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Variation diminution and intervals of sign regular matrices

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ABSTRACT

A sign regular matrix is a matrix having the property that its non-zero minors of all orders have, for each order, an identical sign. Such matrices arise in a wide range of applications. In this paper, intervals of real matrices with respect to the usual entry-wise partial ordering are considered. Using variation diminution, it is shown that all matrices in such an interval are sign-regular with the same signature of their minors if a specified finite set of element matrices in the interval has this property.

1. Introduction

A real $m \times n$ matrix A is *variation diminishing* if for any n-vector x, the vector Ax has no more sign changes than x. This property is intimately connected with the following property. A real $m \times n$ matrix is called *sign regular*[1] (abbreviated *SR*) if all its non-vanishing minors of the same order are all positive or all negative. It is termed *strictly sign regular* (*SSR*) if it is *SR* and all its minors are non-zero. Such matrices arise, e.g., in mechanics [1], computer aided geometric design [2], computer vision [3], and dynamic systems [4]. In the most important case, all minors are nonnegative or positive; these *SR* matrices are called *totally nonnegative* (*TN*) and *totally positive* (*TP*), respectively. Properties of these matrices can be found in [1,5,6].

In this paper, we consider intervals of matrices with respect to the usual entry-wise partial ordering. We ask under which conditions all matrices in such a matrix interval are *SR*. The motivation for this question stems from the investigation of the linear complementarity problem [7]. Often properties of this problem like solvability, uniqueness, convexity, and finite number of solutions are reflected by properties of the constraint matrix (for a large collection of respective matrix classes see [8]). In passing, we note that also the *SR* matrices have to be added to these classes [9]. In the case that one considers the linear complementarity problem with uncertain data modeled by intervals [10,11], it is important to know whether the matrices obtained by choosing all possible values in the intervals are in the same matrix class. Then it is an enormous advantage if one could ascertain this containment by checking a finite set of matrices [12,13] - in the ideal case, from only two matrices. Collections of matrix classes which possess such properties can be found in the survey articles [14,15].

The organization of our paper is as follows. In Section 2, we introduce our notation and give some auxiliary results which we use in the subsequent section. In Section 3, we present our main results.

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[1] Named *sign-definite* in the monograph [1].

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2. Notation, definitions, and auxiliary results

2.1. Notation and definitions

For $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$, we denote by $S^-(x)$ the number of sign changes in the sequence $x_1, \ldots, x_n$ after discarding zero components. We set $S^-(0) := 0$. We denote by $S^+(x)$ the maximum possible number of sign changes in the sequence which results when we replace each zero entry by 1 or -1 with the convention $S^+(0) := n$. The *signature* of an $[S]SR$ matrix $A \in \mathbb{R}^{m\times n}$ is the ordered tuple $\epsilon = (\epsilon_1, \ldots, \epsilon_{\min\{m,n\}})$, where ϵ_i is the sign of the non-zero determinants of all $i \times i$ submatrices of A, $i = 1, \ldots, \min\{m, n\}$. We simply say that A is $[S]SR(\epsilon)$. We set $\epsilon_0 := 1$. An SR matrix A is *of oscillatory type* if there is an integer k such that A^k is SSR.

We consider $\mathbb{R}^{m\times n}$ endowed with the usual entry-wise partial ordering, i.e., for $A, B \in \mathbb{R}^{m\times n}$,

$$A \leq B \quad \text{if and only if} \quad a_{ij} \leq b_{ij}, \quad \text{for } i = 1, \ldots, m, j = 1, \ldots, n.$$

Intervals of matrices in $\mathbb{R}^{m\times n}$ with respect to this ordering are denoted by $[A] = [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m\times n} \mid \underline{A} \leq A \leq \overline{A}\}$, with the *corner matrices* $\underline{A} = (\underline{a}_{ij})$ and $\overline{A} = (\overline{a}_{ij})$. The set of all matrix intervals in $\mathbb{R}^{m\times n}$ is denoted by $\mathbb{I}(\mathbb{R}^{m\times n})$. A matrix $A = (a_{ij}) \in [A] = [\underline{A}, \overline{A}]$ with $a_{ij} \in \left\{\underline{a}_{ij}, \overline{a}_{ij}\right\}$, for $i = 1, \ldots, m$, $j = 1, \ldots, n$, is called a *vertex matrix*. Of special interest are the following vertex matrices: Each matrix interval $[A] = [\underline{A}, \overline{A}]$ can be represented as $\left\{A \in \mathbb{R}^{m\times n} \mid |A - A_c| \leq \Delta\right\}$, where $A_c = \frac{1}{2}(\overline{A} + \underline{A})$ is the *midpoint matrix* and $\Delta = \frac{1}{2}(\overline{A} - \underline{A})$ is the *radius matrix*, in particular, $\underline{A} = A_c - \Delta$ and $\overline{A} = A_c + \Delta$.

With $Y_p = \left\{y \in \mathbb{R}^p \mid |y_i| = 1,\ i = 1, \ldots, p\right\}$ and $T_y = \operatorname{diag}(y_1, y_2, \ldots, y_p)$, for integer p, we define the matrices $A_{yz} = A_c - T_y \Delta T_z$ for all $y \in Y_m$, $z \in Y_n$. The definition implies that for all $i = 1, \ldots, m$, $j = 1, \ldots, n$,

$$(A_{yz})_{ij} = (A_c)_{ij} - y_i (\Delta)_{ij} z_j = \begin{cases} \overline{a}_{ij} & \text{if } y_i z_j = -1, \\ \underline{a}_{ij} & \text{if } y_i z_j = 1, \end{cases}$$

so that all matrices A_{yz}, in particular, $\underline{A}$ and $\overline{A}$, are vertex matrices. We denote by $V([A])$ the set of matrices A_{yz} for all $y \in Y_m$, $z \in Y_n$. Since $A_{-y,-z} = A_{yz}$, the cardinality of $V([A])$ is at most 2^{m+n-1}. The vertex matrices which are obtained for $y = (1, -1, 1, \ldots, (-1)^{m+1})$ and $z = (1, -1, 1, \ldots, (-1)^{n+1})$, $z = (-1, 1, -1, \ldots, (-1)^n)$, are the matrices

$$\downarrow A = \begin{bmatrix} \underline{a}_{11} & \overline{a}_{12} & \underline{a}_{13} & \cdots \\ \overline{a}_{21} & \underline{a}_{22} & \overline{a}_{23} & \cdots \\ \underline{a}_{31} & \overline{a}_{32} & \underline{a}_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\uparrow A = \begin{bmatrix} \overline{a}_{11} & \underline{a}_{12} & \overline{a}_{13} & \cdots \\ \underline{a}_{21} & \overline{a}_{22} & \underline{a}_{23} & \cdots \\ \overline{a}_{31} & \underline{a}_{32} & \overline{a}_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

The matrix interval $[A] \in \mathbb{I}(\mathbb{R}^{m\times n})$ can also be represented as a matrix interval with respect to the *checkerboard partial ordering* on $\mathbb{R}^{m\times n}$ given by

$$A \leq^* B \ \text{ if } \ D_m A D_n \leq D_m B D_n, \ \text{ where } \ D_p := \operatorname{diag}(1, -1, 1, \ldots, (-1)^{p+1}), p \in \{m, n\},$$

with the matrices $\downarrow A$ and $\uparrow A$ as corner matrices.

2.2. Auxiliary results

The following proposition is an application of the Cauchy–Binet Formula, see, e.g., [6, Theorem 1.1.1].

Proposition 2.1 (*[5, Theorem 3.1]*). *If $A, B \in \mathbb{R}^{n\times n}$ are SR with signatures $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ and $\delta = (\delta_1, \ldots, \delta_n)$, respectively, then AB is SR with signature $(\epsilon_1\delta_1, \ldots, \epsilon_n\delta_n)$.*

Theorem 2.2 (*[16, Corollary 3.5]*). *Let $[A] = [\underline{A}, \overline{A}] \in \mathbb{I}(\mathbb{R}^{n\times n})$ and assume that $\underline{A}$ and $\overline{A}$ are nonsingular with $\underline{A}^{-1}, \overline{A}^{-1} \geq 0$. Then all matrices $A \in [A]$ are nonsingular and $\overline{A}^{-1} \leq A^{-1} \leq \underline{A}^{-1}$.*

The key to the proofs of our results are the following characterizations.

Theorem 2.3 (*[17, Theorem C]*). *Given $A \in \mathbb{R}^{m\times n}$ and $\epsilon = (\epsilon_1, \ldots, \epsilon_{\min\{m,n\}})$, the following statements are equivalent:*

(i) *A is $SR(\epsilon)$.*
(ii) *For all $x \in \mathbb{R}^n$, we have $S^-(Ax) \le S^-(x)$. Moreover, for all $x \in \mathbb{R}^n$ with $Ax \neq 0$ and $S^-(Ax) = S^-(x) = r$, if $0 \le r \le \min\{m,n\} - 1$, then the sign of the first (last) non-zero component of Ax agrees with $\epsilon_r \epsilon_{r+1}$ times the sign of the first (last) non-zero component of x.*

Theorem 2.4 ([17, *Theorem B]*). *Given $A \in \mathbb{R}^{m\times n}$ and $\epsilon = (\epsilon_1, \dots, \epsilon_{\min\{m,n\}})$, the following statements are equivalent:*

(i) *A is $SSR(\epsilon)$.*
(ii) *For all $0 \neq x \in \mathbb{R}^n$, we have $S^+(Ax) \le S^-(x)$. Further, for all $x \in \mathbb{R}^n$ with $Ax \neq 0$ and $S^+(Ax) = S^-(x) = r$, if $0 \le r \le \min\{m,n\}-1$, then the sign of the first (last) component of Ax (if zero, the unique sign given in determining $S^+(Ax)$) agrees with $\epsilon_r \epsilon_{r+1}$ times the sign of the first (last) non-zero component of x.*

As shown in [17], one can reduce in Theorems 2.3 and 2.4 the set of test vectors x, i.e., the entire set $\mathbb{R}^n \backslash \{0\}$, to a single test vector for each square submatrix (in case of Theorem 2.4, formed from consecutive rows and columns) of A.

The next theorem specifies a theorem by Gantmacher and Krein to the situation in this paper.

Theorem 2.5 ([1, *Theorem 9 in Chapter V]*). *Let $A \in \mathbb{R}^{n\times n}$ be $SR(\epsilon)$ of oscillatory type. Then the eigenvalues λ_i of A are real and their absolute values are distinct. If they are ordered such that*

$$|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n| > 0,$$

then

$$\operatorname{sign}(\lambda_i) = \frac{\epsilon_i}{\epsilon_{i-1}}, \quad i = 1, \dots, n.$$

3. Main results

Theorem 3.1. *Let $[A] \in \mathbb{I}(\mathbb{R}^{m\times n})$ and let $\epsilon = (\epsilon_1, \dots, \epsilon_{\min\{m,n\}})$ be a fixed signature. Then the following statements hold:*

(i) *All matrices in $[A]$ are $SR(\epsilon)$ if and only if all matrices in $V([A])$ are $SR(\epsilon)$. In addition, for $m = n$, all matrices in $[A]$ are nonsingular if and only if the vertex matrices $\downarrow A$ and $\uparrow A$ are nonsingular.*
(ii) *If $m = n$, all matrices in $[A]$ are $SR(\epsilon)$ of oscillatory type if and only if all matrices $A \in V([A])$ are $SR(\epsilon)$ of oscillatory type.*

Proof. Let $[A] = [\underline{A}, \overline{A}] \in \mathbb{I}(\mathbb{R}^{m\times n})$. For simplicity, we write V for $V([A])$. We have to show only the necessity, and assume that all matrices in V are $SR(\epsilon)$.

(i) Let $A = (a_{ij}) \in [A]$ and $x \in \mathbb{R}^n$. We construct the vertex matrix $B = (b_{ij}) \in V$ row by row ($i = 1, \dots, n$) as follows.

If $0 \le (Ax)_i$, then

$$b_{ij} = \begin{cases} \overline{a}_{ij}, & \text{if } x_j \ge 0, \\ \underline{a}_{ij}, & \text{if } x_j < 0; \end{cases} \tag{1}$$

and if $0 > (Ax)_i$, then

$$b_{ij} = \begin{cases} \underline{a}_{ij}, & \text{if } x_j \ge 0, \\ \overline{a}_{ij}, & \text{if } x_j < 0. \end{cases} \tag{2}$$

Then we obtain, in the case $0 \le (Ax)_i$ that

$$0 \;\le\; (Ax)_i = \sum_{j=1}^{n} a_{ij}\, x_j \;\le\; \sum_{x_j \ge 0} \overline{a}_{ij}\, x_j + \sum_{x_j<0} \underline{a}_{ij}\, x_j = (Bx)_i,$$

and similarly, in the case $0 > (Ax)_i$ that

$$0 \;>\; (Ax)_i \ge \sum_{x_j \ge 0} \underline{a}_{ij}\, x_j + \sum_{x_j<0} \overline{a}_{ij}\, x_j = (Bx)_i.$$

Therefore, we get

$$S^-(Ax) \;\le\; S^-(Bx) \;\le\; S^-(x), \tag{3}$$

since $B \in V$, and we conclude that $S^-(Ax) \;\le\; S^-(x)$, i.e., we have shown the first part of (ii) in Theorem 2.3.

To show the remaining part, let $x \in \mathbb{R}^n$ with $Ax \neq 0$ and assume that $S^-(Ax) = S^-(x) = r$, where $0 \le r \le \min\{m,n\} - 1$. It follows from (3) that $S^-(Bx) = S^-(x) = r$. Since $Ax \neq 0$, there is an index i, $1 \le i \le m$, with $(Ax)_i \neq 0$ which implies $(Bx)_i \neq 0$.

Without loss of generality, we may assume that $1 \le r$. Assume that we discard all zero components in Ax and call the resulting vector v. Then we discard the non-zero components in Bx and call the resulting vector w. If there is a sign change in v at indices i_1 and i_2 with $i_1 < i_2$ and no sign change between them, then w has also a sign change at i_1 and i_2 and no sign change between them, and vice versa since $S^-(Ax) = S^-(Bx)$. Assume that Bx and Ax have their first non-zero components at indices j_1 and j_2, respectively. Then $j_1 \le j_2$. By the definition of B, $j_1 < j_2$ is only possible if $(Ax)_{j_2} > 0$, because in the case that $(Ax)_{j_2} < 0$, if $(Bx)_{j_1} < 0$, then also $(Ax)_{j_1} < 0$ and j_2 is not the first non-zero component of Ax, and in the case that $(Bx)_{j_1} > 0$, we obtain a contradiction to $S^-(Ax) = S^-(Bx)$. By the last argument, $(Ax)_{j_2} > 0$ implies that $(Bx)_j \ge 0$, for all $j_1 \le j < j_2$. So the sign of the first non-zero component of Ax equals the sign of the first non-zero component of Bx, which in turn agrees by Theorem 2.3 with $\epsilon_r\epsilon_{r+1}$ times the sign of the first non-zero component of x. The arguments for the last non-zero components of Ax and x are similar.

Assume now that for $m = n$, the matrices $\downarrow A$ and $\uparrow A$ are nonsingular; their inverses have a checkerboard signed pattern. Multiplication of $\downarrow A^{-1}$ and $\uparrow A^{-1}$ by D_n from the left and from the right results in entry-wise nonnegative or nonpositive matrices according to the sign of $\epsilon_{n-1}\epsilon_n$. By application of Theorem 2.2, we obtain that all matrices $A \in [A]$ are nonsingular.

(ii) Let $m = n$. By Proposition 2.1 it follows that for an $SR(\epsilon)$ matrix A, its even powers are TN and its odd powers are $SR(\epsilon)$. Assume that for each matrix $A \in V$, A is $SR(\epsilon)$ and there is an integer k_A such that A^{k_A} is SSR. Set $k := \max\{k_A \mid A \in V\}$. Then it follows from the Cauchy–Binet Formula that for each matrix $A \in V$, A^k is SSR, and without loss of generality, we may assume that A^k is TP.

Assume first that $\epsilon_1 = 1$, i.e., $0 \le \underline{A}$. Let $A = (a_{ij}) \in [A]$. Then we obtain with

$$\underline{A}^k = (\underline{a}_{ij}^{(k)}), \quad A^k = (a_{ij}^{(k)}), \quad \overline{A}^k = (\overline{a}_{ij}^{(k)}),$$

the relation

$$0 < \underline{A}^k \le A^k \le \overline{A}^k.$$

Let $x \in \mathbb{R}^n, x \neq 0$. Then $A^k x \neq 0$. Choose a fixed assignment of ∓ 1 to the zero components of $A^k x$, which results in $S^+(A^k x)$. The following inequalities hold for $i = 1, \dots, n$:

$$\sum_{x_j \ge 0} \underline{a}_{ij}^{(k)} x_j + \sum_{x_j < 0} \overline{a}_{ij}^{(k)} x_j \le \sum_{j=1}^{n} a_{ij}^{(k)} x_j \le \sum_{x_j \ge 0} \overline{a}_{ij}^{(k)} x_j + \sum_{x_j < 0} \underline{a}_{ij}^{(k)} x_j.$$

We form the vertex matrix $B = (b_{ij}) \in V$ row by row ($i = 1, \dots, n$) by using (1), (2), and $\underline{a}_{ij}$ and $\overline{a}_{ij}$ replaced by $\underline{a}_{ij}^{(k)}$ and $\overline{a}_{ij}^{(k)}$, respectively. If $0 < (A^k x)_i$, then take (1), and if $0 > (A^k x)_i$, then use (2). If $(A^k x)_i = 0$, then take (1) and (2) if the sign according to the assignment above is 1 or -1, respectively. If then $(Bx)_i = 0$, choose the sign according to the assignment above. By definition of B, we get $S^+(A^k x) = S^+(Bx) \le S^-(x)$ by [6, Theorem 4.3.5] because B is TP. Now assume $S^+(A^k x) = S^-(x) = r$, where $0 \le r \le min\{m, n\} - 1$. It follows that $Bx \neq 0$ and $S^+(Bx) = S^-(x) = r$. By definition of B, the sign of $(A^k x)_1$ (if it is zero, the unique sign according to the assignment above) equals the sign of $(Bx)_1$ which in turn agrees by Theorem 2.4 with the sign of the first non-zero component of x. The arguments for the last non-zero components of Ax and x are similar. It follows from Theorem 2.4 that A^k is TP and thus A is of oscillatory type.

We proceed similarly if $\epsilon_1 = -1$, i.e., $0 \ge \overline{A}$. ■

Remark 3.2.

(1) *Reduction of the set $V([A])$:* The following vertex matrices are not needed. The vertex matrices B with either $b_{ij} = \underline{a}_{ij}, i = 1, \dots, m$, or $b_{ij} = \overline{a}_{ij}, i = 1, \dots, m, j = 1, \dots, n$, are chosen if $S^-(x) = 0$. Multiplication of an entry-wise nonnegative or nonpositive matrix by a vector without any sign change results in turn in such a single-signed vector.

(2) In the proof of part (ii) of Theorem 3.1, it follows from $S^+(A^k x) \le S^-(x)$ for all $x \in R^n, x \neq 0$, by [17, Theorem A] that A^k is SSR. As an alternative to the use of the second part of (ii) in Theorem 2.3, in order to show that A^k is TP, too, one can proceed as follows: Assume that A is $SSR(\delta)$ but not TP. By Theorem 2.5, the following inequalities hold for A:

$$\delta_1 \lambda_1(A) > \frac{\delta_2}{\delta_1}\lambda_2(A) > \cdots > \frac{\delta_n}{\delta_{n-1}}\lambda_n(A) > 0.$$

Let q, $1 < q < n-1$, be the smallest index such that $\delta_q = -1$. Take a continuous path $\varphi(t), t \in [0,1]$, with $\varphi(0) = \underline{A}$ and $\varphi(1) = A$. Since all matrices in $[A]$ are nonsingular, $\lambda_q(\varphi(t))$ is non-zero for all $t \in [0,1]$. Thus the inequality

$$\frac{\delta_q}{\delta_{q-1}}\lambda_q(A) = -\lambda_q(\varphi(1)) > 0$$

provides by $\lambda_q(\varphi(0)) = \lambda_q(\underline{A}) > 0$ a contradiction.

We mention a related result which can be proven by an obvious extension of the proof of Theorem 1 in [18]; see also Theorem B in [19].

Theorem 3.3. *Let $[A] \in \mathbb{I}(R^{m\times n})$ and $\epsilon = (\epsilon_1, \dots, \epsilon_{min\{m,n\}})$ be a fixed signature. Then all matrices in $[A]$ are SSR(ϵ) if and only if the vertex matrices $\downarrow A$ and $\uparrow A$ are SSR(ϵ).* □

4. Conclusions

We conclude this paper with a discussion of similar results obtained so far for intervals of special *SR* matrices. Firstly, we note a large gap in the number of vertex matrices required to conclude that all matrices in the interval are *SR* with a certain signature and possibly additional properties. On one side, we have classes of nonsingular *SR* matrices for which the two vertex matrices $\downarrow A$ and $\uparrow A$ suffice. These include the matrices which are *SSR*, see Theorem 3.3, nonsingular and almost *SSR* (a class between the nonsingular *SR* and the *SSR* matrices) [20, Theorem 6.5], and nonsingular, tridiagonal *SR* [20, Theorem 6.11]. A list of nonsingular *SR* matrices with 16 periodic signatures (possibly with an additional (strict) sign condition for one entry) can be found in [21]. On the other hand, Theorem 3.1 requires the set $V([A])$ of vertex matrices. A related result is that the classes of the matrices having all their minors up to a certain order either nonnegative [19, Theorem D] or nonpositive [22, Theorem 5.5] require also the set $V([A])$ of vertex matrices. If $n = m$, an open question, see [14, Conjecture 3.1], is whether for nonsingular *SR* matrices the set $V([A])$ of vertex matrices can be further reduced — in the ideal case to the vertex matrices $\downarrow A$ and $\uparrow A$. For general, not necessarily nonsingular SR matrices, $\downarrow A$ and $\uparrow A$ do not suffice, as the counterexamples for $n = 3$ in [6, Section 3.2] and for $n = 4$ in [18] show.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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