SPEEDING UP AN ALGORITHM FOR CHECKING ROBUST STABILITY OF POLYNOMIALS

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Abstract: The paper considers the robust stability verification of polynomials with polynomial parameter dependency. A new algorithm is presented which relies on the expansion of a multivariate polynomial into Bernstein polynomials and is based on the inspection of the value set of the family of polynomials on the imaginary axis. It is shown how an initial interval on the imaginary axis through which zero crossing of members of the family is only possible is Obtained by applying known bounds for the positive zeros of a polynomial. This interval can be improved by using again Bernstein expansion, thereby speeding up the algorithm considerably.

Keywords: Robust stability, nonlinear parameter variation, value set analysis, bounding method, zero set

1. INTRODUCTION

In the past, robustness analysis of polynomials in the presence of parametric uncertainties has focused on the case of affine and multiaffine parameter dependency, see, e.g. (Ackermann, 1993; Barmish, 1994), and the references therein. However, these cases do not cover most real-life problems. Therefore, this paper is concerned with the far more general case of *polynomial* dependency.

The Robust Stability Problem: Let the parameter set Q be an l-dimensional box, i.e. $Q = [\underline{q}_1, \overline{q}_1] \times \cdots \times [\underline{q}_l, \overline{q}_l]$.
Let a family of polynomials be given by

$$p(s, \mathbf{q}) = \sum_{k=0}^{m} a_{m-k}(\mathbf{q}) s^k, \qquad (1)$$

where the coefficients are depending polynomially on parameters q_i, $i = 1, \ldots, l$, $\mathbf{q} = (q_1, \ldots, q_l)$, i.e. for $k = 0, \ldots, m$

$$a_k(\mathbf{q}) = \sum_{i_1, \ldots, i_l = 0}^{d} a^{(k)}_{i_1 \ldots i_l} q_1^{i_1} \ldots q_l^{i_l}. \qquad (2)$$

Question: *Is the family of polynomials (robustly) stable for Q, i.e. are the polynomials $p(\mathbf{q})$ stable for all $\mathbf{q} \in Q$?*

Here stability is meant in the sense of Hurwitz or of asymptotical stability, i.e. one has to show that $p(s, \mathbf{q}) \neq 0$ for all $s \in \mathbf{C}$ with $Re\, s \geq 0$, $\mathbf{q} \in Q$. To avoid dropping in degree it is assumed throughout this paper that $a_0(\mathbf{q}) \neq 0$ for all $\mathbf{q} \in Q$.

Unfortunately, most of the methods known from literature can only treat problems with polynomial dependency with only a few parameters and/or polynomials of lower degree. For references the reader is referred to (Zettler and Garloff,

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1997); more recent references include (Malan *et al.*, 1996) and (Vehi *et al.*, 1996). (Zettler and Garloff, 1997) presented a new algorithm, the so-called Convex Hull Bernstein Algorithm, which can be used to solve larger robust stability problems. This algorithm is based on the expansion of the even and odd parts of the family (1) into Bernstein polynomials and on the Zero-Exclusion-Theorem, e.g. (Ackermann, 1993; Barmish, 1994) by inspecting the value set of the family (1) on the imaginary axis.

The organization of the paper is as follows: In Sect.2 the Bernstein expansion of a multivariate polynomial is recalled and the sweep procedure is explained which is fundamental for the algorithm. In Sect.3 the new algorithm is presented. A fundamental idea of the new algorithm is to join to the parameter box Q a compact interval Ω [2] enclosing all real numbers ω for which zero crossing over the imaginary axis is possible through $j\omega$. The Bernstein expansion in the variables $q_1, q_2, \ldots, q_l, \omega$ is now applied to the resulting $(l+1)$-dimensional box. The interval Ω is obtained by bounding the range of the coefficient functions $a_k(\mathbf{q})$ (2) over Q and using known bounds on the positive zeros of a polynomial. The resulting bounds are often too pessimistic so that one starts with an interval Ω which is too large causing unnecessarily long computing times. Therefore, emphasis is put on improving upon Ω. This is achieved again by Bernstein expansion. To demonstrate the improvement a numerical example of a large robust control problem is presented in Sect.4. Brief conclusions and directions for further research are given in Sect.5

In applications, e.g. in the robustness analysis of sampled-data control systems, cf. (Ackermann, 1993), one is often faced with exponential parameter dependencies. If these dependencies are not too complicated it is possible to give the range of parts of the coefficient functions over the parameter set Q exactly. Then by introducing new variables the problem can often recast as one involving polynomial parameter dependency.

2. BERNSTEIN EXPANSION

A *multi-index* I is as an ordered l-tupel of non-negative integers $(i_1, \ldots, i_l)$. For simplicity the brackets will be omitted sometimes. Multi-indices will be used e.g. to shorten power products: For $\mathbf{x} = (x_1, \ldots, x_l) \in \mathbf{R}^l$ $\mathbf{x}^I$ stands for $x_1^{i_1} x_2^{i_2} \cdot \ldots \cdot x_l^{i_l}$. Further, let $S = \{I : I \leq N\}$, where $I \leq N$ if $N = (n_1, \ldots, n_l)$ and if $0 \leq i_k \leq n_k$, $k = 1, \ldots, l$. Then an l-variate polynomial p can be written in the form

$$p(\mathbf{x}) = \sum_{I \in S} \mathbf{a}_I \mathbf{x}^I, \quad \mathbf{x} \in \mathbf{R}^l; \tag{3}$$

N is referred to as the *degree of p. The* total degree of polynomial (3) is

$$\hat{n} = \max\{n_i : i = 1, \ldots, l\}. \tag{4}$$

Also, I/N stands for $(i_1/n_1, \ldots, i_l/n_l)$ and $\binom{N}{I}$ for $\binom{n_1}{i_1} \cdot \ldots \cdot \binom{n_l}{i_l}$. With $I = (i_1, \ldots, i_r, \ldots, i_l)$ the index $I_{r,k}$ is associated given by $I_{r,k} = (i_1, \ldots, i_r + k, \ldots, i_l)$, where $0 \leq i_r + k \leq n_r$.

2.1 *Bernstein Transformation of a Polynomial*

In this subsection a given multivariate polynomial (3) is expanded into Bernstein polynomials to obtain bounds for its range over an l-dimensional box, for references see (Garloff, 1993). Without loss of generality the unit box $U = [0,1]^l$ is considered since any nonempty box of $\mathbf{R}^l$ can be mapped affinely onto this box.

For $\mathbf{x} = (x_1, \ldots, x_l) \in \mathbf{R}^l$ the Ith *Bernstein polynomial* of degree N is defined as

$$B_{N,I}(\mathbf{x}) = b_{n_1,i_1}(x_1) b_{n_2,i_2}(x_2) \cdot \ldots \cdot b_{n_l,i_l}(x_l),$$

where for $i_j = 0, \ldots, n_j$, $j = 1, \ldots, l$

$$b_{n_j,i_j}(x_j) = \binom{n_j}{i_j} x_j^{i_j} (1 - x_j)^{n_j - i_j}.$$

The transformation of a polynomial from its power form (3) into its *Bernstein form* results in

$$p(\mathbf{x}) = \sum_{I \in S} b_I(U) B_{N,I}(\mathbf{x}),$$

where the *Bernstein coefficients* $b_I(U)$ of p over U are given by

$$b_I(U) = \sum_{J \leq I} \frac{\binom{I}{J}}{\binom{N}{J}} a_J, \quad I \in S.$$

The Bernstein coefficients are collected in an array $B(U)$, i.e. $B(U) = (b_I(U))_{I \in S}$. A similar notation will be employed for other sets of related coefficients. For an efficient calculation of the Bernstein coefficients see (Garloff, 1986).

An important property of the Bernstein coefficients is their *convex hull property*:

$$\begin{aligned}&\mathrm{Conv}\{(\mathbf{x}, p(\mathbf{x})) : \mathbf{x} \in U\} \\ &\qquad \subseteq \mathrm{Conv}\{(I/N, b_I(U)) : I \in S\}.\end{aligned} \tag{5}$$

[2] In Sect. 3 we consider instead of Ω also the related interval Σ due to the substitution $\sigma = \omega^2$.

2.2 *Sweep Procedure*

A sweep in the rth direction ($1 \le r \le l$) is defined as recursively applied linear interpolation. Let $D = [\underline{d}_1, \overline{d}_1] \times \ldots \times [\underline{d}_l, \overline{d}_l]$ be any subbox of U generated by sweep operations (at the beginning one has $D = U$). Starting with $B^{(0)}(D) = B(D)$ one sets for $k = 1, \ldots, n_r$

$$b_I^{(k)}(D) = \begin{cases} b_I^{(k-1)} : i_r < k \\ (b_{I_{r,-1}}^{(k-1)}(D) + b_I^{(k-1)}(D))/2 : \\ k \le i_r. \end{cases}$$

To obtain the new coefficients, the above formula is applied for $i_j = 0, \ldots, n_j$, $j = 1, \ldots, r-1, r+1, \ldots, l$.

Then the Bernstein coefficients on D_0, where the subbox D_0 is given by

$$D_0 = [\underline{d}_1, \overline{d}_1] \times \ldots \times [\underline{d}_r, \hat{d}_r] \times \ldots \times [\underline{d}_l, \overline{d}_l]$$

are obtained as $B(D_0) = B^{(n_r)}(D)$, where $\hat{d}_r$ denotes the midpoint of $[\underline{d}_r, \overline{d}_r]$. The Bernstein coefficients $B(D_1)$ on the neighbouring subbox D_1

$$D_1 = [\underline{d}_1, \overline{d}_1] \times \cdots \times [\hat{d}_r, \overline{d}_r] \times \cdots \times [\underline{d}_l, \overline{d}_r]$$

are intermediate values of this computation since for $k = 0, \ldots, n_r$ the following relation holds (Garloff, 1993)

$$b_{i_1,\ldots,n_r-k,\ldots,i_l}(D_1) = b_{i_1,\ldots,n_r,\ldots,i_l}^{(k)}(D).$$

In analogy to CAGD the arrays of Bernstein coefficients $B(D_0)$ and $B(D_1)$ are called *patches*. A sweep needs $O(\hat{n}^{l+1})$ additions and binary shifts.

2.3 *Selection Procedures*

Selection of the Sweep Direction

The definition of the sweep procedure shows that one is free in choosing the sweep direction in the computation of the Bernstein coefficients. The rules for selecting the sweep direction are based on an upper bound associated with the first ($h = 1$) or second ($h = 2$) partial derivative of a polynomial in Bernstein form. As sweep direction that r is chosen for which

$$\tilde{I}_r^{(h)} = \max_{I \le N_{r,-h}} \{\frac{n_r!}{(n_r - h)!} ||\Delta_r^{(h)} b_I(D)||_2\}$$

achieves its maximum. Here the difference operator Δ_r is defined recursively by

$$\Delta_r^{(0)} b_I(D) = b_I(D) \quad \text{and for } k = 1, \ldots, h$$
$$\Delta_r^{(k)} b_I(D) = \Delta_r^{(k-1)} b_{I_{r,1}}(D) - \Delta_r^{(k-1)} b_I(D).$$

Patch Selection

After having swept along an axis one has still the choice between two patches. Let $p_i^{(w)}$, $i = 1, \ldots, \mu_w$, generate $\operatorname{Conv} B(D_w)$, cf. p.131 in (Barmish, 1994). Then that patch $B(D_w)$ is chosen for which $(|p_1^{(w)}| + \ldots + |p_{\mu_w}^{(w)}|)/\mu_w$, $w = 0, 1$, is minimal.

3. THE CONVEX HULL BERNSTEIN ALGORITHM AND ITS SPEED-UP

3.1 *The Algorithm*

The case of real coefficient functions (2) is considered first under the assumption that $a_0(\mathbf{q}) > 0$ for all $\mathbf{q} \in Q$. It is assumed furthermore that there is a stable member of the family of polynomials (1) and that $a_m(\mathbf{q}) > 0$ for all $\mathbf{q} \in Q$, otherwise the family (1) would not be robustly stable for Q. To explore the value set $\mathcal{P}(\Omega) = \{p(j\omega, \mathbf{q}) : \omega \in \Omega = [0, \infty) \wedge \mathbf{q} \in Q\}$ of the family of polynomials (1) one splits the polynomial $p(j\omega, \mathbf{q})$ into its even and odd parts

$$p(j\omega, \mathbf{q}) = p_e(\omega^2, \mathbf{q}) + j\omega p_o(\omega^2, \mathbf{q}), \qquad (6)$$

where

$$p_e(\omega^2, \mathbf{q}) = \sum_{k=0}^{\mu} (-1)^k a_{m-2k}(\mathbf{q}) \omega^{2k}$$
$$p_o(\omega^2, \mathbf{q}) = \sum_{k=0}^{\mu} (-1)^k a_{m-2k-1}(\mathbf{q}) \omega^{2k}$$

with $\mu = [m/2]$ for the even and $\mu = [(m-1)/2]$ for the odd part [ ] denoting the integer part.

Then one substitutes $\sigma = \omega^2$ and considers the pair $(p_e(\sigma, \mathbf{q}), p_o(\sigma, \mathbf{q}))$. Under the above assumptions the family (1) is robustly stable for Q if and only if the polynomials $p_e(\mathbf{q})$ and $p_o(\mathbf{q})$ do not have a positive real zero in common for all $\mathbf{q} \in Q$. In the next subsections it will be shown how one can restrict the search for common zeros to a compact interval Σ'. For simplicity, one writes $\mathbf{x} = (x_1, \ldots, x_l) = (q_1, \ldots, q_{l-1}, \sigma)$. After having transformed $Q \times \Sigma'$ to U one checks whether the set $\mathcal{P}(U) = \{(p_e(\mathbf{x}), p_o(\mathbf{x})) : \mathbf{x} \in U\}$ contains the origin.

By expanding $p_e(\mathbf{x})$ and $p_o(\mathbf{x})$ simultaneously into their Bernstein forms one obtains a set of points $(b_I^{(e)}(U), b_I^{(o)}(U))$ in the plane, denoted by $b_I(U)$. Then one computes their convex hull what can be done in optimal time using $O(\nu \log \nu)$ operations, e.g. (Mulmuley, 1994; Preparata and Shamos, 1990), where ν denotes the number of points. Then one checks whether the origin of the plane is contained in the convex hull since Conv $\mathcal{P}(U) \subseteq$ Conv $B(U)$ holds true. If it is outside, the family of polynomials is robustly stable. Otherwise an inclusion test given by (Zettler and Garloff, 1997) is performed. If it fails, i.e. it can not be verified that the origin is in the value set, the sweep procedure is applied splitting the domain to obtain two new patches on which one proceeds as before. If no patch remains and all inclusion tests have failed the family of polynomials is robustly stable. Otherwise, if an inclusion test is successful the algorithm aborts immediately because an unstable polynomial is found.

3.2 *Embracing Possible Zero Crossing over the Imaginary Axis*

Bounds for the Range of the Coefficient Functions: To obtain bounds for the positive zeros of the family of polynomials $p_e(\sigma, \mathbf{q})$ and $p_o(\sigma, \mathbf{q})$ known bounds for the positive zeros of a polynomial are used (see below). However, these bounds are developed for polynomials with constant coefficients. To extend such bounds to families of polynomials interval arithmetic, e.g. (Alefeld and Herzberger, 1983; Neumaier, 1990) is applied giving compact intervals $A_k = [\underline{a}_k, \overline{a}_k]$ with

$$\{a_k(\mathbf{q}) : \mathbf{q} \in Q\} \subseteq A_k, \quad k = 0, \ldots, m.$$

Crude intervals A_k can be computed by naive application of interval arithmetic: substituting for q_i the given interval $[\underline{q}_i, \overline{q}_i]$, $i = 1, \ldots, l$, and replacing the real operations by their corresponding interval operations. More refined techniques for range enclosing, cf. (Alefeld and Herzberger, 1983; Neumaier, 1990; Ratschek and Rokne, 1984), provide tighter intervals. Since $a_k(\mathbf{q})$ is an l-variate polynomial in $q_1, \ldots, q_l$ the basic Bernstein Algorithm (Garloff, 1993) can be applied to obtain tight bounds at the expense of higher computational effort. However, if for a $k \in \{0, \ldots, m\}$ $a_k(\mathbf{q})$ is of low degree and $0 \notin A_k$ the enclosure A_k computed by naive application of interval arithmetic is sufficient.

Bounds for the Zeros of a Polynomial with Parameter Dependent Coefficients: Let $\varphi(\sigma)$ be a real polynomial,

$$\varphi(\sigma) = \sum_{k=0}^{m-1} a_{m-k}\sigma^k + \sigma^m, \tag{7}$$

and let $N_- := \{k \in \mathbf{N} : k < m \wedge a_k < 0\}$ be nonempty. The following upper bounds for the positive zeros of φ are known (only bounds are listed which can easily be extended to the case of parameter dependent coefficients):

$$\delta_1 = 1 + \max_{k \in N_-} |a_k|, \tag{8}$$

$$\delta_2 = 1 + (\max_{k \in N_-} |a_k|)^{1/\min N_-},$$

$$\delta_3 = 2 \max_{k \in N_-} |a_k|^{1/k},$$

$$\delta_4 = \max_{k \in N_-} (card(N_-)|a_k|)^{1/k},$$

where $card(N_-)$ denotes the cardinality of N_-.
The bound δ_1 can be found in (Cajori, 1969) and in (Kioustelidis, 1986), δ_2 is given in (Burnside and Panton, 1960) and in (Householder, 1970), and δ_3 is in (Kioustelidis, 1986) and in (Westerfield, 1933); (Householder, 1970) and (Kioustelidis, 1986) are also references for δ_4.

Application of each of the above upper bounds for the positive zeros of φ to the polynomial $\sigma^m \varphi(1/\sigma)$ constitutes also a lower bound for the positive zeros of φ.

To extend these bounds to the case of parameter dependent coefficients $a_k(\mathbf{q})$ (2), one replaces the constant coefficients a_k by the intervals A_k and evaluates the bounds by using interval arithmetic. E.g. a simple upper bound for the positive zeros of the family $p_e(\mathbf{q})$ with odd m and $\underline{a}_1 > 0$ is given by (cf. (8)) $\overline{\sigma}_e = 1 + \alpha$, where $\alpha = \max\{-\underline{a}_{m-2k}/\underline{a}_1 : k = 0, \ldots, (m-1)/2 \wedge \underline{a}_{m-2k} < 0\}$.

In this way intervals $[\underline{\sigma}_e, \overline{\sigma}_e]$ and $[\underline{\sigma}_o, \overline{\sigma}_o]$ are obtained which enclose the positive zeros of the families $p_e(\mathbf{q})$ and $p_o(\mathbf{q})$, respectively. Then it is sufficient to check $p_e(\mathbf{q})$ and $p_o(\mathbf{q})$ for common zeros only in the interval

$$\Sigma := [\underline{\sigma}_e, \overline{\sigma}_e] \cap [\underline{\sigma}_o, \overline{\sigma}_o]. \tag{9}$$

3.3 *Complex Coefficient Functions*

It will be shown that the case of complex coefficient functions (2)

$$a_k(\mathbf{q}) = b_k(\mathbf{q}) + jc_k(\mathbf{q}) = \sum_{i_1, \ldots, i_l = 0}^{d} (b^{(k)}_{i_1 \ldots i_l} + j\, c^{(k)}_{i_1 \ldots i_l}) q_1^{i_1} \ldots q_l^{i_l},$$

$k = 0, \ldots, m$, can be treated similarly.
As in the real case one splits the polynomial $p(j\omega, \mathbf{q})$ into two real polynomials $p_e(\omega, \mathbf{q})$ and $p_o(\omega, \mathbf{q})$

$$p(j\omega, \mathbf{q}) = p_e(\omega, \mathbf{q}) + jp_o(\omega, \mathbf{q}),$$

where

$$p_e(\omega,\mathbf{q}) = b_0(\mathbf{q}) - c_1(\mathbf{q})\omega - b_2(\mathbf{q})\omega^2 + c_3(\mathbf{q})\omega^3 + b_4(\mathbf{q})\omega^4 - c_5(\mathbf{q})\omega^5 - + \dots$$
$$p_o(\omega,\mathbf{q}) = c_0(\mathbf{q}) + b_1(\mathbf{q})\omega - c_2(\mathbf{q})\omega^2 - b_3(\mathbf{q})\omega^3 + c_4(\mathbf{q})\omega^4 + b_5(\mathbf{q})\omega^5 - + \dots \quad .$$

Under the assumption that there is a stable member of the polynomial family (1) p is robustly stable for Q iff p_e and p_o do not have a real zero in common. As in the real case one determines an interval Ω^+ enclosing all possible common positive zeros of p_e and p_o. By passing to $p_e(-\omega)$ and $p_o(-\omega)$ one obtains similarly an interval Ω^- enclosing all possible common negative zeros of p_e and p_o. Then $\Omega := \Omega^+ \cup \Omega^-$ contains all possible common real zeros of p_e and p_o. The procedure for improving Σ given in the next subsection can be applied similarly to the enclosure Ω.

In passing it is noted that the case that one has to check the polynomial family p for robust Γ-stability, where Γ is a sector with vertex at the origin centered around the negative real axis with aperture $\pi - 2\delta, 0 < \delta < \pi/2$, can be reduced to the case of complex coefficient functions, cf. Sect. 6.8. in (Barmish, 1994). If the coefficient functions of p are all real then (again under the assumption that there is a Γ-stable member) it suffices to check the value set of the transformed polynomial family $p(se^{-j\delta},\mathbf{q}), \mathbf{q} \in Q$, for zero-exclusion over the imaginary axis. If the coefficient functions are complex, the value set of the family $p(se^{j\delta},\mathbf{q}), \mathbf{q} \in Q$ has to be tested additionally.

3.4 *Improving the Bounds*

The interval Σ defined in (9) can be improved. In order to use the previous notation it is assumed that $\dim Q = l - 1$. One shifts the polynomials p_e and p_o from $Q \times \Sigma$ to U and computes the Bernstein coefficients of the shifted polynomials which are denoted again by p_e and p_o. Now by successively sweeping (only in σ-direction) one calculates the Bernstein coefficients of p_e and p_o on $Q\times[0,1/2], Q\times[1/2,1], Q\times[0,1/4], Q\times[3/4,1], \dots$. If at one of these intervals the Bernstein coefficients of p_e or p_o are all positive or all negative this interval can not contain a common zero of p_e and p_o by the convex hull property (5). Then this interval is replaced by the neighbouring interval of the same width to the right or to the left, respectively, on which one already knows the Bernstein coefficients of p_e and p_o, cf. Subsect. 2.2. Continuing in this way one ends up after a few iterates (denoted as *level* in the numerical example in Sect. 4) with a compact interval which corresponds to an interval $\Sigma' \subseteq \Sigma$ again enclosing all common positive zeros of p_e and p_o.

4. EXAMPLE

The example is taken from (Ackermann and Sienel, 1990) and has a multiaffine parameter dependency, where thirteen parameters are involved.

$$p_1(s,c_1,d_1,m_1) = m_1 s^2 + d_1 s + c_1 + 1$$
$$p_2(s,c_2,d_2,m_2) = m_2 s^2 + d_2 s + c_2 + 1$$

$$N_c(s,\mathbf{b}) = b_3 s^3 + b_2 s^2 + b_1 s + b_0$$
$$D_c(s,\mathbf{a}) = s^3 + a_2 s^2 + a_1 s + a_0 .$$

The family of polynomials to be checked for zero exclusion is given by $(p_1 p_2 - 1)D_c + N_c$. The intervals are given by

$$m_1 \in [1,3]\ ,\quad d_1 \in [0.5,2]\ ,\quad c_1 \in [1,2]\ ,$$
$$m_2 \in [2,5]\ ,\quad d_2 \in [0.5,2]\ ,\quad c_2 \in [2,4]\ ,$$

$$a_0 \in [17100, 20900],\ b_0 \in [212062.5, 259187.5]\ ,$$
$$a_1 \in [1305, 1595],\ b_1 \in [805837.5, 984912.5]\ ,$$
$$a_2 \in [55.8, 68.2],\ b_2 \in [721012.5, 881237.5]\ ,$$
$$b_3 \in [424125.0, 518375.0].$$

The algorithm reported that the polynomial family is not robustly stable. In Table 1 the results of the procedure for improving the interval Σ (9) to Σ', are given for different prescribed subdivision levels displayed in the first column. The number *0* stands for no improvement, i.e. the algorithm starts with interval Σ, *1* means the partition of $[0,1]$ into $[0,1/2], [1/2,1]$, etc. In the column labeled *sweeps* the number of sweeps required for the stability check are given. In the last column the computing times on a HP Workstation 9000/755 are presented for the case that the selection of the sweep direction is based on the second partial derivative, cf. Subsect. 2.3. For the selection procedure based on the first partial derivative the results differ only slightly. At the levels 7, 9 the initial interval $\Sigma = [0.48, 4879.76]$ was improved to $\Sigma' = [0.48, 286.38]$. A further improvement based on the procedure described in §3.4 seems not be possible.
In all cases the number of sweeps was identical with the maximum number of sweeps performed on one patch. This documents the efficiency of the selection rules.

Table 1. Results for different levels

level	sweeps	t/sec.
0	6	28
1	4	20
3	2	12
5	2	12
7	1	8
9	1	8

5. CONCLUSIONS

An algorithm for checking robust stability of a family of polynomials has been presented. It has been discussed how to speed up the algorithm by tightening the starting interval for the frequencies relevant for the value set analysis. Further research will be directed to prevent the algorithm to follow inefficient paths in the search tree and to reduce the storage needed since the application of the algorithm for to very large problems is complicated by its excessive memory requirement.

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6. REFERENCES

Ackermann, J. (1993). *Robust Control, Systems with Uncertain Physical Parameters.* Springer, London.

Ackermann, J. and W. Sienel (1990). What is a "large" number of parameters in robust systems?. In: *Proc. 29th Conf. Decision Contr.*, pp. 3496–3497.

Alefeld, G. and J. Herzberger (1983). *Introduction to Interval Computations.* Academic Press, New York.

Barmish, B.R. (1994). *New Tools for Robustness of Linear Systems.* Macmillan, New York.

Burnside, W.S. and A.W. Panton (1960). *The Theory of Equations.* Vol. 1, pp. 180–181. Dover Publ., New York.

Cajori, F. (1969). *An Introduction to the Theory of Equations.* p. 43, Dover Publ., New York.

Garloff, J. (1986). Convergent bounds for the range of multivariate polynomials. In: *Interval Mathematics 1985* (K. Nickel, Ed.), Lecture Notes in Computer Science, Vol. 212, pp. 37–56. Springer, Berlin.

Garloff, J. (1993). The Bernstein algorithm. *Interval Computations,* **2**, 154–168.

Householder, A.S. (1970). *The Numerical Treatment of a Single Nonlinear Equation.* pp. 70. McGraw-Hill, New York.

Kioustelidis, J.B. (1986). Bounds for positive roots of polynomials. *J. Comput. Appl. Math.*, **16**, 241–244.

Malan, S., M. Milanese and M. Taragna (1996). Robust analysis and design of control systems using interval arithmetics. In: *Proc. 13th IFAC World Congr. 1996,* Vol. H (R. Bitmead, I.R. Petersen, H.F. Chen and G. Picci, Eds.), pp. 25–30. Pergamon.

Mulmuley, K. (1994). *Computational Geometry, An Introduction through Randomized Algorithms.* Prentice Hall, Englewood-Cliffs.

Neumaier, A. (1990). *Interval Methods for Systems of Equations.* Cambridge Univ. Press, Cambridge.

Preparata, F.P. and M.I. Shamos (1990). *Computational Geometry: An Introduction.* 3rd ed.. Springer, New York.

Ratschek, H. and J. Rokne (1984). *Computer Methods for the Range of Functions.* Ellis Horwood, Chichester.

Vehi, J., J. Armengol and M. A. Sainz (1996). Robust stability and interval plants with polynomial coefficients using modal interval arithmetic. Rep. No. 96/09–EIA., Systems Engineering and Automatic Control Group, University of Girona.

Westerfield, E.C. (1933). A new bound for the zeros of polynomials. *Amer. Math. Monthly,* **40**, 18–23.

Zettler, M. and J. Garloff (1997). Robustness analysis of polynomials with polynomial parameter dependency using Bernstein expansion. Fachhochschule Konstanz, Fachbereich Informatik, Techn. Rep. No. 9601, to appear in: *IEEE Trans. Automat. Contr.,* Nov. 97.