-1} + \cdots + b_n, \qquad b_i \in \mathbb{R},\ i = 0, 1, \ldots, n, \tag{3.3}$$

so that $\deg p = n$ and $\deg q \leqslant n$. If the greatest common divisor of p and q has degree l, then the rational function R has exactly $r = n - l$ poles.

Definition 3.1. A rational function R is called *R-function* if it maps the upper half-plane of the complex plane to the lower half-plane[1]:
$$\operatorname{Im} z > 0 \Rightarrow \operatorname{Im} R(z) < 0. \tag{3.4}$$

By now, these functions, as well as their meromorphic analogues, have been considered by many authors and have acquired various names. For instance, these functions are called *strongly real functions* in the monograph [32] due to their property to take real values only for real values of the argument (a more general and detailed discussion can be found in [9], see also [19]).

Let us associate to the rational function (3.1)–(3.3) the following matrix:
If $\deg q < \deg p$, that is, if $b_0 = 0$, then

$$H(p,q) = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & \dots \\ 0 & b_1 & b_2 & b_3 & b_4 & b_5 & \dots \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & \dots \\ 0 & 0 & b_1 & b_2 & b_3 & b_4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}; \tag{3.5}$$

if $\deg q = \deg p$, that is, $b_0 \neq 0$, then

$$H(p,q) = \begin{pmatrix} b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & \dots \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & \dots \\ 0 & b_0 & b_1 & b_2 & b_3 & b_4 & \dots \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3.6}$$

Definition 3.2. The matrix $H(p,q)$ is called the *infinite matrix of Hurwitz type.* We denote its leading principal minor of order j, $j = 1, 2, \ldots$, by $\eta_j(p,q)$.

In [19] it was noticed that if $g = \gcd(p,q)$, then $\deg g = l$ if and only if the following holds
$$\eta_{n-l}(p,q) \neq 0 \quad \text{and} \quad \eta_j(p,q) = 0, \quad j > n - l. \tag{3.7}$$

In this case, the matrix $H(p,q)$ can be factorized as follows [19].

[1] In [19] such functions are called R-functions of *negative type*.

Theorem 3.3. ([19, Theorem 1.43]) *If $g(z) = g_0 z^l + g_1 z^{l-1} + \cdots + g_l$, then*

$$H(p \cdot g, q \cdot g) = H(p,q)\mathcal{T}(g), \tag{3.8}$$

where $\mathcal{T}(g)$ is the infinite upper triangular Toeplitz matrix formed from the coefficients of the polynomial g:

$$\mathcal{T}(g) = \begin{pmatrix} g_0 & g_1 & g_2 & g_3 & g_4 & \dots \\ 0 & g_0 & g_1 & g_2 & g_3 & \dots \\ 0 & 0 & g_0 & g_1 & g_2 & \dots \\ 0 & 0 & 0 & g_0 & g_1 & \dots \\ 0 & 0 & 0 & 0 & g_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3.9}$$

Here we set $g_i := 0$ for all $i > l$.

Moreover, the following two theorems on properties of R-functions were established in [19], Corollary 3.28 and Theorems 3.44.

Theorem 3.4. ([19]) *The function* (3.1) *is an R-function with exactly k poles, all of which are negative, if and only if*

$$\eta_j(p,q) > 0, \qquad j = 1, 2, \dots, k, \tag{3.10}$$

$$\eta_j(p,q) = 0, \qquad j > k, \tag{3.11}$$

where k is odd if $\deg q < \deg p$, and k is even if $\deg q = \deg p$, and $\eta_j(p,q)$ is the $j \times j$ leading principal minor of the matrix $H(p,q)$ defined in (3.5)–(3.6).

Theorem 3.5. (**total nonnegativity of the infinite Hurwitz matrix**, [19]) *The following statements are equivalent:*

1) *The polynomials p and q defined by* (3.2)–(3.3) *have only nonpositive zeros*[2], *and the function $R = q/p$ is either an R-function or identically zero.*

2) *The infinite matrix of Hurwitz type $H(p,q)$ defined by* (3.5)–(3.6) *is totally nonnegative.*

Thus, the inequalities (3.10) and equations (3.11) constitute a necessary and sufficient condition for total nonnegativity of the matrix $H(p,q)$. We use these facts to describe some properties of stable and quasi-stable polynomials.

Finally, we remind of a remarkable result established in a more general form in [3, 4] (see also [21, 31]).

Theorem 3.6. *The polynomial*

$$g(z) = g_0 z^l + g_1 z^{l-1} + \cdots + g_l, \quad g_0 g_l \neq 0,$$

has only negative zeros if and only if its Toeplitz matrix $\mathcal{T}(g)$ defined by (3.9) *is totally nonnegative.*

Together with the infinite matrix $H(p,q)$, we consider its finite submatrices:

[2] Here we include the case when $q(z) \equiv 0$.

Definition 3.7. Let the polynomials p and q be given by (3.2)–(3.3). If $\deg q < \deg p = n$, let $\mathcal{H}_{2n}(p,q)$ denote the following $2n \times 2n$-matrix:

$$\mathcal{H}_{2n}(p,q) = \begin{pmatrix} b_1 & b_2 & b_3 & \dots & b_n & 0 & 0 & \dots & 0 & 0 \\ a_0 & a_1 & a_2 & \dots & a_{n-1} & a_n & 0 & \dots & 0 & 0 \\ 0 & b_1 & b_2 & \dots & b_{n-1} & b_n & 0 & \dots & 0 & 0 \\ 0 & a_0 & a_1 & \dots & a_{n-2} & a_{n-1} & a_n & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_0 & a_1 & a_2 & \dots & a_n & 0 \\ 0 & 0 & 0 & \dots & 0 & b_1 & b_2 & \dots & b_n & 0 \\ 0 & 0 & 0 & \dots & 0 & a_0 & a_1 & \dots & a_{n-1} & a_n \end{pmatrix}. \tag{3.12}$$

If $\deg q = \deg p = n$, let $\mathcal{H}_{2n+1}(p,q)$ denote the following $(2n{+}1) \times (2n{+}1)$-matrix

$$\mathcal{H}_{2n+1}(p,q) = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} & a_n & 0 & \dots & 0 & 0 \\ b_0 & b_1 & b_2 & \dots & b_{n-1} & b_n & 0 & \dots & 0 & 0 \\ 0 & a_0 & a_1 & \dots & a_{n-2} & a_{n-1} & a_n & \dots & 0 & 0 \\ 0 & b_0 & b_1 & \dots & b_{n-2} & b_{n-1} & b_n & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_0 & a_1 & a_2 & \dots & a_n & 0 \\ 0 & 0 & 0 & \dots & b_0 & b_1 & b_2 & \dots & b_n & 0 \\ 0 & 0 & 0 & \dots & 0 & a_0 & a_1 & \dots & a_{n-1} & a_n \end{pmatrix}. \tag{3.13}$$

Both matrices $\mathcal{H}_{2n}(p,q)$ and $\mathcal{H}_{2n+1}(p,q)$ are called *finite matrices of Hurwitz type.* The leading principal minors of these matrices are denoted by[3] $\Delta_j(p,q)$.

Analogously to (3.8), one can factorize finite Hurwitz matrices.

Theorem 3.8. *If* $\deg p = \deg q + 1 = n$ *and* $\deg g = m$*, then*

$$\mathcal{H}_{2n+2m}(p \cdot g, q \cdot g) = \mathcal{H}_{2n+2m}(p,q)\mathcal{T}_{2n+2m}(g), \tag{3.14}$$

where $\mathcal{H}_{2n+2m}(p,q)$ *is the principal submatrix of* $H(p,q)$ *of order* $2n+2m$ *indexed by rows (and columns)* 2 *through* $2n+2m+1$*, and the matrix* $\mathcal{T}_{2n+2m}(g)$ *is the leading principal submatrix of the matrix* $\mathcal{T}(g)$ *of order* $2n+2m$*.*

Moreover, if $\det[\mathcal{H}_{2n}(p,q)] \neq 0$ *and* $p(0) \neq 0$*, then rank of* $\mathcal{H}_{2n+2m}(p \cdot g, q \cdot g)$ *equals* $2n+m$*.*

Proof. Multiplication of the matrices $\mathcal{H}_{2n+2m}(p,q)$ and $\mathcal{T}_{2n+2m}(g)$ shows that the factorization is true.

By the Cauchy-Binet formula, rank of the matrix $\mathcal{H}_{2n+2m}(p \cdot g, q \cdot g)$ equals to rank of the matrix $\mathcal{H}_{2n+2m}(p,q)$, since the matrix $\mathcal{T}_{2n+2m}(g)$ is nonsingular as a triangular matrix with nonzero diagonal. At the same time, $\mathcal{H}_{2n+2m}(p,q)$ has m zero columns, so its rank is at most $2n+m$. However, if $p(0) \neq 0$ and $\det[\mathcal{H}_{2n}(p,q)] \neq 0$, then the determinant of $\mathcal{H}_{2n+2m}(p,q)$ of order $2n+m$ formed with the columns $1, 2, \dots, 2n{+}m$, and with the rows $1, 2, \dots, 2n, 2n{+}2, 2n{+}4, \dots, 2n{+}2m$, equals $\det[\mathcal{H}_{2n}(p,q)] \cdot [p(0)]^m$ which is nonzero. $\square$

In the same way as above, one can establish the following fact.

Theorem 3.9. *If* $\deg p = \deg q = n$ *and* $\deg g = m$*, then*

$$\mathcal{H}_{2n+2m+1}(p \cdot g, q \cdot g) = \mathcal{H}_{2n+2m+1}(p,q)\mathcal{T}_{2n+2m+1}(g), \tag{3.15}$$

[3] That is, $\Delta_j(p,q)$ is the leading principal minor of the matrix $\mathcal{H}_{2n}(p,q)$ of order j if $\deg q < \deg p$. Otherwise (when $\deg q = \deg p$), $\Delta_j(p,q)$ denotes the leading principal minor of the matrix $\mathcal{H}_{2n+1}(p,q)$ of order j.

where $\mathcal{H}_{2n+2m+1}(p,q)$ is the principal submatrix of $H(p,q)$ of order $2n+2m+1$ indexed by rows (and columns) 2 through $2n+2m+2$, and the matrix $\mathcal{T}_{2n+2m+1}(g)$ is the leading principal submatrix of the matrix $\mathcal{T}(g)$ of order $2n+2m+1$.

Moreover, if $\det[\mathcal{H}_{2n+1}(p,q)] \neq 0$ *and* $p(0) \neq 0$, *then rank of* $\mathcal{H}_{2n+2m+1}(p\cdot g, q\cdot g)$ *equals* $2n+m+1$.

Note that the factorizations (3.14)–(3.15) are simply extended versions of the factorization (3.70) in [19].

4. Quasi-stable polynomials and total nonnegativity of Hurwitz matrices

Consider a real polynomial

$$p(z) = a_0z^n + a_1z^{n-1} + \cdots + a_{n-1}z + a_n, \qquad a_0 > 0, \quad a_n \neq 0. \tag{4.1}$$

Throughout this section we use the following notation

$$l = \left[\frac{n}{2}\right], \tag{4.2}$$

where $n = \deg p$, and $[\rho]$ denotes the largest integer not exceeding ρ.

The polynomial p can always be represented as follows

$$p(z) = p_0(z^2) + zp_1(z^2), \tag{4.3}$$

where

for $n = 2l$,

$$\begin{aligned} p_0(u) &= a_0u^l + a_2u^{l-1} + \ldots + a_n, \\ p_1(u) &= a_1u^{l-1} + a_3u^{l-2} + \ldots + a_{n-1}, \end{aligned} \tag{4.4}$$

and for $n = 2l+1$,

$$\begin{aligned} p_0(u) &= a_1u^l + a_3u^{l-1} + \ldots + a_n, \\ p_1(u) &= a_0u^l + a_2u^{l-1} + \ldots + a_{n-1}. \end{aligned} \tag{4.5}$$

We introduce the following function[4]

$$\Phi(u) = \frac{p_1(u)}{p_0(u)}. \tag{4.6}$$

Definition 4.1. We call Φ the *function associated with the polynomial* p.

Note that the infinite Hurwitz matrix associated with the function Φ, is the matrix $H_\infty(p)$ defined in (1.3), since $H_\infty(p) = H(p_0,p_1)$. We denote the leading principal minors of the matrix $H_\infty(p)$ as $\eta_j(p)$, $j = 1, 2, \ldots$.

The corresponding finite Hurwitz matrix related to the rational function Φ is the matrix $\mathcal{H}_n(p)$ defined in (1.2), since $\mathcal{H}_n(p) = H_{2l}(p_0,p_1)$ if $n = 2l$, and $\mathcal{H}_n(p) = H_{2l+1}(p_0,p_1)$ if $n = 2l+1$.

Definition 4.2. The leading principal minors of the matrix $\mathcal{H}_n(p)$ are denoted by $\Delta_j(p)$,

$$\Delta_j(p) = \begin{vmatrix} a_1 & a_3 & a_5 & a_7 & \ldots & a_{2j-1} \\ a_0 & a_2 & a_4 & a_6 & \ldots & a_{2j-2} \\ 0 & a_1 & a_3 & a_5 & \ldots & a_{2j-3} \\ 0 & a_0 & a_2 & a_4 & \ldots & a_{2j-4} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & a_j \end{vmatrix}, \quad j = 1, \ldots, n, \tag{4.7}$$

[4]In the book [14, Chapter XV], F. Gantmacher used the function $-\dfrac{p_1(-u)}{p_0(-u)}$.

with the convention that $a_i = 0$ for $i > n$, and are called the *Hurwitz determinants* or the *Hurwitz minors* of the polynomial p. For simplicity, we set $\Delta_0(p) \equiv 1$.

4.1. Stable polynomials

In this section, we remind the reader some basic and well known facts about stable polynomials.

Definition 4.3. A polynomial is called *(Hurwitz) stable* if all its zeros lie in the open left half-plane of the complex plane.

It is well-known [14, Chapter XV] that the polynomial p is stable if and only if the polynomials $p_0(u)$ and $p_1(u)$ have simple, negative, and interlacing zeros, that is between two zeros of one polynomial there lies exactly one zero of the other polynomial. This fact together with some properties of R-functions (see, e.g., [19, Theorem 3.4]) implies the following result [14, Chapter XV] whose proof (due to Barkovsky) is given in Appendix to make the paper self-contained.

Proposition 4.4. *The polynomial p defined in* (4.1) *is stable if and only if its associated function* Φ *defined in* (4.6) *is an R-function with exactly l poles, all of which are negative, and the limit* $\lim\limits_{u\to\pm\infty}\Phi(u)$ *is positive whenever $n = 2l + 1$, where the number l is defined in* (4.2).

Proposition 4.4 together with Theorems 3.4 and 3.5 imply the next theorem which, besides providing other properties, completely characterizes the total nonnegativity of the Hurwitz matrices of stable polynomials.

Theorem 4.5. *Given a polynomial p of degree n as in* (4.1), *the following statements are equivalent:*

1) The polynomial p is stable;

2) all Hurwitz minors $\Delta_j(p)$ are positive:

$$\Delta_1(p) > 0,\ \Delta_2(p) > 0, \ldots,\ \Delta_n(p) > 0; \tag{4.8}$$

3) the determinants $\eta_j(p)$ are positive up to order $n + 1$:

$$\eta_1(p) > 0,\ \eta_2(p) > 0, \ldots,\ \eta_{n+1}(p) > 0; \tag{4.9}$$

4) the matrix $\mathcal{H}_n(p)$ defined in (1.2) *is nonsingular and totally nonnegative;*

5) the matrix $H_\infty(p)$ defined in (1.3) *is totally nonnegative with the minor $\eta_{n+1}(p)$ being nonzero.*

Note that the equivalence of 1) and 2) is the famous Hurwitz criterion of stability [20] (see also [14, Ch. XV]). The implications 1) $\Longrightarrow$ 4) and 1) $\Longrightarrow$ 5) were proved in [6, 23]. The implication 4) $\Longrightarrow$ 1) was, in fact, proved in [6]. However, the implication 5) $\Longrightarrow$ 1) was only mentioned in [11] as a consequence of [19, Theroem 3.44].

4.2. Quasi-stable polynomials

In this section, we deal with polynomials whose zeros lie in the *closed* left half-plane.

Definition 4.6. A polynomial p of degree n defined in (4.1) is called *quasi-stable* with *the stability index m*, $0 \leqslant m \leqslant n$, if all its zeros lie in the *closed* left half-plane of the complex plane and the number of zeros of p on the imaginary axis, counting multiplicities, equals $n - m$. We call the number $n - m$ *the degeneracy index* of the quasi-stable polynomial p.

Obviously, any stable polynomial is quasi-stable with zero degeneracy index, that is, it has the smallest degeneracy index and the largest stability index (which equals the degree of the polynomial).

Remark 4.7. Note that the degeneracy index $n-m$ is always *even* due to the condition $p(0)\neq 0$ adopted in (4.1).

Throughout this section we use the following notation

$$r=\left[\frac{m}{2}\right], \tag{4.10}$$

where m is the stability index of the polynomial p.

Moreover, if p is a quasi-stable polynomial, then

$$p(z)=p_0(z^2)+zp_1(z^2)=g(z^2)q(z)=g(z^2)\left[q_0(z^2)+zq_1(z^2)\right],$$

where q is a stable polynomial, while $g(u)=\gcd(p_0,p_1)$ has only *negative* zeros. Using this representation of quasi-stable polynomials, one can extend almost all results of Section 4.1 to quasi-stable polynomials in the same way.

The next theorem is an extended version of Proposition 4.4.

Theorem 4.8. *The polynomial p defined in* (4.1) *is quasi-stable with the stability index m if and only if its associated function Φ defined in* (4.6) *is an R-function with exactly r poles all of which are negative, and $\lim\limits_{u\to\pm\infty}\Phi(u)$ is positive whenever n is odd. The number r is defined in* (4.10).

Now it is also easy to extend Theorem 4.5 to quasi-stable polynomials.

Theorem 4.9. *Given a polynomial p of degree n as in* (4.1)*, the following statements are equivalent:*

1) The polynomial p is quasi-stable with the stability index m;

2) the Hurwitz minors $\Delta_j(p)$ are positive up to order m:

$$\Delta_1(p)>0,\ \Delta_2(p)>0,\ldots,\ \Delta_m(p)>0,\ \Delta_{m+1}(p)=\ldots=\Delta_n(p)=0, \tag{4.11}$$

and $g(u)=\gcd(p_0,p_1)$ has only negative zeros;

3) the determinants $\eta_j(p)$ are positive up to order $m+1$:

$$\eta_1(p)>0,\ \eta_2(p)>0,\ldots,\ \eta_{m+1}(p)>0,\ \eta_{m+i}(p)=0,\quad i=2,3,\ldots, \tag{4.12}$$

and $g(u)=\gcd(p_0,p_1)$ has only negative zeros;

4) the matrix $H_\infty(p)$ is totally nonnegative and

$$\eta_{m+1}(p)\neq 0,\ \eta_{m+i}(p)=0,\quad i=2,3,\ldots.$$

The implication 1) $\Longrightarrow$ 4) was proved in [6, 23]. The implication 4) $\Longrightarrow$ 1) is a simple consequence of Theorems 3.5 and 4.8 as it was noticed in [11]. Thus, the equivalence 1) $\Longleftrightarrow$ 4) can be rewritten in the following form.

Theorem 4.10. *A polynomial is quasi-stable if and only if its infinite Hurwitz matrix is totally nonnegative.*

Note that in the conditions 2) and 3) of Theorem 4.9 we cannot circumvent the condition that the gcd of the polynomials $p_0(u)$ and $p_1(u)$, the even and odd parts of the polynomial p, has only negative zeros. Indeed, the implications 1) $\Longrightarrow$ 2) and 1) $\Longrightarrow$ 3) follow from results in [6] and [23], respectively, since the inequalities (4.11) and (4.12) follow from the total nonnegativity of the Hurwitz matrix $H_\infty(p)$ of a quasi-stable polynomial (for properties of totally nonnegative matrices, see, e.g., [5, 12, 21, 28]).

However, the inequalities (4.11) or (4.12) without any additional condition imply only that the polynomial

$$q(z) = \frac{p(z)}{g(z^2)}, \tag{4.13}$$

where $g(u) = \gcd(p_0, p_1)$, is stable and is of degree m. They provide no information about the root location of the polynomial $g(u)$ at all. For example, the polynomial $p(z) = (z+1)(z^4+1)$ satisfies the inequalities (4.11) with $m = 1$. Its finite Hurwitz matrix has the form

$$\mathcal{H}_5(p) = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

But this matrix is not totally nonnegative.

Taking into account Theorem 4.10, one can suppose that the assumption of the total nonnegativity of the finite Hurwitz matrix of a polynomial can imply the stability of the polynomial. As we announced in the introduction, this supposition is true, and the total nonnegativity of the finite Hurwitz matrix put some restrictions on the roots of the polynomial. But it also does not imply quasi-stability of the polynomial. As we mentioned in the introduction, Asner [6] provided as a counterexample the polynomial $p(z) = z^4 + 198z^2 + 10201$ with zeros $\pm 1 \pm i10$ whose finite Hurwitz matrix

$$\mathcal{H}_4(p) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 198 & 10201 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 198 & 10201 \end{pmatrix}$$

is totally nonnegative. Note that p satisfies the conditions of Theorem 2.3.

The main reason of this phenomenon is the following. As we show below (see (5.4)), the finite Hurwitz matrix of the given polynomial p can be factorized as follows $\mathcal{H}_n(p) = H_n(q)\mathcal{T}_n(g)$, where $H_n(q)$ is a truncation of the infinite Hurwitz matrix of the polynomial q defined in (4.13), while $\mathcal{T}_n(g)$ is a finite triangular Toeplitz matrix consisting of the coefficients of the polynomial $g(u) = \gcd(p_0, p_1)$; both matrices are to be defined in Section 5. As Asner's example shows, it is possible to find a polynomial p such that the matrix $\mathcal{T}_n(g)$ is totally nonnegative and q is stable. In this case, $\mathcal{H}_n(p)$ is totally nonnegative as a product of matrices of such a type while the infinite Hurwitz matrix $H_\infty(p)$ is not totally nonnegative if $g(u)$ has positive or/and non-real zeros. However, if $\deg g = 1$, $g(u)$ cannot have nonnegative roots in the case when $\mathcal{H}_n(p)$ is totally nonnegative. So we can generalize Theorem 4.5 as follows.

Theorem 4.11. *Given a polynomial p of degree n as in (4.1), the following statements are equivalent:*

1) The polynomial p is quasi-stable with the stability index at least $n-2$;

2) the matrix $\mathcal{H}_n(p)$ is totally nonnegative and $\Delta_{n-2}(p) \neq 0$.

In the next section we find the location of the roots of polynomials whose finite Hurwitz matrices are totally nonnegative. Throughout the next sections by

$$A\begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix}$$

we denote the minor of a matrix A (finite or infinite) formed with its rows $i_1, i_2, \ldots, i_k$ and columns $j_1, j_2, \ldots, j_k$. We suppose here that $1 \leqslant i_1 < i_2 < \cdots < i_k$, and $1 \leqslant j_1 < j_2 < \cdots < j_k$.

5. Proofs of Theorems 2.1 and 2.3

In this section, we find necessary and sufficient conditions for the total nonnegativity of the finite Hurwitz matrix of a given polynomial. Throughout the section we suppose additionally that the size of the last nonzero leading principal minor of the finite Hurwitz matrix is known in advance, and establish all results under this additional condition. Then Theorems 2.1 and 2.3 follow from the results of this section by putting the size of the last nonzero leading principal minor to be maximal or minimal.

The first of our results deals with rank of finite Hurwitz matrices. Obviously, given a polynomial p, rank of the matrix $\mathcal{H}_n(p)$ equals n whenever $\Delta_n(p) \neq 0$. Let us generalize this fact.

Lemma 5.1. *Let p be the polynomial defined in* (4.1)*. If $\Delta_m(p) \neq 0$ and $\Delta_j(p) = 0$, $j = m+1, \ldots, n$, for some number m, $0 \leqslant m \leqslant n$, then rank of the finite Hurwitz matrix $\mathcal{H}_n(p)$ is $\dfrac{n+m}{2}$.*

Proof. The case $m = n$ was just mentioned above. If $m = 0$, then the claim of the lemma is trivial.

Now the assertion of the theorem in the case $1 \leqslant m \leqslant n-1$ follows from Theorems 3.8 and 3.9. $\square$

Remark 5.2. Note that the number $\dfrac{n+m}{2} = m + \dfrac{n-m}{2}$ is always integer, since $n-m$ is an even number according to Remark 4.7.

For our next result, we use the following auxiliary definition and facts.

Definition 5.3. Given a polynomial g, if all minors of order less than or equal to r of the infinite Toeplitz matrix $\mathcal{T}(g)$ defined in (3.9) are nonnegative, then the sequence of the coefficients of the polynomial g and the matrix $\mathcal{T}(g)$ are called *r-times nonnegative* or *r-nonnegative*. If $\mathcal{T}(g)$ is totally nonnegative, then the sequence of the coefficients of the polynomial g is called *totally nonnegative* [30, 31][5]. The class r-nonnegative (totally nonnegative) sequences is usually denoted by PF_r (PF_∞), see [21, p. 393].

For polynomials in the class PF_r, I. Schoenberg established the following theorem.

Theorem 5.4. (Schoenberg [31, Theorem 2]). *Given a polynomial $g(u) = g_0u^l + \cdots + g_l$, $g_0 > 0$, the sequence $\{g_j\}_{j=0}^n$ belongs to the class PF_r if and only if the $r \times (r+l)$ matrix*

$$T_r \overset{def}{=} \begin{pmatrix} g_0 & \cdots & g_l & & 0 \\ & \ddots & & \ddots & \\ 0 & & g_0 & \cdots & g_l \end{pmatrix}$$

is totally nonnegative.

Moreover, it is easy to see that the total nonnegativity of the matrix T_r is equivalent to the nonnegativity of the minors of order r of T_r, since every minor of the matrix T_r of order less than r formed with consecutive rows is either zero or a product of a minor of order r and a positive constant of the form g_0^{-k}, $k \in \mathbb{N}$. Indeed, for any s, $1 \leqslant s \leqslant r$, and $1 \leqslant j_1 < j_2 < \cdots < j_s \leqslant r$ one has

$$T_r \begin{pmatrix} i & i+1 & \cdots & i+s-1 \\ j_1 & j_2 & \cdots & j_s \end{pmatrix} = 0,$$

for any $i > j_1$. If $i \leqslant j_1$, the following identity holds

$$T_r \begin{pmatrix} i & i+1 & \cdots & i+s-1 \\ j_1 & j_2 & \cdots & j_s \end{pmatrix} = T_r \begin{pmatrix} i+t & i+t+1 & \cdots & r \\ j_1+t & j_2+t & \cdots & j_s+t \end{pmatrix}, \tag{5.1}$$

[5]In [30, 31], such sequences are called *r-positive* and *totally positive*, respectively.

where $t = r + 1 - i - s$. Now since

$$T_r \begin{pmatrix} 1 & 2 & \cdots & k \\ 1 & 2 & \cdots & k \end{pmatrix} = g_0^k, \qquad k = 1, \dots, r,$$

we finally can conclude from (5.1) that

$$T_r \begin{pmatrix} i & i+1 & \cdots & i+s-1 \\ j_1 & j_2 & \cdots & j_s \end{pmatrix} = \frac{1}{g_0^{r+s}} \cdot T_r \begin{pmatrix} 1 & 2 & \cdots & i+t-1 & i+t & i+t+1 & \cdots & r \\ 1 & 2 & \cdots & i+t-1 & j_1+t & j_2+t & \cdots & j_s+t \end{pmatrix}. \tag{5.2}$$

This formula together with Theorem 5.4 and [5, Theorem 2.1] imply the following fact.

Lemma 5.5. *The sequence of coefficients of the polynomial $g(u) = g_0 u^l + \cdots + g_l$, $g_0 \neq 0$, belongs to the class PF_r if and only if all the minors of the matrix T_r of order r are nonnegative.*

Now we are in a position to establish a fact which is a basic tool in the proofs of our main results[6].

Lemma 5.6. *Let p be the polynomial defined in (4.1). Its finite Hurwitz matrix $\mathcal{H}_n(p)$ is totally nonnegative with*

$$\Delta_m(p) \neq 0 \quad \textit{and} \quad \Delta_{m+1}(p) = 0^7, \tag{5.3}$$

if and only if $p(z) = q(z)g(z^2)$ with $\deg q = m$, where q is a stable polynomial and the sequence of the coefficients of the polynomial g belongs to the class $PF_{\frac{n+m}{2}}$.

Proof. Let $p(z) = q(z)g(z^2)$, where

$$q(z) = b_0 z^m + b_1 z^{m-1} + \cdots + b_m, \qquad b_0 > 0,$$

is a stable polynomial, and the sequence of the coefficients of the polynomial g belongs to the class $PF_{\frac{n+m}{2}}$. The inequalities (5.3) follow from (3.7). Furthermore, by Theorems 3.8 and 3.9 the matrix $\mathcal{H}_n(p)$ can be factorized as follows

$$\mathcal{H}_n(p) = H_n(q)\mathcal{T}_n(g), \tag{5.4}$$

where $H_n(q)$ is the $n \times n$ principal submatrix of the infinite Hurwitz matrix $H_\infty(q)$ indexed by rows (and columns) 2 through $n+1$, and the matrix $\mathcal{T}_n(g)$ is the $n \times n$ leading principal submatrix of the matrix $\mathcal{T}(g)$. By Theorem 4.10, the matrix $H_\infty(q)$ is totally nonnegative, so is its submatrix $H_n(q)$. Moreover, all minors of $\mathcal{T}(g)$ of order $\leqslant \frac{n+m}{2}$ are nonnegative by assumption. Therefore, by the Binet-Cauchy formula, all the minors of $\mathcal{H}_n(p)$ of order less than or equal to $\frac{n+m}{2}$ are nonnegative. Now since $\operatorname{rank} \mathcal{H}_n(p) = \frac{n+m}{2}$ according to Lemma 5.1, we obtain that $\mathcal{H}_n(p)$ is totally nonnegative.

Conversely, suppose that $\mathcal{H}_n(p)$ is totally nonnegative and condition (5.3) holds. By [14, Chapter XIII, Lemma 5] (see also [21, Chapter 2, Corollary 9.1]), one obtains

$$\begin{aligned} &\Delta_j(p) > 0, \qquad j = 1, \dots, m, \\ &\Delta_j(p) = 0, \qquad j = m+1, \dots, n. \end{aligned} \tag{5.5}$$

As we mentioned in Section 3, this means that p can be factorized as follows

$$p(z) = q(z)g(z^2)$$

[6]In fact, Lemma 5.6 proves Conjecture 3.49 in [19].

[7]In fact, this implies that $\Delta_i(p) = 0$ for $i = m+1, \dots, n$, as it follows from some properties of totally nonnegative matrices [13].

with $\deg q = m$ and $\deg g = \frac{n-m}{2}$. Now from the factorization (5.4) we obtain that

$$\mathcal{H}_n(p)\begin{pmatrix}1 & 2 & \cdots & k\\ 1 & 2 & \cdots & k\end{pmatrix} = g_0^k \cdot H_n(q)\begin{pmatrix}1 & 2 & \cdots & k\\ 1 & 2 & \cdots & k\end{pmatrix}, \quad k = 1, \ldots, n, \tag{5.6}$$

due to the special structure of $\mathcal{T}_n(g)$. Consequently, we have that $\Delta_i(q) > 0$, $i = 1, \ldots, m$, since we can always choose g_0, the leading coefficient of $g(u)$, to be positive. So the polynomial q is stable, and the matrix $H_\infty(q)$ is totally nonnegative according to Theorem 4.5. By Lemma 5.5, it suffices to show now that

$$\mathcal{T}_n(g)\begin{pmatrix}1 & 2 & \cdots & \frac{n+m}{2}\\ j_1 & j_2 & \cdots & j_{\frac{n+m}{2}}\end{pmatrix} \geqslant 0,$$

for any j_i such that $1 \leqslant j_1 < j_2 < \cdots < j_{\frac{n+m}{2}} \leqslant n$.

Since the only nonzero minor of the matrix $H_n(q)$ formed with rows 1, 2, …, m, $m+2$, $m+4$, …, n, is the one which is formed with columns 1, 2, …, $\frac{n+m}{2}$, the Binet-Cauchy formula implies

$$\begin{aligned}\mathcal{T}_n(g)\begin{pmatrix}1 & 2 & \cdots & \frac{n+m}{2}\\ j_1 & j_2 & \cdots & j_{\frac{n+m}{2}}\end{pmatrix} &= \frac{\mathcal{H}_n(p)\begin{pmatrix}1 & 2 & \cdots & m & m+2 & m+4 & \cdots & n\\ j_1 & j_2 & \cdots & j_m & j_{m+1} & j_{m+2} & \cdots & j_{\frac{n+m}{2}}\end{pmatrix}}{H_n(q)\begin{pmatrix}1 & 2 & \cdots & m & m+2 & m+4 & \cdots & n\\ 1 & 2 & \cdots & m & m+1 & m+2 & \cdots & \frac{n+m}{2}\end{pmatrix}}\\ &= \frac{\mathcal{H}_n(p)\begin{pmatrix}1 & 2 & \cdots & m & m+2 & m+4 & \cdots & n\\ j_1 & j_2 & \cdots & j_m & j_{m+1} & j_{m+2} & \cdots & j_{\frac{n+m}{2}}\end{pmatrix}}{b_m^{\frac{n-m}{2}}\,\Delta_m(q)} \geqslant 0,\end{aligned} \tag{5.7}$$

where $\Delta_m(q) > 0$ due to the stability of the polynomial q. Thus, the coefficients of the polynomial g belongs to the class $PF_{\frac{n+m}{2}}$, as desired.

□

To obtain our final results of this section which imply Theorems 2.1 and 2.3, we remind the reader the following remarkable theorems due to Schoenberg [30, 31].

Theorem 5.7. (Schoenberg [31]) *Given a real polynomial $g(u)$ of degree r, if the matrix $\mathcal{T}(g)$ is k-times nonnegative, then $g(u)$ has no zeros in the sector*

$$|\arg u| < \frac{\pi k}{r+k-1}\,. \tag{5.8}$$

The constant in (5.8) *is sharp.*

Indeed, in [31] it was shown that the sequence of the coefficients of the polynomial

$$g(z) = \prod_{j=1}^{r}\left(z + e^{i\theta(r-2j+1)}\right), \qquad \theta = \frac{\pi}{r+k-1}, \tag{5.9}$$

belongs to the class PF_k, but $g(z)$ has zeros on the border of the sector (5.8). These zeros are $-e^{\pm i\theta(r-1)}$.

Theorem 5.8. (Schoenberg [30]) *Given a real polynomial $g(u)$, if all zeros of $g(u)$ lie in the sector*

$$|\pi - \arg u| \leqslant \frac{\pi}{k+1}\,, \tag{5.10}$$

then the sequence of the coefficients of the polynomial g is k-times nonnegative. The constant in (5.10) *is sharp.*

In [30, 31] it was shown that the sequence of the coefficients of a polynomial $g(u)$ of the form

$$(u+ce^{i\theta})(u+ce^{-i\theta}), \quad c>0, \tag{5.11}$$

belongs to PF_k if and only if

$$0 \leqslant \theta \leqslant \frac{\pi}{k+1}. \tag{5.12}$$

Thus, the sequence of the coefficients of any product of polynomials of the form (5.11) with θ exterior to the interval (5.12) does not belong to PF_k.

Now we are in a position to prove the main results of this section.

Theorem 5.9. *Let p be the polynomial of degree n given in* (4.1). *If its finite Hurwitz matrix $\mathcal{H}_n(p)$ is totally nonnegative and $\Delta_m(p) \neq 0$, $\Delta_{m+1}(p)=0$ for some m, $0 \leqslant m \leqslant n-2$, then p has no zeros in the sector*

$$|\arg z| < \frac{\pi}{4}\cdot\frac{n+m}{n-1}. \tag{5.13}$$

The constant in (5.13) *is sharp.*

Proof. Let the finite Hurwitz matrix $\mathcal{H}_n(p)$ of the polynomial p be totally nonnegative and $\Delta_m(p) \neq 0$, $\Delta_{m+1}(p)=0$ for some m, $0 \leqslant m \leqslant n-2$. Then by Lemma 5.6, one has $p(z)=q(z)g(z^2)$, where q is stable and the sequence of the coefficients of g belongs to $PF_{\frac{n+m}{2}}$. Consequently, by Theorem 5.7 all zeros of the polynomial $g(u)$ lie outside the sector

$$|\arg u| < \frac{\pi}{2}\cdot\frac{n+m}{n-1}, \tag{5.14}$$

so all the zeros of $g(z^2)$ lie outside the sector (5.13). Moreover, for any m, $0 \leqslant m \leqslant n-2$, the finite Hurwitz matrix of the polynomial

$$p(z) = (z+1)^m \prod_{j=1}^{\frac{n-m}{2}} \left(z^2 + e^{i\frac{\pi}{2}\frac{n-m-4j+2}{n-1}}\right)$$

whose zeros lie outside the sector (5.13) and on the border of this sector, is totally nonnegative. This follows from Lemma 5.6 and from Schoenberg's example (5.9). Thus, the angle on the right-hand side of (5.13) cannot be improved. ☐

Note that in the case $m=n-2$ the polynomial p is quasi-stable that corresponds to Theorem 4.11. So in the following theorem we suppose that $m \leqslant n-4$.

Theorem 5.10. *Let the polynomial of degree $n \geqslant 4$ given in* (4.1) *have no zeros in the sector*

$$|\arg z| < \frac{\pi}{2}\cdot\frac{n+m}{n+m+2}, \tag{5.15}$$

where $m = n-2\deg\gcd(p_0,p_1)$, $0 \leqslant m \leqslant n-4$, and the polynomials $p_0(u)$ and $p_1(u)$ are defined in (4.4)–(4.5). *If p satisfies the "reflection property": $p(-\lambda)=0$ for any λ such that $p(\lambda)=0$ and $\operatorname{Re}\lambda>0$, then its finite Hurwitz matrix $\mathcal{H}_n(p)$ is totally nonnegative with $\Delta_m(p)\neq 0$ and $\Delta_{m+1}(p)=0$. The constant in* (5.15) *is sharp.*

The polynomial p of degree n, $1 \leqslant n \leqslant 3$, is quasi-stable if and only if $\mathcal{H}_n(p)$ is totally nonnegative.

Proof. From the condition of the theorem, it follows that p can be factorized as $p(z) = q(z)g(z^2)$, where q is stable, $\deg q = m$, and $g(z^2)$ has no zeros in the sector (5.15). This means that $g(u)$ has all zeros in the sector

$$|\pi - \arg u| \leqslant \frac{2\pi}{n+m+2}, \tag{5.16}$$

so by Theorem 5.8, $g \in PF_{\frac{n+m}{2}}$. Now from Lemma 5.6 it follows that $\mathcal{H}_n(p)$ is totally nonnegative.

Furthermore, if there exists a number λ, $\operatorname{Re}\lambda > 0$, such that $p(\lambda) = 0$ and $p(-\lambda) \neq 0$, then[8] we have $p(z) = q(z)g(z^2)$ and $q(\lambda) = 0$. So, q is not stable, and there exists $\Delta_k(q) < 0$ or $\Delta_k(q) = 0$ and $\Delta_{k+1}(q) \neq 0$ for some k, $1 \leqslant k \leqslant m-1$. Since $\Delta_k(p) = g_0^k \Delta_k(q)$ by (5.6), we obtain that $\mathcal{H}_n(p)$ is not totally nonnegative, a contradiction.

Also, if we suppose that the polynomial p satisfies the reflection property, and that it has zeros in the sector (5.15) for some number m, $1 \leqslant m \leqslant n-2$, then $p(z) = q(z)g(z^2)$, where q is stable, but $g(u)$ has roots outside the sector (5.16). Thus, according to Theorem 5.8, the sequence of the coefficients of g does not belong to $PF_{\frac{n+m}{2}}$, so $\mathcal{H}_n(p)$ is not totally nonnegative by Lemma 5.6. Therefore, the angle in (5.15) is sharp.

Finally, for $1 \leqslant n \leqslant 3$, the statement of the theorem follows immediately from Lemma 5.6 and Theorem 4.11. □

For example, the polynomial

$$p(z) = (z+1)^m (z^2+1)^{r_1} (z^4 + 2z^2\cos\theta + 1)^{r_2},$$

where $r_1 = \frac{n-m}{2} - 2\left\lfloor \frac{n-m}{4} \right\rfloor$, $r_2 = \left\lfloor \frac{n-m}{4} \right\rfloor$, and

$$\theta = \frac{\pi}{n+m+2} + \varepsilon,$$

with $\varepsilon > 0$ arbitrarily small, has zeros inside the sector (5.16) but arbitrary close to its border (due to ε). According to results of Schoenberg [30] and Lemma 5.6, the finite Hurwitz matrix of p is not totally nonnegative. Thus, the angle on the right-hand side of (5.16) cannot be improved.

If we take $m = 0$ for even n, and $m = 1$ for odd n in Theorem 5.9, we get Theorem 2.1. Also, putting $m = n-4$ in Theorem 5.10, we get Theorem 2.3.

6. Spectral properties of totally nonnegative finite Hurwitz matrices

In this section, we extend the results of Asner [6] and Lehnigk [25] on the spectrum of totally nonnegative finite Hurwitz matrices and on their eigenspaces.

Theorem 6.1. *Let p be the polynomial defined in (4.1) and let its finite Hurwitz matrix $\mathcal{H}_n(p)$ be totally nonnegative. If for some m, $0 \leqslant m \leqslant n$, the following holds*

$$\Delta_m(p) \neq 0 \quad \text{and} \quad \Delta_{m+1}(p) = 0, \tag{6.1}$$

then $\mathcal{H}_n(p)$ has a zero eigenvalue of algebraic and geometric multiplicities $\frac{n-m}{2}$. All other other eigenvalues are positive with geometric multiplicity one. Moreover, all positive eigenvalues of $\mathcal{H}_n(p)$ have algebraic multiplicity one with the possible exception of exactly one which is $p(0)$ and has algebraic multiplicity 2.

[8] $g(u)$ can be a constant.

Proof. The case $m = n$ was considered in [6] and [25], so in what follows we consider $m \leqslant n-2$.

If $\mathcal{H}_n(p)$ is totally nonnegative, then from [14, Lemma 5, Ch. XIII] it follows that there exists a number m, $0 \leqslant m \leqslant n$, such that the condition (6.1) holds. Now by Lemma 5.1, rank of the matrix $\mathcal{H}_n(p)$ equals $\frac{n+m}{2}$. Thus, the polynomial p can be factorized as $p(z) = q(z)g(z^2)$, where $q(z) = b_0z^m + b_1z^{m-1} + \cdots + b_m$, $b_0 > 0$, is stable, and the sequence of the coefficients of $g(u)$ belongs to $PF_{\frac{n+m}{2}}$. By Theorem 5.7, the polynomial $g(u)$ has no zeros in the sector (5.14), in particular, $g(u)$ is stable, and $\Delta_j(g) > 0$, $j = 1, \ldots, \frac{n-m}{2}$. So from (5.4) and (5.7) one has

$$
\begin{aligned}
&\mathcal{H}_n(p)\begin{pmatrix} 1 & 2 & \cdots & m & m+2 & m+4 & \cdots & n \\ 1 & 2 & \cdots & m & m+2 & m+4 & \cdots & n \end{pmatrix} = \\
&b_m^{\frac{n-m}{2}}\Delta_m(q)\mathcal{T}_n(g)\begin{pmatrix} 1 & 2 & \cdots & m & m+1 & m+2 & \cdots & \frac{n+m}{2} \\ 1 & 2 & \cdots & m & m+2 & m+4 & \cdots & n \end{pmatrix} = \\
&b_m^{\frac{n-m}{2}}\cdot\Delta_m(q)\cdot g_0^m\cdot\mathcal{T}_n(g)\begin{pmatrix} m+1 & m+2 & \cdots & \frac{n+m}{2} \\ m+2 & m+4 & \cdots & n \end{pmatrix} = \\
&b_m^{\frac{n-m}{2}}\cdot g_0^m\cdot\Delta_m(q)\mathcal{T}_n(g)\begin{pmatrix} 1 & 2 & \cdots & \frac{n-m}{2} \\ 2 & 4 & \cdots & n-m \end{pmatrix} = b_m^{\frac{n-m}{2}}\cdot g_0^m\cdot\Delta_m(q)\cdot\Delta_{\frac{n-m}{2}}(g) > 0.
\end{aligned}
\quad. \tag{6.2}
$$

Therefore, the size of the largest nonsingular principal submatrix of $\mathcal{H}_n(p)$ equals $\frac{n+m}{2}$, since $\operatorname{rank}\mathcal{H}_n(p) = \frac{n+m}{2}$. Thus, we obtain that the algebraic multiplicity of the zero eigenvalue coincides with its geometric multiplicity and equals $\frac{n-m}{2} = n - \frac{n+m}{2}$ (see, e.g., [12, p.107]). Consequently, the totally nonnegative matrix $\mathcal{H}_n(p)$ has exactly $\frac{n+m}{2}$ positive eigenvalues, counting their algebraic multiplicities.

In the same way as in [6] and [25], one can prove that only one positive eigenvalue of $\mathcal{H}_n(p)$ can be algebraic multiple. It must be equal $p(0)$ (if any) with algebraic multiplicity 2. Indeed, the matrix $\mathcal{H}_n^{(n-1)}(p)$ obtained from $\mathcal{H}_n(p)$ by deleting the last row and column is an irreducible totally nonnegative matrix, and so its positive eigenvalues are simple and distinct by [12, Theorem 5.4.5]. Moreover, the spectrum of $\mathcal{H}_n(p)$ equals the spectrum of $\mathcal{H}_n^{(n-1)}(p)$ together with $a_n = p(0)$. Therefore, only a_n can be a multiple eigenvalue of $\mathcal{H}_n(p)$ (if any). Its algebraic multiplicity equals two. Let us prove that its geometric multiplicity is one. To do this, we need to prove that rank of the matrix

$$
B_n = \mathcal{H}_n(p) - a_nI, \tag{6.3}
$$

is $n-1$, where I denotes the identity matrix. Consider the minor

$$
B_n\begin{pmatrix} 2 & 3 & \cdots & n \\ 1 & 2 & \cdots & n-1 \end{pmatrix} = a_0B_n\begin{pmatrix} 3 & 4 & \cdots & n \\ 2 & 3 & \cdots & n-1 \end{pmatrix}. \tag{6.4}
$$

It is easy to see that the minor (6.4) can be expressed as the sum of 2^{n-3} determinants. One of these determinants is $\Delta_{n-2}(p)$. Let us look at the remaining $2^{n-3}-1$ determinants. Each one of them can be expressed as a product of an integral power of a_n and a minor of the matrix $\mathcal{H}_n(p)$. Since $\mathcal{H}_n(p)$ is totally nonnegative, all its minors are nonnegative, and all the coefficients of p are positive [29]. From the structure of the matrix (6.3) it follows that in each of the aforementioned $2^{n-3}-1$ determinants, a_n stands at a place such that its cofactor has the sign factor -1. Let us prove that one of these determinants is positive, that is, that the minor on the right-hand side of (6.4) has the form

$$
B_n\begin{pmatrix} 3 & 4 & \cdots & n \\ 2 & 3 & \cdots & n-1 \end{pmatrix} = \Delta_{n-2}(p) + (2^{n-3}-2)\text{ nonnegative terms} + \delta_n, \tag{6.5}
$$

where the minor δ_n is positive. We prove it by induction, and to do this, we need to write explicitly the minors δ_n for $n = 4, \ldots, 8$. They have the form

$$
\delta_4 = a_4a_0, \qquad \delta_5 = a_5a_1^2, \qquad \delta_6 = a_6^2a_0a_1, \qquad \delta_7 = a_7^3a_0a_1, \qquad \delta_8 = a_8^4a_0^2.
$$

It is easy to check that such minors exist in the Laplace expansion of the minor (6.5) for the corresponding n. Now we prove that in general, the following recursive formulæ hold for $l \geqslant 5$

$$\begin{aligned} \delta_{2l-1} &= a_{2l-1}^{l-2} a_1 \delta_l, \\ \delta_{2l} &= a_{2l}^{l-1} a_0 \delta_l. \end{aligned} \tag{6.6}$$

The first formula is true for $n = 9, 10$ as it can easily be seen. Suppose that it is true for all $n (\geqslant 9)$ up to some number $N-1$, and consider $n = N$. Let $N = 2k-1 \geqslant 11$. From the structure of the minor (6.5) it can be seen that in its Laplace expansion there exists a minor of the form

$$a_N^{k-2} a_1 B_N \begin{pmatrix} 3 & 4 & \cdots & k \\ 2 & 3 & \cdots & k-1 \end{pmatrix}.$$

But the considered minor of the matrix B_N has the same structure as the corresponding minor of the matrix B_k with a_N instead of a_k, so we obtain

$$B_N \begin{pmatrix} 3 & 4 & \cdots & N \\ 2 & 3 & \cdots & N-1 \end{pmatrix} = a_N^{k-2} a_1 B_N \begin{pmatrix} 3 & 4 & \cdots & k \\ 2 & 3 & \cdots & k-1 \end{pmatrix} = a_N^{k-2} a_1 \delta_k + \text{nonnegative terms}.$$

Thus, the first formula in (6.6) is proved. The second one can be proved in a similar way for $n = 2l \geqslant 12$. Consequently, the matrix (6.3) has rank $n-1$, as required. □

Let us illustrate the theorem by an example of a polynomial whose finite Hurwitz matrix has a double non-zero eigenvalue. The polynomial

$$p(z) = z^5 + (2+\sqrt{2})z^4 + \sqrt{2}z^3 + (2+2\sqrt{2})z^2 + z + 2 + \sqrt{2},$$

has the zeros $-2-\sqrt{2}$, $\pm\exp\left(\pm i\dfrac{3\pi}{8}\right)$, so $m = 1$. The finite Hurwitz matrix of the polynomial p has the form

$$\mathcal{H}_5(p) = \begin{pmatrix} 2+\sqrt{2} & 2+2\sqrt{2} & 2+\sqrt{2} & 0 & 0 \\ 1 & \sqrt{2} & 1 & 0 & 0 \\ 0 & 2+\sqrt{2} & 2+2\sqrt{2} & 2+\sqrt{2} & 0 \\ 0 & 1 & \sqrt{2} & 1 & 0 \\ 0 & 0 & 2+\sqrt{2} & 2+2\sqrt{2} & 2+\sqrt{2} \end{pmatrix}.$$

By Lemma 5.1, rank of this matrix is equal to 3, and according to Theorem 5.10, it is totally nonnegative. The Jordan form of the matrix $\mathcal{H}_5(p)$ is the following

$$\begin{pmatrix} 3+3\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 2+\sqrt{2} & 1 & 0 & 0 \\ 0 & 0 & 2+\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

so the double positive eigenvalue $2+\sqrt{2}$ has exactly one Jordan block that agrees with Theorem 6.1.

7. Conclusions

In this paper, we have presented necessary and sufficient conditions on the containment of the zeros of a given real polynomial outside of a sector in the right-half plane of the complex plane for its finite Hurwitz matrix to be totally nonnegative. We have shown that the enclosing sectors are sharp. As a by-product, we have extended some results known from literature on the spectrum and the eigenspaces of the finite Hurwitz matrix and proved the conjecture posed in [19, Conjecture 3.49].

In closing, we mention an open problem related to our results. In [16], see also [28, Section 4.8], the *generalized Hurwitz matrix*

$$H_\infty^M(p) = (a_{Mj-i})_{i,j=1}^\infty \tag{7.1}$$

for a polynomial p given in (4.1) and a natural number $M \leqslant n$ was introduced. Here we use the convention that $a_k = 0$ for $k < 0$ and $k > n$.

Note that for $M = 1$, this matrix is just the matrix $\mathcal{T}(p)$ in (3.9), so $\mathcal{T}(p) = H_\infty^1(p)$, and for $M = 2$ the infinite Hurwitz matrix $H_\infty(p)$ in (1.3), so $H_\infty(p) = H_\infty^2(p)$. In [18], it was established that if p has only positive coefficients and the matrix $H_\infty^M(p)$ is totally nonnegative, then the polynomial p has no zeros in the sector $|\arg z| < \frac{\pi}{M}$. This result generalizes the results by Aissen-Edrei-Schoenberg-Whitey [3] ($M = 1$), see Theorem 3.6, and the necessary condition in Theorem 4.10 ($M = 2$), and is related to the theorem by Cowling-Thron [10] ($M = n$). A challenging problem is to find suitable conditions on the location of the zeros of p for the total nonnegativity of the generalized Hurwitz matrix. Such a fact was established in [16, Theorems 2.1 and 2.4]: Suppose that p has degree $n \leqslant (M-1)m + 1$ for some integer $m \geqslant 1$. If p has all its zeros in the sector

$$|\pi - \arg z| < \frac{\pi}{m+1},$$

then the matrix $H_\infty^M(p)$ is totally nonnegative[9]. However, if the degree of p increases, this sector becomes smaller and smaller. Therefore, a sufficient condition which does not depend on the degree of p is desired. In [18], it was shown that the condition that the polynomial p is stable does not suffice for general M, $M \neq 2$, (but is suffices if M is even). Furthermore, nothing seems to be known about the eigenstructure of the finite generalized Hurwitz matrices for general M.

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AppendixA. Proof of Proposition 4.4

As announced in Section 4, we give here a proof of Proposition 4.4 which seems to explicitly appear the first time in [8] (and implicitly in [14, 20]). The proof we present here is due to Yu. Barkovsky [7], and is based on properties of R-functions. This proof seems to be new [10], and we allow ourselves to reproduce it here.

In the sequel, we need the following property of R-functions.

[9] In fact, it is even almost strictly totally positive

[10] One more proof distinct from this one and from Gantmacher's proof (see [14, Ch. XV]) and more close to Hurwitz's proof [20] was presented by Barkovsky in [8].

Theorem AppendixA.1. *Let h and f be real polynomials such that $\deg h - 1 \leqslant \deg f = n$. For the real rational function*

$$R = \frac{h}{f},$$

with exactly r poles, the following conditions are equivalent:

1) *R is an R-function:*

2) *The function F can be represented in the form*

$$R(z) = -\alpha z + \beta + \sum_{j=1}^{r} \frac{\gamma_j}{z+\omega_j}, \quad \alpha \geqslant 0, \;\; \beta, \omega_j \in \mathbb{R}, \tag{A.1}$$

where

$$\gamma_j = \frac{h(\omega_j)}{f'(\omega_j)} > 0, \quad j = 1, \ldots, r. \tag{A.2}$$

According to Definition 3.1, R-functions satisfy the condition (3.4).

Now we provide some additional relations between the polynomials (4.4)–(4.5) and the function (4.6). Let the polynomial p be given as in (4.1). Then the polynomials $p_0(z^2)$ and $p_1(z^2)$ satisfy the following identities:

$$\begin{aligned} p_0(z^2) &= \frac{p(z)+p(-z)}{2}, \\ p_1(z^2) &= \frac{p(z)-p(-z)}{2z}. \end{aligned} \tag{A.3}$$

From (A.3) and (4.6) one can derive the relation

$$z\Phi(z^2) = \frac{p(z)-p(-z)}{p(z)+p(-z)} = \frac{1-\dfrac{p(-z)}{p(z)}}{1+\dfrac{p(-z)}{p(z)}}. \tag{A.4}$$

Let us recall the following simple necessary condition for polynomials to be stable. This condition is usually called the *Stodola condition* [14] (see also [8]).

Theorem AppendixA.2. (Stodola) *If the polynomial p is stable, then all its coefficients are positive*[11].

Now we are in a position to prove Proposition 4.4.

Proposition 4.4 ([8, 14]) *The polynomial p defined in* (4.1) *is stable if and only if its associated function Φ defined in* (4.6) *is an R-function with exactly l poles, all of which are negative, and the limit $\lim\limits_{u\to\pm\infty} \Phi(u)$ is positive whenever $n = 2l+1$. The number l is defined in* (4.2).

Proof. Let the polynomial p be stable. First, we show that

$$\left|\frac{p(-z)}{p(z)}\right| < 1, \quad \forall z : \operatorname{Re} z > 0. \tag{A.5}$$

[11] More precisely, the coefficients must be of the same sign, but $a_0 > 0$ by (4.1).

Note that the polynomials $p(z)$ and $p(-z)$ have no common zeros if p is stable, so the function $\dfrac{p(-z)}{p(z)}$ has exactly n poles. The stable polynomial p can be represented in the form

$$p(z) = a_0 \prod_k (z - \lambda_k) \prod_j (z - \xi_j)\left(z - \overline{\xi}_j\right),$$

where $\lambda_k < 0, \operatorname{Re}\xi_j < 0$ and $\operatorname{Im}\xi_j \neq 0$. Then we have

$$\left|\frac{p(-z)}{p(z)}\right| = \prod_k \frac{|z+\lambda_k|}{|z-\lambda_k|} \prod_j \frac{|z+\xi_j|\,\left|z+\overline{\xi}_j\right|}{\left|z-\overline{\xi}_j\right|\,|z-\xi_j|}. \tag{A.6}$$

It is easy to see that the function of type $\dfrac{z+a}{z-\overline{a}}$, where $\operatorname{Re} a < 0$, maps the right half-plane to the unit disk. In fact,

$$\left|\frac{z+a}{z-\overline{a}}\right|^2 = \frac{(\operatorname{Re} z + \operatorname{Re} a)^2 + (\operatorname{Im} z + \operatorname{Im} a)^2}{(\operatorname{Re} z - \operatorname{Re} a)^2 + (\operatorname{Im} z + \operatorname{Im} a)^2} < 1,$$

whenever $\operatorname{Re} z > 0$ and $\operatorname{Re} a < 0$.

Now from (A.6) it follows that the function $\dfrac{p(-z)}{p(z)}$ also maps the right half-plane to the unit disk as a product of functions of such a type. Thus, the inequality (A.5) is valid.

At the same time, the fractional linear transformation $z \mapsto \dfrac{1-z}{1+z}$ conformally maps the unit disk to the right half-plane:

$$|z| < 1 \implies \operatorname{Re}\left(\frac{1-z}{1+z}\right) = \frac{1-|z|^2}{|1+z|^2} > 0. \tag{A.7}$$

Consequently, from the relations (A.4), (A.5), and (A.7) we obtain that the function $z\Phi(z^2)$ maps the right half-plane to itself, so the function $-z\Phi(-z^2)$ maps the upper half-plane of the complex plane to the lower half-plane:

$$\operatorname{Im} z > 0 \implies \operatorname{Re}(-iz) > 0 \implies \operatorname{Re}\left[-iz\Phi(-z^2)\right] =$$

$$\operatorname{Im}\left[z\Phi(-z^2)\right] > 0 \implies \operatorname{Im}\left[-z\Phi(-z^2)\right] < 0.$$

Since p is stable by assumption, the polynomials $p(z)$ and $p(-z)$ have no common zeros, therefore, p_0 and p_1 also have no common zeros, and $p_0(0) \neq 0$ by (4.3). Moreover, by Theorem AppendixA.2 we have $a_0 > 0$ and $a_1 > 0$, so $\deg p_0 = l$ (see (4.4) and (4.5)). Thus, the number of poles of the function $-z\Phi(-z^2)$ equals the number of the zeros of the polynomial $p_0(-z^2)$, i.e., exactly $2l$.

So according to Theorem AppendixA.1, the function $-z\Phi(-z^2)$ can be represented in the form (A.1), where all poles are located symmetrically with respect to 0 and $\beta = 0$, since $-z\Phi(-z^2)$ is an odd function. Denote the poles of $-z\Phi(-z^2)$ by $\pm\nu_1, \ldots, \pm\nu_l$ such that

$$0 < \nu_1 < \nu_2 < \ldots < \nu_l.$$

Note that $\nu_1 \neq 0$, since $p_0(0) \neq 0$ as we mentioned above.

Thus, the function $-z\Phi(-z^2)$ can be represented in the following form

$$-z\Phi(-z^2) = -\alpha z + \sum_{j=1}^{l} \frac{\gamma_j}{z-\nu_j} + \sum_{j=1}^{l} \frac{\gamma_j}{z+\nu_j} = -\alpha z + \sum_{j=1}^{l} \frac{2\gamma_j z}{z^2 - \nu_j^2},$$

where

$$\alpha \geqslant 0,\ \gamma_j, \nu_j > 0.$$

By dividing this equality by $-z$ and changing variables as follows $-z^2 \to u$, $2\gamma_j \to \beta_j$, $\nu_j^2 \to \omega_j$, we obtain the following representation of the function Φ:

$$\Phi(u) = \frac{p_1(u)}{p_0(u)} = \alpha + \sum_{j=1}^{l} \frac{\beta_j}{u+\omega_j}, \tag{A.8}$$

where $\alpha \geqslant 0$, $\beta_j > 0$ and

$$0 < \omega_1 < \omega_2 < \ldots < \omega_l.$$

Here $\alpha = 0$ whenever $n = 2l$, and $\alpha = \dfrac{a_0}{a_1} > 0$ whenever $n = 2l+1$. Since Φ can be represented in the form (A.8), we have that by Theorem AppendixA.1, Φ is an R-function with exactly l poles, all of which are negative, and $\lim\limits_{u\to\pm\infty} \Phi(u) = \alpha > 0$ as $n = 2l+1$.

Conversely, let the polynomial p be defined as in (4.1) and let its associated function Φ be an R-function with exactly l poles, all of which are negative, and $\lim\limits_{u\to\pm\infty} \Phi(u) > 0$ as $n = 2l+1$. We will show that p is stable.

By Theorem AppendixA.1, Φ can be represented in the form (A.8), where $\alpha = \lim\limits_{u\to\pm\infty} \Phi(u) \geqslant 0$ such that $\alpha > 0$ if $n = 2l+1$, and $\alpha = 0$ if $n = 2l$. Thus, the polynomial p_0 has only negative zeros, and the polynomials p_0 and p_1 have no common zeros. Together with (4.3) and (4.6), this implies that the set of zeros of the polynomial p coincides with the set of solutions of the equation

$$z\Phi(z^2) = -1. \tag{A.9}$$

Let λ be a zero of the polynomial p and therefore, a solution of the equation (A.9). Then from (A.8) and (A.9) we obtain

$$-1 = \operatorname{Re}\left[\lambda\Phi(\lambda^2)\right] = \left[\alpha + \sum_{j=1}^{l} \beta_j \frac{|\lambda|^2 + \omega_j}{|\lambda^2+\omega_j|^2}\right] \operatorname{Re}\lambda,$$

where $\alpha \geqslant 0$, and β_j, $\omega_j > 0$ for $j = 1, \ldots, l$. Thus, if λ is a zero of p, then $\operatorname{Re}\lambda < 0$, and so p is stable. □

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