Linear Algebra and its Applications 514 (2017) 222–233

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Invariance of total nonnegativity of a matrix under entry-wise perturbation and subdirect sum of totally nonnegative matrices

Mohammad Adm [a,b], Jürgen Garloff [c,d,*]

[a] Department of Applied Mathematics and Physics, Palestine Polytechnic University, Hebron, Palestine
[b] Zukunftskolleg/Department of Mathematics and Statistics, University of Konstanz, D-78464 Konstanz, Germany
[c] Department of Mathematics and Statistics, University of Konstanz, D-78464 Konstanz, Germany
[d] Institute for Applied Research, University of Applied Sciences/HTWG Konstanz, D-78405 Konstanz, Germany

ARTICLE INFO

Article history:
Received 24 May 2016
Accepted 1 November 2016
Available online 11 November 2016
Submitted by R. Brualdi

MSC:
15B48

Keywords:
Totally nonnegative matrix
Entry-wise perturbation
k-subdirect sum

ABSTRACT

A real matrix is called totally nonnegative if all of its minors are nonnegative. In this paper, the minors are determined from which the maximum allowable entry perturbation of a totally nonnegative matrix can be found, such that the perturbed matrix remains totally nonnegative. Also, the total nonnegativity of the first and second subdirect sum of two totally nonnegative matrices is considered.

* Corresponding author.
E-mail addresses: mjamathe@yahoo.com, moh_95@ppu.edu (M. Adm), garloff@htwg-konstanz.de, Juergen.Garloff@htwg-konstanz.de (J. Garloff).

http://dx.doi.org/10.1016/j.laa.2016.11.001
0024-3795/

1. Introduction

A real matrix is called totally nonnegative (positive) if all of its minors are nonnegative (positive). For properties of these matrices the reader is referred to the two monographs [8,14] and the survey paper [6]. In the first part of our paper (taken from the dissertation [1] of the first author) we consider the question of how much the entries of a totally nonnegative matrix can be perturbed without losing the property of total nonnegativity. This question is solved in [3] for tridiagonal totally nonnegative matrices, in [5] for totally positive matrices, and in [9], see also [8, Section 9.5], for a few specified entries. A related question is whether all matrices lying between two totally nonnegative matrices, with respect to a suitable partial ordering, are totally nonnegative as well. The second author conjectured in 1982 [10] that this is true for the nonsingular totally nonnegative matrices and the so-called checkerboard ordering, see also [8, Section 3.2] and [14, Section 3.2]. In [2] we applied the so-called Cauchon algorithm to settle this conjecture.

In the second part of our paper we use the so-called condensed form of the Cauchon algorithm, see [1,4], to study special cases of the k-subdirect sum of totally nonnegative matrices [7]. We first give a short proof for the fact that the 1-subdirect sum of totally nonnegative matrices is in turn totally nonnegative [7]. The 2-subdirect sum of totally nonnegative matrices is studied in [12]. We present a counterexample which shows that a result in [12] does not hold. Finally we derive a necessary and sufficient condition for two totally nonnegative matrices given as in [12] that their 2-subdirect sum is totally nonnegative.

The organization of our paper is as follows. In the next section we explain our notation and collect some auxiliary results. In Section 3 we present our main results on the perturbation of the entries of totally nonnegative matrices. In Section 4 we give our results on the 2-subdirect sum of totally nonnegative matrices.

2. Notation and preliminary results

The set of n-by-m real matrices will be denoted by $\mathbb{R}^{n,m}$. For κ, n we denote by $Q_{\kappa,n}$ the set of all strictly increasing sequences of κ integers chosen from $\{1, 2, \ldots, n\}$. For $\alpha \in Q_{\kappa,n}$ we define $\alpha^c := \{1, \ldots, n\} \setminus \alpha$, where $\alpha^c \in Q_{n-\kappa,n}$.

For $A \in \mathbb{R}^{n,m}$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\kappa) \in Q_{\kappa,n}$, and $\beta = (\beta_1, \beta_2, \ldots, \beta_\mu) \in Q_{\mu,m}$, we denote by $A[\alpha|\beta]$ the κ-by-μ submatrix of A lying in the rows indexed by $\alpha_1, \alpha_2, \ldots, \alpha_\kappa$ and columns indexed by $\beta_1, \beta_2, \ldots, \beta_\mu$. We suppress the brackets when we enumerate the indices explicitly. By $A(\alpha|\beta)$ we denote the $(n-\kappa)$-by-$(m-\mu)$ submatrix $A[\alpha^c|\beta^c]$ of A. When $\alpha = \beta$, the *principal submatrix* $A[\alpha|\alpha]$ is abbreviated to $A[\alpha]$ and $\det A[\alpha]$ is called a *principal minor*, with the similar notation $A(\alpha)$ for the complementary principal submatrix.

A matrix $A \in \mathbb{R}^{n,m}$ is called *totally nonnegative* (*totally positive*) (abbreviated TN (TP)) if $\det A[\alpha|\beta] \geq 0$ (> 0), for all $\alpha \in Q_{\kappa,n}$, $\beta \in Q_{\kappa,m}$, $\kappa = 1, \ldots, n' := \min\{n, m\}$. In passing, we note that if A is TN then so are its transpose and $A^\# := T_n A T_n$, where

$T_n = (t_{ij})$ is the permutation matrix of order n with $t_{i,n-i+1} = 1$, for $i = 1, \ldots, n$, the backward identity matrix, see, e.g., [8, Theorem 1.4.1]. The n-by-n matrix whose only nonzero entry is a one in the (i,j)th position is denoted by E_{ij}.

We denote the set of the n-by-m TN matrices by $TN_{n,m}$, and by $\mathcal{F}$ any family of pairs $[\alpha|\beta]$ for some $\alpha \in Q_{k,n}$, $\beta \in Q_{k,m}$, and $k \leq n'$, i.e.,

$$\mathcal{F} := \{[\alpha|\beta] \;\mid\; \alpha \in Q_{k,n},\; \beta \in Q_{k,m} \text{ for some } k \leq n'\}.$$

The set $TN_{n,m}$ admits a partition into so-called totally nonnegative cells as follows [11, 13]. The *totally nonnegative cell* (abbreviated TN *cell*) *associated with the family of minors* $\mathcal{F}$, denoted by $\mathcal{S}_{\mathcal{F}}$, is defined as

$$\mathcal{S}_{\mathcal{F}} := \{A \in TN_{n,m} \mid \det A[\alpha|\beta] = 0 \text{ if and only if } [\alpha|\beta] \in \mathcal{F}\}. \tag{1}$$

There is a parametrization of the nonempty TN cells by using Cauchon diagrams. In fact, there is a one to one correspondence between these diagrams and the TN cells [11,13].

Definition 2.1. [13, Definition 2.1] An n-by-m *Cauchon diagram* C is an n-by-m grid consisting of $n \cdot m$ squares colored black and white, where each black square has the property that either every square to its left (in the same row) or every square above it (in the same column) is black. The set of n-by-m Cauchon diagrams is denoted by $\mathcal{C}_{n,m}$.

Following [11], we identify an n-by-m Cauchon diagram with the set of coordinates of its black squares, i.e., we fix positions in a Cauchon diagram in the following way: For $C \in \mathcal{C}_{n,m}$ and $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, m\}$, we say that $(i,j) \in C$ if the square in row i and column j is black.

Definition 2.2. [13, Definition 2.2] Let $A = (a_{ij}) \in \mathbb{R}^{n,m}$ and $C \in \mathcal{C}_{n,m}$. We say that A is a *Cauchon matrix associated with the Cauchon diagram* C if for all (i,j), $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, m\}$, we have $a_{ij} = 0$ if and only if $(i,j) \in C$. If A is a Cauchon matrix associated with an unspecified Cauchon diagram, we just say that A is a *Cauchon matrix*.

If A is a Cauchon matrix, then we also say that C is a *Cauchon diagram associated to the Cauchon matrix* A, denoted by C_A, if A is a Cauchon matrix associated with the Cauchon diagram C.

Now we present the condensed form of the Cauchon Algorithm, which can be used to check a given matrix for total nonnegativity, and for membership in a specified TN cell, see [11,13]. The following algorithm called *the condensed form of the Cauchon Algorithm* is introduced in [4]; it reduces the number of the required arithmetic operations of the original Cauchon Algorithm from $O(n^4)$ to $O(n^3)$ (n being the order of the matrices which we consider for simplicity as square).

Algorithm 2.1. [4, Algorithm 3.2], [1, Algorithm 3.3] *Let* $A = (a_{ij}) \in \mathbb{R}^{n,m}$. *Set* $A^{(n)} := A$.
For $k = n-1, \ldots, 1$ *define* $A^{(k)} = (a_{ij}^{(k)}) \in \mathbb{R}^{n,m}$ *as follows:*
For $i = 1, \ldots, k$,
for $j = 1, \ldots, m-1$
set $u_j := \min \left\{ h \in \{j+1, \ldots, m\} \mid a_{k+1,h}^{(k+1)} \neq 0 \right\}$ *(we set* $u_j := \infty$ *if this set is empty)*

$$a_{ij}^{(k)} := \begin{cases} a_{ij}^{(k+1)} - \frac{a_{k+1,j}^{(k+1)} a_{iu_j}^{(k+1)}}{a_{k+1,u_j}^{(k+1)}} & \text{if } u_j < \infty, \\ a_{ij}^{(k+1)} & \text{if } u_j = \infty, \end{cases}$$

for $i = k+1, \ldots, n$, $j = 1, \ldots, m$, *and* $i = 1, \ldots, k$, $j = m$

$$a_{ij}^{(k)} := a_{ij}^{(k+1)}.$$

Put $\tilde{A} := A^{(1)}$; $\tilde{A}$ *is called* the matrix obtained from A (by the condensed form of the Cauchon Algorithm).

Note that in the formula of the above algorithm the entry $a_{ij}^{(k)}$ is obtained from the minor $A^{(k+1)}[i, k+1|j, u_j]$.

Theorem 2.1. [13, Theorem 2.6] *Let* $A \in \mathbb{R}^{n,m}$. *Then* A *is* TN *if and only if* $\tilde{A}$ *is a nonnegative Cauchon matrix.*

Proposition 2.1. [2, Proposition 2.8] *Let* $A \in TN_{n,n}$. *Then* A *is nonsingular if and only if* $\tilde{a}_{ii} > 0$, $i = 1, \ldots, n$.

In [13], it was shown that the Cauchon Algorithm represents a bijection between the TN cells and the set of Cauchon diagrams. We denote by S_C^0 the TN cell associated with the Cauchon diagram C. The following special type of finite sequences will play a fundamental role in characterizing TN cells.

Definition 2.3. [13, Definition 3.1] Let $C \in \mathcal{C}_{n,m}$. We say that a sequence

$$\gamma := ((i_k, j_k), k = 0, 1, \ldots, p) \tag{2}$$

which is strictly increasing in both arguments is a *lacunary sequence with respect to* C if the following conditions hold:

1. $(i_k, j_k) \notin C$, $k = 1, \ldots, p$.
2. $(i, j) \in C$ for $i_p < i \leq n$ and $j_p < j \leq m$.
3. Let $s \in \{0, \ldots, p-1\}$. Then $(i, j) \in C$ if

(a) either for all (i,j), $i_s < i < i_{s+1}$ and $j_s < j$,
or for all (i,j), $i_s < i < i_{s+1}$ and $j_0 \leq j < j_{s+1}$

and

(b) either for all (i,j), $i_s < i$ and $j_s < j < j_{s+1}$,
or for all (i,j), $i < i_{s+1}$, and $j_s < j < j_{s+1}$.

In the sequel, let $\delta_{ij} := \det A[i_0, i_1, \ldots, i_p | j_0, j_1, \ldots, j_p]$ be the minor of A associated to the sequence given by (2), which starts at position $(i,j) = (i_0, j_0)$, and which is constructed by the following procedure. We describe only the construction of the sequence from the starting pair (i_0, j_0) to the next pair (i_1, j_1) with $0 < \delta_{i_1,j_1}$, since for a given matrix A, the determinantal test is performed by moving row by row from the bottom to the top row. Once we have found the next index pair (i_1, j_1), we append to (i_0, j_0) the sequence starting at (i_1, j_1).

Procedure 2.1. *[4, Procedure 5.2], [1, Procedure 3.2] Construction of the sequence γ given by (2) starting at (i_0, j_0) to the next index pair (i_1, j_1) for the n-by-m TN matrix A.*

If *$i_0 = n$ or $j_0 = m$ or $\mathcal{U} := \{(i,j) \mid i_0 < i \leq n,\ j_0 < j \leq m,$ and $0 < \delta_{ij}\}$ is void*
then *terminate with $p := 0$;*
else
  if *$\delta_{ij_0} = 0$ for all $i = i_0 + 1, \ldots, n$* **then** *put $(i_1, j_1) := \min \mathcal{U}$ with respect to the colexicographic order*
  else
  put *$i' := \min\{k \mid i_0 < k \leq n$ such that $0 < \delta_{kj_0}\}$,*
    $J := \{k \mid j_0 < k \leq m$ such that $0 < \delta_{i',k}\}$;
   if *J is not void* **then** *put $(i_1, j_1) := (i', \min J)$*
   else *put $(i_1, j_1) := \min \mathcal{U}$ with respect to the lexicographic order;*
   end if
  end if
end if.

Theorem 2.2. [13, Theorem 4.4] *Let $A \in \mathbb{R}^{n,m}$ and $C \in \mathcal{C}_{n,m}$. Then the following two statements are equivalent:*

(i) *The matrix A is TN and $A \in \mathcal{S}_C^0$.*
(ii) *For each (i_0, j_0), $i_0 \in \{1, \ldots, n\}$, $j_0 \in \{1, \ldots, m\}$, fix a lacunary sequence γ given by (2) (with respect to C) starting at (i_0, j_0). Then*

$$\det A[i_0, i_1, \ldots, i_p | j_0, j_1, \ldots, j_p] \begin{cases} = 0 & \text{if } (i,j) \in C, \\ > 0 & \text{if } (i,j) \notin C. \end{cases}$$

We conclude this section by recalling the definition of the k-subdirect sum of two matrices. Let $0 \le k \le n'$ and $A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{m,m}$ be partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where $A_{22}, B_{11} \in \mathbb{R}^{k,k}$. Then we call

$$C = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} + B_{11} & B_{12} \\ 0 & B_{21} & B_{22} \end{bmatrix}$$

the *k-subdirect sum* of A and B, which is denoted by $A \oplus_k B$.

3. Perturbation of totally nonnegative matrices

In this section the perturbation of the single entries of a TN matrix is investigated. We will not give explicit bounds as in [3,5] since the situation is different from one TN matrix to another. Instead, we determine the minors from which such bounds can be calculated. The TN cells and the determinantal tests that are presented in Section 2 play a fundamental role in our analysis.

The entry in position $(1,1)$ of a given n-by-n TN matrix is easily handled. It can be arbitrary increased since it enters positively in any minor containing it. The bound for adding a negative value to the $(1,1)$ entry, when the principal minor which is formed by deleting the first row and column is positive, is presented in the following lemma. The entry in position (n,n) can be handled analogously.

Lemma 3.1. *[9, Lemma 4.1], [8, Lemma 9.5.2] Let $A \in TN_{n,n}$ with $\det A(1) \neq 0$. Then $A - \tau E_{11}$ is TN for all $\tau \in \left[0, \dfrac{\det A}{\det A(1)}\right]$.*

The following theorem generalizes the above lemma to the case that the principal minor which is formed by deleting the first row and column of a given TN matrix is nonnegative.

Theorem 3.1. *Let $A \in TN_{n,n}$. Then $A - \tau E_{11}$ is TN for all $\tau \in [0, \tilde{a}_{11}]$, where $\tilde{A} = (\tilde{a}_{ij})$ is the matrix obtained by the application of Algorithm 2.1 to A.*

Proof. Let $B := A - \tau E_{11}$. Then the application of Algorithm 2.1 to B yields the matrix $\tilde{B} = (\tilde{b}_{ij})$, which coincides with $\tilde{A}$ in all of its entries, with the exception of $\tilde{b}_{11} = \tilde{a}_{11} - \tau$. Since A is TN, we have by Theorem 2.1 that $\tilde{A}$ is a nonnegative Cauchon matrix, and so $\tilde{B}$ is a Cauchon matrix with nonnegative entries for all $\tau \in [0, \tilde{a}_{11}]$. Hence by Theorem 2.1 the claim follows. □

Remark 3.1. If in the above theorem $0 < \det A(1)$, then by application of Proposition 2.1 to $A(1)$ we obtain that the main diagonal of $\tilde{A}(1)$ is positive. Hence the sequence $((1,1),(2,2),\ldots,(n,n))$ is a lacunary sequence with respect to the Cauchon diagram $C_{\tilde{A}}$. Therefore by [13, Proposition 4.1] we have

$$\tilde{a}_{11} = \frac{\det A}{\det A(1)},$$

and so Lemma 3.1 becomes a special case of the above theorem.

Now we turn to the other entries and derive our main results. Let $A \in TN_{n,n}$. Then by Theorem 2.1 $\tilde{A}$ is a nonnegative Cauchon matrix. Suppose that we want to perturb the entry in position (i,j) for some $i,j \in \{1,\ldots,n\}$ such that the resulting matrix is TN. Let $A_{\tau\pm} := A \pm \tau E_{ij}$. Then we want to find bounds on τ such that $A_{\tau\pm}$ is TN. If we apply Procedure 2.1 to $A_{\tau\pm}$ then the sequences that are starting at the positions (k,l), $i < k$, $l = 1,\ldots,n$, and $k = 1,\ldots,n$, $j < l$, coincide with the corresponding lacunary sequences which we obtain when we apply Procedure 2.1 to A and hence they are lacunary. Thus we may concentrate on the sequences that are starting from the positions (k,l), $k = 1,\ldots,i$, $l = 1,\ldots,j$.

In order to find the values of τ such that $A_{\tau\pm}$ is TN, we distinguish the following two cases:

(i) A and $A_{\tau\pm}$ belong to the same TN cell, or
(ii) $A_{\tau\pm}$ transits possibly to another TN cell.

Case (i): If we want the matrix $A_{\tau\pm}$ to stay in the same TN cell as A, i.e., $C_{\tilde{A}} = C_{\tilde{A}_{\tau\pm}}$, then by Theorem 2.2 the lacunary sequences that are starting at the positions (k,l), $k = 1,\ldots,i$, $l = 1,\ldots,j$, with respect to $C_{\tilde{A}_{\tau\pm}}$ must coincide with the corresponding lacunary sequences with respect to $C_{\tilde{A}}$. Hence we have to find the values of τ that preserve the positivity or the vanishing of the minors associated with these lacunary sequences. Thus we have at most $i \cdot j$ minors to consider. The following minors can be discarded from further consideration since they do not impose any restriction on the values of τ:

(1) The minors which do not involve the entry $a_{ij} \pm \tau$, and
(2) the minors which involve the entry $a_{ij} \pm \tau$ but in such a way that the minors that are obtained from the corresponding submatrices by deleting row i and column j in A are zero.

In [5, Theorem 4.1] we give the minimal conditions that are required for the maximum allowable perturbation of a TP matrix. Hence by employing the approach that has been used in the proof of [5, Theorem 4.1] we may discard further superfluous conditions (minors).

Case (ii): If the matrix $A_{\tau\pm}$ is allowed to transit to another TN cell, then the analysis is more involved. First we consider the entries in the first column, i.e., $A_{\tau\pm} = A \pm \tau E_{i1}$, $i = 2, \ldots, n$. We have to treat the following two cases:

- Suppose that $\tilde{a}_{i1} = 0$. Then we conclude that $\tau = 0$ in $A_{\tau-}$ since the minor that corresponds to the lacunary sequence that starts from the position $(i, 1)$ with respect to $C_{\tilde{A}}$ becomes negative while in $A_{\tau+}$ we have one of the following two cases:
  - $\tilde{a}_{il} = 0$ for all $l = 2, \ldots, h$ for some $2 \le h \le n$. Then we distinguish the following two cases:
    * $\tilde{a}_{kl} = 0$ for all $k = 1, \ldots, i-1$ and $l = 2, \ldots, h$. Then τ can be calculated by using the minors that correspond to the lacunary sequences which start from the positions $(k, 1)$, $k = 1, \ldots, i$, and contain an entry (i, t) for some $t > 1$.
    * Otherwise there exists $k_0 \in \{1, \ldots, i-1\}$ and $l_0 \in \{2, \ldots, h\}$ such that $0 < \tilde{a}_{k_0, l_0}$; since $\tilde{A}_{\tau+}$ must be a Cauchon matrix we conclude that $\tau = 0$.
  - $0 < \tilde{a}_{i2}$. Then τ can be calculated by using the minors that correspond to the lacunary sequences which start from the positions $(k, 1)$, $k = 1, \ldots, i$, and contain an entry (i, t) for some $t > 1$.
- $0 < \tilde{a}_{i1}$, then τ in $A_{\tau\pm}$ can be calculated by using the minors that correspond to the lacunary sequences which start from the positions $(k, 1)$, $k = 1, \ldots, i$, and contain an entry (i, t) for some $t > 1$.

In the sequel we assume that $1 < i, j < n$ since otherwise we consider A^T for the first row, $A^{\#}$ for the last column, and $A^{\#^T}$ for the last row of A.

Suppose that $\tilde{a}_{ij} = 0$. Then we conclude that $\tau = 0$ in $A_{\tau-}$ while in $A_{\tau+}$ we distinguish the following two cases:

- Suppose that $\tilde{a}_{i,t_1} = 0$ for all $t_1 = j+1, \ldots, h_1$ for some $j + 1 \le h_1 \le n$ and there exists $0 < \tilde{a}_{i_0,t_0}$ for some $i_0 \in \{1, \ldots, i-1\}$, $t_0 \in \{j+1, \ldots, h_1\}$, or $\tilde{a}_{t_2,j} = 0$ for all $t_2 = i+1, \ldots, h_2$ for some $i + 1 \le h_2 \le n$ and there exists $0 < \tilde{a}_{i',t'}$ for some $i' \in \{i+1, \ldots, h_2\}$, $t' \in \{1, \ldots, j-1\}$. Then $\tau = 0$ which follows from the fact that $\tilde{A}_{\tau+}$ must be a Cauchon matrix.
- Otherwise τ can be calculated by using the minors corresponding to the lacunary sequences that start from the positions (k, l), $k = 1, \ldots, i$, $l = 1, \ldots, j$, and contain the entries (i, t) and (h, j) for $t, h \in \{1, \ldots, n\}$.

If $0 < \tilde{a}_{ij}$ then we have to consider all the lacunary sequences that start from (k, l), $k = 1, \ldots, i$, $l = 1, \ldots, j$, and find the restrictions on the values of τ in $A_{\tau\pm}$.

4. Subdirect sum of totally nonnegative matrices

In this section we consider the k-subdirect sum of TN matrices with $k = 1, 2$. We start with $k = 1$ and present a shorter proof for a result given in [7].

Theorem 4.1. [7, Theorem 5.1], see [8, Theorem 10.2.3] *Let*

$$C = \begin{bmatrix} C_{11} & c_{12} & 0 \\ c_{21} & c_{22} & c_{23} \\ 0 & c_{32} & C_{33} \end{bmatrix} \in \mathbb{R}^{n+m-1,n+m-1}, \tag{3}$$

where $C_{11} \in \mathbb{R}^{n-1,n-1}$, $c_{12} \in \mathbb{R}^{n-1,1}$, $c_{21} \in \mathbb{R}^{1,n-1}$, $c_{23} \in \mathbb{R}^{1,m-1}$, $c_{32} \in \mathbb{R}^{m-1,1}$, $C_{33} \in \mathbb{R}^{m-1,m-1}$, *and* c_{22} *is a constant. Then* C *is* TN *if and only if* C *can be written as* $C = A \oplus_1 B$, *where* A *and* B *are* TN.

Proof. Suppose that C can be written as $C = A \oplus_1 B$, where A and B are TN. Then application of Algorithm 2.1 to C results in

$$\tilde{C} = \begin{bmatrix} \tilde{C}_{11} & \tilde{c}_{12} & 0 \\ \tilde{c}_{21} & \tilde{c}_{22} & \tilde{c}_{23} \\ 0 & \tilde{c}_{32} & \tilde{C}_{33} \end{bmatrix}, \tag{4}$$

where $\tilde{C}_{33} = \tilde{B}[2,\dots,m]$, $\tilde{c}_{23} = \tilde{B}[1|2,\dots,m]$, $\tilde{c}_{32} = \tilde{B}[2,\dots,m|1]$, $\tilde{c}_{22} = a_{nn} + \tilde{b}_{11}$, $\tilde{c}_{21} = \tilde{A}[n|1,\dots,n-1]$, $\tilde{c}_{12} = \tilde{A}[1,\dots,n-1|n]$, and $\tilde{C}_{11} = \tilde{D}[1,\dots,n-1]$, where $D := A + \tilde{b}_{11}E_{nn}$. By the discussion in the paragraph preceding Lemma 3.1 and Theorem 2.1 we have $\tilde{A}$, $\tilde{B}$, and $\tilde{D}$ are nonnegative Cauchon matrices. Hence $\tilde{C}$ is a nonnegative Cauchon matrix. Therefore by Theorem 2.1 C is TN.

Let C given by (3) be a TN matrix. Then define A and B as follows:

$$A[1,\dots,n-1] := C_{11},\quad A[1,\dots,n-1|n] := c_{12},\quad A[n|1,\dots,n-1] := c_{21},$$
$$a_{nn} := \tilde{c}_{22},\quad b_{11} := c_{22} - \tilde{c}_{22}, B[1|2,\dots,m] := c_{23},\quad B[2,\dots,m|1] := c_{32},\ \text{and}$$
$$B[2,\dots,m] := C_{33}.$$

Obviously $C = A \oplus_1 B$ holds. Application of Algorithm 2.1 to A and B results in $\tilde{A}$ and $\tilde{B}$, respectively, where

$$\tilde{A}[1,\dots,n-1] = \tilde{C}_{11},\quad \tilde{A}[1,\dots,n-1|n] = \tilde{c}_{12},\quad \tilde{A}[n|1,\dots,n-1] = \tilde{c}_{21},$$
$$\tilde{a}_{nn} = \tilde{c}_{22},$$

and

$$\tilde{b}_{11} = 0,\quad \tilde{B}[2,\dots,m|1] = \tilde{c}_{23},\quad \tilde{B}[1|2,\dots,m] = \tilde{c}_{32},\quad \tilde{B}[2,\dots,m] = \tilde{C}_{33}.$$

Therefore, since $\tilde{C}$ is a nonnegative Cauchon matrix we obtain that $\tilde{A}$ and $\tilde{B}$ are so. Hence by Theorem 2.1, A and B are TN matrices. This completes the proof. □

Next we turn to the 2-subdirect sum of TN matrices. In [12, Theorem 4.2], a result is presented which states that the 2-subdirect sum of TN matrices is TN under some conditions. This result seems not to be true as the following example shows.

Example 4.1. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

Then A and B are TN matrices and satisfy the conditions of [12, Theorem 4.2] with $k_1 = 1$. By direct computations we have

$$C := A \oplus_2 B = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

but $\det C[3, 4, 5] = -1$.

The following theorem gives necessary and sufficient conditions for two given special matrices (as in [12]) that their 2-subdirect sum is TN.

Theorem 4.2. *Let*

$$C = \begin{bmatrix} C_{11} & c_{12} & 0 & 0 \\ c_{21} & c_{22} & c_{23} & 0 \\ 0 & c_{32} & c_{33} & c_{34} \\ 0 & 0 & c_{43} & C_{44} \end{bmatrix} \in \mathbb{R}^{n+m-2,n+m-2}, \tag{5}$$

such that $C_{11} \in \mathbb{R}^{n-2,n-2}$, $c_{12} \in \mathbb{R}^{n-2,1}$, $c_{21} \in \mathbb{R}^{1,n-2}$, $c_{34} \in \mathbb{R}^{1,m-2}$, $c_{43} \in \mathbb{R}^{m-2,1}$, $C_{44} \in \mathbb{R}^{m-2,m-2}$, and $c_{22}, c_{23}, c_{32}, c_{33}$ are constants. Then C is TN if and only if C can be written as $C = A \oplus_2 B$, where A and B are TN and $\eta \geq 0$, where η is the determinant of the following matrix

$$\begin{bmatrix} b_{11} & a_{n-1,n} + b_{12} \\ a_{n,n-1} + b_{21} & a_{nn} + \tilde{b}_{22} \end{bmatrix}. \tag{6}$$

Proof. Suppose that C can be written as $C = A \oplus_2 B$, where A and B are TN and $\eta \geq 0$. Then application of Algorithm 2.1 to C results in

$$\tilde{C} = \begin{bmatrix} \tilde{C}_{11} & \tilde{c}_{12} & 0 & 0 \\ \tilde{c}_{21} & \tilde{c}_{22} & \tilde{c}_{23} & 0 \\ 0 & \tilde{c}_{32} & \tilde{c}_{33} & \tilde{c}_{34} \\ 0 & 0 & \tilde{c}_{43} & \tilde{C}_{44} \end{bmatrix}, \tag{7}$$

where

$$\tilde{C}_{44} = \tilde{B}[3,\ldots,m], \quad \tilde{c}_{43} = \tilde{B}[3,\ldots,m|2], \quad \tilde{c}_{34} = \tilde{B}[2|3,\ldots,m], \quad \tilde{c}_{33} = a_{nn} + \tilde{b}_{22},$$
$$\tilde{c}_{32} = a_{n,n-1} + b_{21}, \quad \tilde{c}_{23} = a_{n-1,n} + b_{12}, \quad \tilde{c}_{22} = \begin{cases} a_{n-1,n-1} + b_{11} & \text{if } \tilde{c}_{33} = 0, \\ \frac{\zeta}{\tilde{c}_{33}} & \text{if } \tilde{c}_{33} > 0, \end{cases}$$
$$\tilde{c}_{21} = \tilde{A}[n-1|1,\ldots,n-2], \quad \tilde{c}_{12} = \tilde{A}[1,\ldots,n-2|n-1],$$
$$\tilde{C}_{11} = \tilde{D}[1,\ldots,n-2] \text{ with } D = A[1,\ldots,n-1] + \vartheta E_{n-1,n-1}.$$

Herein ζ, ϑ denote

$$\zeta = \det \begin{bmatrix} a_{n-1,n-1} + b_{11} & a_{n-1,n} + b_{12} \\ a_{n,n-1} + b_{21} & a_{nn} + \tilde{b}_{22} \end{bmatrix}, \quad \vartheta = \begin{cases} b_{11} & \text{if } \tilde{c}_{33} = 0, \\ \frac{\eta}{\tilde{c}_{33}} & \text{if } \tilde{c}_{33} > 0. \end{cases}$$

Hence $\tilde{C}$ is a nonnegative Cauchon matrix since $\tilde{A}$ and $\tilde{B}$ are so and $\eta \geq 0$ (and consequently also $\zeta \geq 0$).

Let C given by (5) be a TN matrix. Then define A and B as follows:

$$A[1,\ldots,n-2] := C_{11}, \quad A[1,\ldots,n-2|n-1] := c_{12}, \quad A[1,\ldots,n-2|n] := 0,$$
$$A[n-1|1,\ldots,n-2] := c_{21}, \quad A[n|1,\ldots,n-2] := 0,$$
$$\begin{cases} a_{n-1,n-1} := c_{22} - \frac{c_{23}c_{32}}{\tilde{c}_{33}}, \quad b_{11} := \frac{c_{23}c_{32}}{\tilde{c}_{33}} & \text{if } \tilde{c}_{33} > 0, \\ a_{n-1,n-1} := c_{22}, \quad b_{11} := 0 & \text{if } \tilde{c}_{33} = 0, \end{cases}$$
$$a_{n-1,n} := c_{23}, \quad b_{12} := 0, \quad a_{n,n-1} := 0, \quad b_{21} := c_{32}, \quad a_{nn} := \tilde{c}_{33}, \quad b_{22} := c_{33} - \tilde{c}_{33},$$
$$B[1|3,\ldots,m] := 0, \quad B[2|3,\ldots,m] := c_{34}, \quad B[3,\ldots,m|1] := 0,$$
$$B[3,\ldots,m|2] := c_{43}, \quad B[3,\ldots,m] := C_{44}.$$

Obviously $C = A \oplus_2 B$ holds. To complete the proof we have to show that A and B are TN matrices and $\eta \geq 0$.

Application of Algorithm 2.1 to A and B yields

$$\tilde{A}[1,\ldots,n-2] = \tilde{C}_{11}, \quad \tilde{A}[1,\ldots,n-2|n-1] = \tilde{c}_{12}, \quad \tilde{A}[1,\ldots,n-2|n] = 0,$$
$$\tilde{A}[n-1|1,\ldots,n-2] = \tilde{c}_{21}, \quad \tilde{A}[n|1,\ldots,n-2] = 0,$$
$$\tilde{a}_{n-1,n-1} = a_{n-1,n-1} (= \tilde{c}_{22}), \quad \tilde{a}_{n-1,n} = c_{23}, \quad \tilde{a}_{n,n-1} = 0, \quad \tilde{a}_{nn} = \tilde{c}_{33},$$

and

$$\tilde{b}_{11} = b_{11}, \quad \tilde{b}_{12} = 0, \quad \tilde{b}_{21} = c_{32}, \quad \tilde{b}_{22} = 0, \quad \tilde{B}[1|3,\ldots,m] = 0,$$
$$\tilde{B}[2|3,\ldots,m] = \tilde{c}_{34}, \quad \tilde{B}[3,\ldots,m|1] = 0, \quad \tilde{B}[3,\ldots,m|2] = \tilde{c}_{43},$$
$$\tilde{B}[3,\ldots,m] = \tilde{C}_{44}.$$

Therefore, since $\tilde{C}$ is a nonnegative Cauchon matrix we obtain that $\tilde{A}$ and $\tilde{B}$ are so. Hence by Theorem 2.1, A and B are TN matrices. By direct calculations we obtain

$$\eta = \begin{cases} c_{23}c_{32} - c_{23}c_{32} & \text{if } \tilde{c}_{33} > 0, \\ -c_{23}c_{32} & \text{if } \tilde{c}_{33} = 0. \end{cases}$$

Not only in the first case but also in the second case $\eta = 0$ holds since $\tilde{C}$ is a nonnegative Cauchon matrix and so in the case $\tilde{c}_{33} = 0$ we must have $c_{23} = 0$ or $c_{32} = 0$. This completes the proof. □

Acknowledgements

The first author gratefully acknowledges support by the Zukunftskolleg/Universität Konstanz.

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