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# On the spectral properties of nonsingular matrices that are strictly sign-regular for some order with applications to totally positive discrete-time systems

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ABSTRACT

A matrix is called strictly sign-regular of order $k$ (denoted by $SSR_k$) if all its $k \times k$ minors are non-zero and have the same sign. For example, totally positive matrices, i.e., matrices with all minors positive, are $SSR_k$ for all $k$. Another important subclass are those that are $SSR_k$ for all odd $k$. Such matrices have interesting sign variation diminishing properties, and it has been recently shown that they play an important role in the analysis of certain nonlinear cooperative dynamical systems. In this paper, the spectral properties of nonsingular matrices that are $SSR_k$ for a specific value $k$ are studied. One of the results is that the product of the first $k$ eigenvalues is real and of the same sign as the $k \times k$ minors, and that linear combinations of certain eigenvectors have specific sign patterns. It is then shown how known spectral properties for matrices that are $SSR_k$ for several values of $k$ can be derived from these results. Using these theoretical results, the notion of a totally positive discrete-time system (TPDTS) is introduced. This may be regarded as the discrete-time analogue of the important notion of a totally positive differential system. It is shown that TPDTSs can be applied to prove that certain time-varying nonlinear dynamical systems entrain to periodic excitations.



## 1. Introduction

A matrix (not necessarily square) is called *sign-regular of order $k$* (denoted by $SR_k$) if all its minors of order $k$ are non-negative or all are non-positive. It is called *strictly sign-regular of order $k$* (denoted by $SSR_k$) if it is sign-regular of order $k$, and all the minors of order $k$ are non-zero. In other words, all minors of order $k$ are non-zero and have the same sign. A matrix is called *sign-regular* ($SR$) if it is $SR_k$ for all $k$, and *strictly sign-regular* ($SSR$) if it is $SSR_k$ for all $k$.

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The most prominent examples of $SR$ [$SSR$] matrices are totally nonnegative $TN$ [totally positive $TP$] matrices, that is, matrices with all minors nonnegative [positive]. Such matrices have applications in a number of fields including approximation theory, economics, probability theory, computer aided geometric design and more [2,4,12].

A very important property of $SSR$ matrices is that multiplying a vector by such a matrix can only decrease the number of sign variations. To explain this variation diminishing property (VDP), we introduce some notation. We use small letters to denote column vectors. If $y \in \mathbb{R}^n$ is such a vector then $y'$ denotes its transpose. For $y \in \mathbb{R}^n$, we use $s^-(y)$ to denote the number of sign variations in $y$ after deleting all its zero entries, and $s^+(y)$ to denote the maximal possible number of sign variations in $y$ after each zero entry is replaced by either $+1$ or $-1$. For example, for $y = [1 \quad -1 \quad 0 \quad -\pi]'$, $s^-(y) = 1$ and $s^+(y) = 3$. Obviously,

$$0 \leq s^-(y) \leq s^+(y) \leq n-1 \text{ for all } y \in \mathbb{R}^n.$$

The first important results on the VDP of matrices were obtained by Fekete [3] and Schoenberg [14]. Later on, Grantmacher and Krein [4, Chapter V] elaborated rather completely the various forms of VDPs and worked out the spectral properties of $SR$ matrices. Two examples of such VDPs are: if $A \in \mathbb{R}^{n\times m}$ ($m \leq n$) is $SR$ and of rank $m$ then

$$s^-(Ax) \leq s^-(x) \text{ for all } x \in \mathbb{R}^m,$$

whereas if $A$ is $SSR$ then

$$s^+(Ax) \leq s^-(x) \text{ for all } x \in \mathbb{R}^m \setminus \{0\}.$$

The more recent literature on $SR$ matrices focuses on the recognition and factorization of matrices with special signs of their minors, see, e.g., [6] and the references therein.

There is a renewed interest in such VDPs in the context of dynamical systems. Indeed, Reference [10] shows that powerful results on the asymptotic behavior of the solutions of continuous-time nonlinear cooperative and tridiagonal dynamical systems can be derived using the fact that the transition matrix of a certain linear time-varying system (called the variational system) is $TP$ for all time $t$. In other words, the linear system is a totally positive differential system (TPDS) [15]. These transition matrices are real, square, and nonsingular. In a recent paper [1], it is shown that the transition matrix satisfies a VDP with respect to the *cyclic* number of sign variations if and only if it is $SSR_k$ for all odd $k$ and all $t$.

Some of the spectral properties of $SR$ matrices can be extended to matrices which are $SR_k$ for all $k$ up to a certain order [4, Chapter V]. In this paper, we study the spectral properties of nonsingular matrices which are $SSR_k$ for a specific value of $k$. Such matrices are only rarely considered in the literature, see, e.g., [11], where a test for an $n \times k$ matrix with $k < n$ to be $SSR_k$ is presented. Let $\epsilon_k \in \{-1, 1\}$ denote the sign of the minors of order $k$, with convention $\epsilon_0 := 1$. We prove that the strict sign-regularity of order $k$ implies that the product of the first $k$ eigenvalues is real and has the sign $\epsilon_k$, and that certain eigenvectors satisfy a special sign pattern. Note that these eigenvectors are in general complex, but their real and imaginary part satisfy a special sign pattern. Then we show how to extend these results to obtain the spectral properties of matrices that are $SSR_k$ for several values of $k$, for example, for all odd $k$.

The theoretical results are used to derive a new class of dynamical systems called *totally positive discrete-time systems* (TPDTSs). This is the discrete-time analogue of the important notion of a TPDS. We analyze the asymptotic properties of TPDTSs and show how they can be used to show that certain nonlinear time-varying dynamical systems entrain to periodic excitations. This result may be regarded as the analogue of an important result of Smith [17] on entrainment in a continuous-time periodic nonlinear cooperative systems with a special Jacobian.

The remainder of this paper is organized as follows. The next section provides known definitions and reviews results that will be used later on. In Section 3, we present our main results and in Section 4 we apply these results to introduce and analyze TPDTSs. The final section concludes and outlines possible topics for further research.

## 2. Preliminaries

In this section, we collect several known definitions and results that will be used later on.

Given a square matrix $A \in \mathbb{R}^{n\times n}$ and $p \in \{1,\dots,n\}$, consider the $\binom{n}{p}^2$ minors of $A$ of order $p$. Each minor is defined by a set of $p$ row indexes

$$1 \le i_1 < i_2 < \cdots < i_p \le n \tag{1}$$

and $p$ column indexes

$$1 \le j_1 < j_2 < \cdots < j_p \le n. \tag{2}$$

This minor is denoted by $A(\alpha|\beta)$, where $\alpha := \{i_1,\dots,i_p\}$ and $\beta := \{j_1,\dots,j_p\}$ (with a mild abuse of notation, we will regard these sequences as sets). We suppress the curly brackets if we enumerate the indexes explicitly.

The next result, known as Jacobi's identity, provides information on the relation between the minors of a nonsingular matrix $A \in \mathbb{R}^{n\times n}$ and the minors of $A^{-1}$. For a sequence $\alpha = \{i_1,\dots,i_p\}$, let $\bar{\alpha}$ denote the sequence $\{1,\dots,n\} \setminus \alpha$ which we consider as ordered increasingly.

**Proposition 1.** *[12, Section 1.2] Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular matrix, and put $B := A^{-1}$. Pick $p \in \{1,\dots,n-1\}$. Then for any two sequences $\alpha = \{i_1,i_2,\dots,i_p\}$ and $\beta = \{j_1,j_2,\dots,j_p\}$ satisfying (1) and (2) we have*

$$B(\alpha,\beta) = (-1)^s \frac{A(\bar{\beta},\bar{\alpha})}{\det(A)},$$

*with $s := i_1 + \cdots + i_p + j_1 + \cdots + j_p$.*

Let $D_{\pm1} \in \mathbb{R}^{n\times n}$ denote the diagonal matrix with diagonal entries $(1,-1,1,-1,\dots,(-1)^{n+1})$, and let $\operatorname{adj}(A)$ denote the adjugate of $A$. Then $D_{\pm1}\operatorname{adj}(A)D_{\pm1}^{-1}$ is called the unsigned adjugate of $A$. Proposition 1 yields the following result.

**Corollary 1.** *Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular matrix. Suppose that for some $k \in \{1,\dots,n-1\}$ all the minors of order $k$ of $A$ are non-zero and have the same sign. Then all minors of order $n-k$ of the unsigned adjugate of $A$ are non-zero and have the same sign.*

The $SSR_k$ property is closely related to VDPs. The following well-known result demonstrates this.

**Proposition 2.** *[4, Chapter V, Theorem 1] Consider a matrix $U \in \mathbb{R}^{n\times m}$ with $n > m$. The following two conditions are equivalent:*

*(i) For any $x \in \mathbb{R}^m \setminus \{0\}$, we have*

$$s^+(Ux) \le m-1. \tag{3}$$

*(ii) The matrix $U$ is $SSR_m$, that is, all minors of the form*

$$U(i_1 \ \dots \ i_m|1 \ \dots \ m), \ \text{with } 1 \le i_1 < i_2 < \cdots < i_m \le n, \tag{4}$$

*are non-zero and have the same sign.*

Note that the assumption that $n > m$ cannot be weakened. For example, if we take $n = m = 2$ and a square matrix $U \in \mathbb{R}^{2\times 2}$ then condition (i) obviously holds, yet condition (ii) only holds if $U$ is nonsingular, so the conditions are not equivalent in this case.

It was recently shown that for square matrices the $SSR_k$ property is equivalent to a non-standard VDP.

**Theorem 1.** *[1, Theorem 1] Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular matrix. Pick $k \in \{1, \dots, n\}$. Then the following two conditions are equivalent:*

*(i) For any vector $x \in \mathbb{R}^n \setminus \{0\}$ with $s^-(x) \le k-1$, we have*

$$s^+(Ax) \le k-1.$$

*(ii) $A$ is $SSR_k$.*

We emphasize that condition (i) in Theorem 1 does not assert that $s^-(x) \le k-1$ implies that $s^+(Ax) \le s^-(x)$, but only that $s^+(Ax) \le k-1$.

## 3. Main results

We consider from here on a nonsingular matrix $A \in \mathbb{R}^{n\times n}$. We order its eigenvalues such that

$$|\lambda_1| \ge |\lambda_2| \ge \cdots \ge |\lambda_n| > 0, \tag{5}$$

with complex conjugate eigenvalues appearing in consecutive pairs (we say, with a mild abuse of notation, that $z \in \mathbb{C}^n$ is *complex* if $z \ne \bar{z}$, where $\bar{z}$ denotes the complex conjugate of $z$). Let

$$v^1, \ v^2, \dots, v^n \in \mathbb{C}^n \tag{6}$$

denote the eigenvectors corresponding to the $\lambda_i$'s. We always assume that every $v^i$ is not purely imaginary. Indeed, otherwise we can replace $v^i$ by $\mathrm{Im}(v^i)$ that is a real eigenvector. Also, note the fact that $A$ is real means that if $v^i$ is complex then its real and imaginary parts can be chosen as linearly independent.

Define a set of real vectors $u^1, u^2, \dots, u^n \in \mathbb{R}^n$ by going through the $v^i$'s as follows. If $v^1$ is real then we put $u^1 := v^1$ and proceed to examine $v^2$. If $v^1$ is complex (and whence $v^2 = \bar{v}^1$) then we put $u^1 := \mathrm{Re}(v^1)$, $u^2 := \mathrm{Im}(v^1)$ and proceed to examine $v^3$, and so on. Let $U := \begin{bmatrix} u^1 & \dots & u^n \end{bmatrix} \in \mathbb{R}^{n\times n}$.

Suppose that for some $i, k$ the eigenvector $v^i$ is real and $v^k$ is complex. Then is not difficult to show that since $A$ is real and nonsingular, the real vectors $v^i, \mathrm{Re}(v^k), \mathrm{Im}(v^k)$ are linearly independent.

Note that if $v^i, v^{i+1} \in \mathbb{C}^n$ is a complex conjugate pair and $c \in \mathbb{C} \setminus \{0\}$ is complex then

$$cv^i + \bar{c}v^{i+1} = 2(\mathrm{Re}(c)\,\mathrm{Re}(v^i) - \mathrm{Im}(c)\,\mathrm{Im}(v^i)) \in \mathbb{R}^n \setminus \{0\},$$

and by choosing an appropriate complex $c \in \mathbb{C} \setminus \{0\}$ we can get any nonzero real linear combination of the real vectors $\mathrm{Re}(v^i)$ and $\mathrm{Im}(v^i)$.

We say that a set $c_p, \ldots, c_k \in \mathbb{C}$, $p \leq k$, *matches* the set $v^p, \ldots, v^k$ of consecutive eigenvectors (6) if the $c_i$'s are not all zero and for every $i$ if the vector $v^i$ is real then $c_i$ is real, and if $v^i, v^{i+1}$ is a complex conjugate pair then $c_{i+1} = \bar{c}_i$. In particular, this implies that $\sum_{i=p}^{k} c_i v^i \in \mathbb{R}^n$.

In order to prove our main results, we need the following auxiliary result that is a generalization of Proposition 2 to the case of eigenvectors of a nonsingular real matrix. We use the notation $j$ for the imaginary unit ($j^2 = -1$).

**Proposition 3.** *Consider the set of $n$ vectors $v^1, \ldots, v^n \in \mathbb{C}^n$. Define $V \in \mathbb{C}^{n\times n}$ by*

$$V := \begin{bmatrix} v^1 & v^2 & \ldots & v^n \end{bmatrix}.$$

*The following two conditions are equivalent:*

*(i) For any $c_1, \ldots, c_m \in \mathbb{C}$ that match $v^1, \ldots, v^m$, with $m \leq n$, we have*

$$s^+(\sum_{i=1}^{m} c_i v^i) \leq m - 1. \tag{7}$$

*(ii) Let $q \in \mathbb{C}^{\binom{n}{m}}$ be the vector that contains all the minors of order $m$ of $V$ of the form $V(i_1 \ldots i_m|1 \ldots m)$, with $1 \leq i_1 < i_2 < \cdots < i_m \leq n$, arranged in the lexicographic order. Then there exists an integer $k$ such that the vector $j^k q$ is real and all its entries are positive.*

**Proof.** Consider first the case $m = 1$. In this case, the definition of a matching set means that $v^1$ is real. Condition (i) becomes

$$s^+(v^1) = 0 \tag{8}$$

and condition (ii) becomes

$$v^1 = j^k q$$

for some integer $k \geq 0$ and $q$ a real vector with all entries positive. Since $v^1$ is real, this means that all the entries of $v^1$ are either all positive or all negative. Thus, in this case the two conditions are indeed equivalent.

Suppose now that $m = 2$. Then two cases are possible.

*Case 1.* Both $v^1$ and $v^2$ are real. Then by the definition of a matching set, (i) becomes: for any $d_1, d_2 \in \mathbb{R}$, that are not both zero,

$$s^+(d_1 v^1 + d^2 v^2) \leq 1.$$

On the other-hand, (ii) becomes: the vector $q \in \mathbb{C}^{\binom{n}{2}}$ that contains all the minors of order 2 of $V$ of the form $V(i_1\ i_2|1\ 2)$, with $1 \leq i_1 < i_2 \leq n$, arranged in the lexicographic order, is real and all its entries are non-zero and have the same sign. Now the equivalence of the two conditions follows from Proposition 2.

*Case 2.* Both $v^1$ and $v^2$ are complex. In this case, $v^2 = \mathrm{Re}(v^1) - j\,\mathrm{Im}(v^1)$, and (i) becomes: for any $d_1, d_2 \in \mathbb{R}$, that are not both zero,

$$s^+(d_1\,\mathrm{Re}(v^1) + d_2\,\mathrm{Im}(v^1)) \leq 1.$$

On the other-hand, (ii) becomes: let $q \in \mathbb{C}^{\binom{n}{2}}$ be the vector that contains all the minors of order 2 of

$$V = \begin{bmatrix} \mathrm{Re}(v^1) + j\,\mathrm{Im}(v^1) & \mathrm{Re}(v^1) - j\,\mathrm{Im}(v^1) \end{bmatrix}$$

of the form $V(i_1\ i_2|1\ 2)$, with $1 \leq i_1 < i_2 \leq n$, arranged in the lexicographic order. Then there exists an integer $k$ such that the vector $j^k q$ is real, and all its entries are positive.

Let $U := \begin{bmatrix} \mathrm{Re}(v^1) & \mathrm{Im}(v^1) \end{bmatrix}$. Then

$$V(i_1\ i_2|1\ 2) = (-2j)U(i_1\ i_2|1\ 2)$$

for all $1 \leq i_1 < i_2 \leq n$. Since $U$ is a real matrix, (ii) becomes: let $q \in \mathbb{R}^{\binom{n}{2}}$ be the vector that contains all the minors of order 2 of $U$ of the form $U(i_1\ i_2|1\ 2)$, with $1 \leq i_1 < i_2 \leq n$, arranged in the lexicographic order. Then $q$ is real, and all its entries are non-zero and have the same sign. Now the equivalence of the two conditions follows from Proposition 2.

For any $m > 2$ it is easy to show that there exists an integer $k$ such that

$$V(i_1, \dots, i_m|1, \dots, m) = (-2j)^k U(i_1, \dots, i_m|1, \dots, m) \tag{9}$$

by the following operations on $U$:

(i) multiply each column which represents $\mathrm{Im}(v^i)$ for some $i \in \{1, \dots, m\}$ by $j$, and then add it to the column which represents $\mathrm{Re}(v^i)$;

(ii) multiply each column which represents $j\,\mathrm{Im}(v^i)$ by $-2$, and then add to it the respective column obtained in (i).

Then by properties of the determinant, (9) holds and the equivalence of the two conditions follows from Proposition 2. □

### 3.1. Matrices that are $SSR_k$ for some value $k$

We now state our first main result that describes the spectral properties of a nonsingular matrix that is $SSR_k$ for some value $k$.

**Theorem 2.** *Suppose that $A \in \mathbb{R}^{n\times n}$ is nonsingular and $SSR_k$ for some value $k$, with $k \in \{1, \dots, n-1\}$. Then the following properties hold:*

*(i) The product $\lambda_1\lambda_2 \dots \lambda_k$ is real, and*

$$\epsilon_k \lambda_1\lambda_2 \dots \lambda_k > 0. \tag{10}$$

*(ii) The eigenvalues satisfy the inequality*

$$|\lambda_k| > |\lambda_{k+1}|. \tag{11}$$

*(iii) Pick $1 \leq p \leq k$, $k+1 \leq q \leq n$, and $c_p, \dots, c_q \in \mathbb{C}$ such that $c_p, \dots, c_k$ [$c_{k+1}, \dots, c_q$] match the eigenvectors $v^p, \dots, v^k$ [$v^{k+1}, \dots, v^q$] of $A$. Then*

$$s^+(\sum_{i=p}^{k} c_i v^i) \leq k-1, \tag{12}$$

*and*

$$s^-(\sum_{i=k+1}^{q} c_i v^i) \geq k. \tag{13}$$

*(iv) Let $u^1, \dots, u^n$ be the set of real vectors constructed from $v^1, \dots, v^n$ as described above. Then $u^1, \dots, u^k$ are linearly independent. In particular, if $v^1, \dots, v^k$ are real then they are linearly independent.*

**Remark 1.** Roughly speaking, equations (12) and (13) imply that the first $k$ [last $n-k$] eigenvectors of $A$ have a sign pattern with a “small” [“large”] number of sign changes. In particular, for any $i \leq k$ we have that if $v^i \in \mathbb{R}^n$ then $s^+(v^i) \leq k-1$, and if $v^{i-1}, v^i$ is a complex conjugate pair then for any $d_1, d_2 \in \mathbb{R}$, that are not both zero, we have $s^+(d_1 \operatorname{Re}(v^i) + d_2 \operatorname{Im}(v^i)) \leq k-1$.

Similarly, for any $j \geq k+1$ we have that if $v^j \in \mathbb{R}^n$ then $s^-(v^j) \geq k$, and if $v^j, v^{j+1}$ is a complex conjugate pair then for any $d_1, d_2 \in \mathbb{R}$, that are not both zero, we have $s^-(d_1 \operatorname{Re}(v^j) + d_2 \operatorname{Im}(v^j)) \geq k$.

**Proof of Theorem 2.** Let $r := \binom{n}{k}$. Recall that the $k$th *multiplicative compound matrix* $A^{(k)}$ is the $r \times r$ matrix that includes all the minors of order $k$ of $A$ ordered lexicographically. By Kronecker's theorem, see, e.g., [12, p. 132], the eigenvalues $\zeta_i$, $i = 1, \dots, r$, of $A^{(k)}$ are all the $k$ products of $k$ eigenvalues of $A$, that is,

$$\begin{aligned} \zeta_1 &= \lambda_1 \lambda_2 \dots \lambda_{k-1} \lambda_k, \\ \zeta_2 &= \lambda_1 \lambda_2 \dots \lambda_{k-1} \lambda_{k+1}, \\ &\vdots \\ \zeta_r &= \lambda_{n-k+1} \lambda_{n-k+2} \dots \lambda_{n-1} \lambda_n. \end{aligned} \tag{14}$$

Combining this with (5) implies that

$$|\zeta_1| \geq |\zeta_2| \geq \dots \geq |\zeta_r| > 0. \tag{15}$$

Since $A$ is $SSR_k$, all the entries in $A^{(k)}$ are non-zero and have the sign $\epsilon_k$. Thus, all the entries of $\epsilon_k A^{(k)}$ are positive. Perron's theorem implies that $\epsilon_k \zeta_1$ is real and positive with a corresponding entry-wise positive eigenvector $w$ (that is unique up to scaling), and that

$$\epsilon_k \zeta_1 > |\zeta_2| \geq \dots \geq |\zeta_r| > 0.$$

Using (14), we conclude that $\epsilon_k \lambda_1 \lambda_2 \dots \lambda_k$ is real and positive and that

$$|\lambda_k| > |\lambda_{k+1}|.$$

To prove the sign patterns of the eigenvectors, let $V := \begin{bmatrix} v^1 & \dots & v^n \end{bmatrix}$ and $D := \operatorname{diag}(\lambda_1, \dots, \lambda_n)$, so that $AV = VD$. Define $q \in \mathbb{C}^r$ by $q_\alpha := V(\alpha|1, 2, \dots, k)$, where $\alpha$ is running over all $k$-tuples $1 \leq i_1 < \dots < i_k \leq n$, i.e.,

$$q := \begin{bmatrix} V(\alpha_1|1, \dots, k) \\ \vdots \\ V(\alpha_r|1, \dots, k) \end{bmatrix}, \tag{16}$$

and the components of $q$ are ordered lexicographically. Using the fact that

$$A^{(k)} = \begin{bmatrix} A(\alpha_1|\alpha_1) & \dots & A(\alpha_1|\alpha_r) \\ \vdots & \vdots & \vdots \\ A(\alpha_r|\alpha_1) & \dots & A(\alpha_r|\alpha_r) \end{bmatrix}, \tag{17}$$

yields

$$(A^{(k)}q)_\alpha = \sum_\beta A(\alpha,\beta)V(\beta|1,2,\dots,k),$$

where the summation is over all $k$-tuples $\beta$. Let $B := AV$. Applying the Cauchy–Binet formula for the minors of the product of two matrices, e.g., [2, Theorem 1.1.1], yields $(A^{(k)}q)_\alpha = B(\alpha|1,2,\dots,k)$. Since $B = VD$, a further application of the Cauchy–Binet formula results in

$$\begin{aligned}(A^{(k)}q)_\alpha &= \sum_\beta V(\alpha,\beta)D(\beta|1,2,\dots,k) \\ &= V(\alpha|1,2,\dots,k)\lambda_1\lambda_2\dots\lambda_k \\ &= \zeta_1 q_\alpha,\end{aligned}$$

where the second equation follows from the fact that $D$ is a diagonal matrix. Since this holds for any entry of the vector $A^{(k)}q$, we conclude that $q$ is an eigenvector of $\epsilon_k A^{(k)}$ corresponding to its Perron root $\epsilon\zeta_1$. Thus, there exists $\eta \in \mathbb{C}\setminus\{0\}$ such that $q = \eta w$, where $w \in \mathbb{R}^r$ is an entry-wise positive vector. Using the fact that the complex vectors in $V$ appear in conjugate pairs, and using determinantal properties as in the proof of Proposition 3, we have that $\eta = j^k$ for some integer $k$. Now application of Proposition 3 yields that for any $c_1,\dots,c_k \in \mathbb{C}$ that match $v^1,\dots,v^k$ we have

$$s^+\Big(\sum_{i=1}^k c_i v^i\Big) \le k-1$$

which proves (12).

To prove (13), note that $Av^i = \lambda_i v^i$ and the identity $\operatorname{adj}(A)A = \det(A)I$ yield

$$(D_{\pm1}\operatorname{adj}(A)D_{\pm1}^{-1})D_{\pm1}\lambda_i v^i = \det(A)D_{\pm1}v^i.$$

In other words, the eigenvalues $\eta_i$ of the unsigned adjugate of $A$, ordered such that $|\eta_1| \ge |\eta_2| \ge \dots \ge |\eta_n| > 0$, are

$$\eta_1 = \frac{\det(A)}{\lambda_n},\ \eta_2 = \frac{\det(A)}{\lambda_{n-1}},\ \dots,\ \eta_n = \frac{\det(A)}{\lambda_1},$$

with corresponding eigenvectors

$$z^1 := D_{\pm1}v^n,\ z^2 := D_{\pm1}v^{n-1},\ \dots, z^n := D_{\pm1}v^1.$$

Since $A$ is nonsingular and $SSR_k$, Corollary 1 implies that all minors of order $n-k$ of the unsigned adjugate of $A$ are non-zero and have the same sign, that is, $D_{\pm1}\operatorname{adj}(A)D_{\pm1}^{-1}$ is $SSR_{n-k}$. By Equation (12) (that has already been proved), this means that for any $c_1,\dots,c_{n-k} \in \mathbb{C}$ that match $z^1,\dots,z^{n-k}$, we have $s^+(\sum_{i=1}^{n-k} c_i z^i) \le n-k-1$, i.e., $s^+(D_{\pm1}\sum_{i=k+1}^n c_{n-i+1}v^i) \le n-k-1$. Combining this with the identity

$$s^+(D_{\pm 1}x) + s^-(x) = n - 1 \text{ for all } x \in \mathbb{R}^n,$$

see, e.g., [2, p. 88], yields (13).

To prove (iv) note that since $k \le n-1$, (12) yields $s^+(\sum_{i=1}^k c_i v^i) \le n-2$. This means that $\sum_{i=1}^k c_i v^i \ne 0$ for any matching $c_1, \dots, c_k \in \mathbb{C}$, that is, $\sum_{i=1}^k d_i u^i \ne 0$ for any $d_1, \dots, d_k \in \mathbb{R}$ that are not all zero. This completes the proof of Theorem 2. □

**Example 1.** Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix}. \tag{18}$$

It is straightforward to verify that this matrix is nonsingular, and that all minors of order 3 are positive, so $A$ is $SSR_3$ with $\epsilon_3 = 1$ (but not $SSR_1$ nor $SSR_2$). Its eigenvalues are (all numerical values in this paper are to four-digit accuracy)

$$\lambda_1 = 2.7900,\ \lambda_2 = 1.5000 + 1.0790j,\ \lambda_3 = 1.5000 - 1.0790j,\ \lambda_4 = 0.2100,$$

and thus $\lambda_1\lambda_2\lambda_3$ is real and positive, and $|\lambda_3| > |\lambda_4|$. The matrix of corresponding eigenvectors is

$$\begin{aligned} V &= \begin{bmatrix} v^1 & v^2 & v^3 & v^4 \end{bmatrix} \\ &= \begin{bmatrix} 0.79 & -0.5 + 1.079j & -0.5 - 1.079j & -1.79 \\ 0.7071 & -0.7071 & -0.7071 & 0.7071 \\ 1.266 & -0.3536 - 0.763j & -0.3536 + 0.763j & -0.5586 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \end{aligned}$$

and thus

$$\begin{aligned} U &:= \begin{bmatrix} v^1 & \mathrm{Re}(v^2) & \mathrm{Im}(v^2) & v^4 \end{bmatrix} \\ &= \begin{bmatrix} 0.79 & -0.5 & 1.079 & -1.79 \\ 0.7071 & -0.7071 & 0 & 0.7071 \\ 1.266 & -0.3536 & -0.763 & -0.5586 \\ 1 & 1 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Calculating the vector $q$ that contains all minors in the form $V(\alpha|1,2,3)$ yields $q = -jw = j^3 w$, with

$$w := \begin{bmatrix} 1.7049 & 3.0518 & 5.4629 & 2.1580 \end{bmatrix}'.$$

A calculation of all the minors of the form $U(\alpha|1,2,3)$ gives the values $[0.8524 \quad 1.5259 \quad 2.7315 \quad 1.0790]'$. Since these are all positive, application of Proposition 2 to the submatrix containing the first three columns of $U$ gives that for any $d_1, d_2, d_3 \in \mathbb{R}$, that are not all zero,

$$s^+(d_1 v^1 + d_2 \,\mathrm{Re}(v^2) + d_3 \,\mathrm{Im}(v^2)) \le 2,$$

which agrees with (12). Furthermore, it holds that $s^-(v^4) \ge 3$ which agrees with (13). □

So far, we have considered matrices that are $SSR_k$ for a single value of $k$. Our next goal is to demonstrate that Theorem 2 can be used as a basic building block in the analysis of matrices that are $SSR_k$ for *several* values of $k$, by simply applying Theorem 2 for every such $k$.

### 3.2. Matrices that are $SSR_k$ for two consecutive values of $k$

The next result analyzes matrices that are $SSR_k$ for two consecutive values of $k$.

**Corollary 2.** *Suppose that $A \in \mathbb{R}^{n\times n}$ is nonsingular, $SSR_i$ and $SSR_{i+1}$ for some value $i$, with $i \in \{1, \dots, n-2\}$. Then the following properties hold:*

*(i) The signed eigenvalue $\epsilon_i\epsilon_{i+1}\lambda_{i+1}$ is real and positive.*
*(ii) The eigenvalues satisfy the inequalities*

$$|\lambda_i| > |\lambda_{i+1}| > |\lambda_{i+2}|. \tag{19}$$

*(iii) The eigenvector $v^{i+1}$ can be chosen as a real vector and*

$$s^-(v^{i+1}) = s^+(v^{i+1}) = i. \tag{20}$$

*Furthermore, for any $p, q$ with $1 \le p \le i$, $i + 2 \le q \le n$, and $c_p, \dots, c_i$ [$c_{i+2}, \dots, c_q$] $\in \mathbb{C}$ that match the eigenvectors $v^p, \dots, v^i$ [$v^{i+2}, \dots, v^q$] we have*

$$s^+(\sum_{\ell=p}^{i} c_\ell v^\ell) \le i-1,\ s^+(\sum_{\ell=p}^{i+1} c_\ell v^\ell) \le i, \tag{21}$$

*and*

$$s^-(\sum_{\ell=i+1}^{q} c_\ell v^\ell) \ge i,\ s^-(\sum_{\ell=i+2}^{q} c_\ell v^\ell) \ge i+1. \tag{22}$$

*(iv) The vectors $u^1, \dots, u^{i+1}$ are linearly independent.*

**Proof.** By Theorem 2, $\epsilon_i\lambda_1\lambda_2 \dots \lambda_i$ and $\epsilon_{i+1}\lambda_1\lambda_2 \dots \lambda_{i+1}$ are real and positive which yields (i). Thus, $v^{i+1}$ can be chosen as a real vector. Using (11) with $k = i$ and with $k = i+1$ gives (19). Inequalities (12) and (13) imply (21) and (22), and this implies

$$i \le s^-(v^{i+1}) \le s^+(v^{i+1}) \le i.$$

The last statement of the corollary follows immediately from Theorem 2 and this completes the proof. □

### 3.3. Matrices that are $SSR$

Known results on the spectral structure of $TP$ matrices, see, e.g., [2, Chapter 5], [12, Chapter 5], (and, more generally, of $SSR$ matrices [4, Chapter V]), follow immediately from Corollary 2. Indeed, suppose that $A \in \mathbb{R}^{n\times n}$ is $SSR$. Then it is $SSR_i$ and $SSR_{i+1}$ for all $i \in \{1, \dots, n-1\}$ and thus Corollary 2 implies that the eigenvalues $\lambda_i$, $i = 1, \dots, n-1$, are real and that

$$|\lambda_1| > |\lambda_2| > \dots > |\lambda_n| > 0. \tag{23}$$

Since $\det(A) = \lambda_1 \dots \lambda_n$ is real, this implies that $\lambda_n$ is also real. Pick indexes $p, q$ with $1 \le p \le q \le n$, and $c_p, c_{p+1}, \dots, c_q \in \mathbb{R}$ such that $v := \sum_{i=p}^{q} c_i v^i \neq 0$. Then since (21) and (22) hold for any $i$, we conclude that

$$p-1 \le s^-(v) \le s^+(v) \le q-1.$$

In particular, taking $p = q$ yields

$$s^-(v^p) = s^+(v^p) = p-1 \text{ for all } p \in \{1, \dots, n\}.$$

### 3.4. Matrices that are $SSR_k$ for all odd $k$

We now analyze the spectral properties of matrices that are $SSR_k$ for all odd $k$. To explain why such matrices are interesting, we first review their VDPs. For a vector $y \in \mathbb{R}^n$, let

$$s_c^-(y) := \max_{i \in \{1,\dots,n\}} s^-([y_i \quad \dots \quad y_n \quad y_1 \quad \dots \quad y_i]'). \tag{24}$$

The subscript $c$ here stands for "cyclic". Intuitively speaking, this corresponds to placing the entries of $y$ along a circular ring so that $y_n$ is followed by $y_1$, then counting $s^-$ starting from any entry along the ring, and finding the maximal value.

For example, for $y = [0 \quad -1 \quad 0 \quad 2 \quad 0 \quad 3]'$, $s_c^-(y) = s^-([-1 \quad 0 \quad 2 \quad 0 \quad 3 \quad 0 \quad -1]') = 2$. Similarly, let

$$s_c^+(y) := \max_{i \in \{1,\dots,n\}} s^+([y_i \quad \dots \quad y_n \quad y_1 \quad \dots \quad y_i]'),$$

but here if $y_i = 0$ then in the calculation of $s^+([y_i \quad \dots \quad y_n \quad y_1 \quad \dots \quad y_i]')$ *both* $y_i$'s are replaced by either 1 or $-1$. In our above example, we have $s_c^+(y) = 4$. Clearly, $s_c^-(y) \le s_c^+(y)$ for all $y \in \mathbb{R}^n$, and $s_c^-(y), s_c^+(y)$ are invariant under cyclic shifts of the vector $y$.

There is a simple and useful relation between the cyclic and non-cyclic number of sign variations of a vector, namely, for any vector $x$,

$$s_c^-(x) = \begin{cases} s^-(x), & \text{if } s^-(x) \text{ is even}, \\ s^-(x)+1, & \text{if } s^-(x) \text{ is odd}, \end{cases}$$

and, similarly,

$$s_c^+(x) = \begin{cases} s^+(x), & \text{if } s^+(x) \text{ is even}, \\ s^+(x)+1, & \text{if } s^+(x) \text{ is odd}, \end{cases}$$

see, e.g., [7, Chapter 5], where also other useful results of the cyclic variations of sign can be found. This implies in particular that for any vector $x$,

$$s_c^-(x), s_c^+(x) \in \begin{cases} \{0, 2, 4, \dots, n\}, & \text{if } n \text{ is even}, \\ \{0, 2, 4, \dots, n-1\}, & \text{if } n \text{ is odd}. \end{cases} \tag{25}$$

It was shown in [1] that a nonsingular matrix $A \in \mathbb{R}^{n \times n}$ satisfies the *cyclic VDP*, i.e.,

$$s_c^+(Ax) \le s_c^-(x) \text{ for all } x \in \mathbb{R}^n \setminus \{0\}, \tag{26}$$

if and only if $A$ is $SSR_k$ for all *odd* $k$ in the range $\{1, \dots, n\}$.

The proof of the next result uses Theorem 2 to analyze the spectral properties of such matrices.

**Theorem 3.** *Suppose that $A \in \mathbb{R}^{n \times n}$ is nonsingular and $SSR_k$ for all odd $k$ in the range $\{1, \dots, n-1\}$. Then*

*(i)* *The eigenvalue $\lambda_1$ of $A$ is simple, real, with $\epsilon_1\lambda_1 > 0$.*
*(ii)* *The algebraic multiplicity of any eigenvalue of $A$ is not greater than 2, and the eigenvalues satisfy the inequalities $|\lambda_1| > |\lambda_2| \geq |\lambda_3| > |\lambda_4| \geq |\lambda_5| > \dots$.*
*(iii)* *For every even $k$ in the range $\{2,\dots,n-1\}$, the inequality $\epsilon_{k-1}\epsilon_{k+1}\lambda_k\lambda_{k+1} > 0$ holds.*
*(iv)* *If $n$ is even then $\lambda_n$ is real, and $\epsilon_{n-1}\det(A)\lambda_n > 0$.*
*(v)* *For any $i$, the eigenvectors have the following properties: if $v^{2i+1}$ is real then $s^+(v^{2i+1}) \leq 2i$, and if $v^{2i}, v^{2i+1}$ is a complex conjugate pair then for any matching $c_1, c_2 \in \mathbb{C}$*

$$s^+(c_1v^{2i} + c_2v^{2i+1}) \leq 2i.$$

*Also, if $v^{2i}$ is real then $s^-(v^{2i}) \geq 2i-1$, and if $v^{2i}, v^{2i+1}$ is a complex conjugate pair then for any matching $c_1, c_2 \in \mathbb{C}$*

$$s^-(c_1v^{2i} + c_2v^{2i+1}) \geq 2i-1.$$

*Furthermore, the vectors $u^1,\dots,u^p$, with $p$ the largest odd number satisfying $p \leq n$, are linearly independent.*

**Proof.** Statement (i) follows from the fact that $A$ is $SSR_1$ and by Perron's theorem. By (11) it follows that

$$|\lambda_1| > |\lambda_2|,\ |\lambda_3| > |\lambda_4|,\ |\lambda_5| > |\lambda_6|\ \dots$$

which implies (ii). By Theorem 2, the products

$$\epsilon_1\lambda_1,\ \epsilon_3\lambda_1\lambda_2\lambda_3,\ \epsilon_5\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5,\ \dots$$

are all real and positive, which implies $\epsilon_{k-1}\epsilon_{k+1}\lambda_k\lambda_{k+1} > 0$ and thus (iii). To prove (iv), note that if $n$ is even then $n-1$ is odd, so $A$ is $SSR_{n-1}$. This implies that $\beta := \epsilon_{n-1}\lambda_1\lambda_2\dots\lambda_{n-1}$ is real and positive, and from the fact that $\epsilon_{n-1}\det(A) = \beta\lambda_n$ the claim follows. The results on the sign pattern of the eigenvectors follow from (12) and (13), and the linear independence of $u^1,\dots,u^p$ follows from Theorem 2 (iv). □

**Example 2.** Consider a nonsingular matrix $A \in \mathbb{R}^{3\times3}$ with all entries positive. Then $A$ is $SSR_1$, with $\epsilon_1 = 1$, and $SSR_3$. Theorem 3 implies that $\lambda_1$ is positive, and $s^-(v^1) = s^+(v^1) = 0$, and also that only one of the following two cases is possible.
*Case 1.* The eigenvalues $\lambda_2, \lambda_3$ are both real, with

$$\lambda_1 > |\lambda_2| \geq |\lambda_3|$$

and $s^-(v^2) \geq 1$. The eigenvectors $v^1, v^2, v^3$ are linearly independent.
*Case 2.* The eigenvalues $\lambda_2, \lambda_3$ are complex with $\lambda_2 = \bar{\lambda}_3$. Then

$$\lambda_1 > |\lambda_2| = |\lambda_3|$$

and $s^-(d_1\operatorname{Re}(v^2) + d_2\operatorname{Im}(v^2)) \geq 1$ for all $d_1, d_2 \in \mathbb{R}$ that are not both zero. The vectors $v^1, \operatorname{Re}(v^2), \operatorname{Im}(v^2)$ are linearly independent. □

### *3.5. Matrices that are $SSR_k$ for all even $k$*

We start with a result on the spectral properties of matrices which are $SSR_k$ for all even $k$. Its proof is parallel to one of Theorem 3.

**Corollary 3.** *Let $A \in \mathbb{R}^{n,n}$ be nonsingular and $SSR_k$ for all even $k$ in the range $\{2, \dots, n-1\}$. Then*

*(i) The algebraic multiplicity of any eigenvalue of $A$ is not greater than 2, and the eigenvalues satisfy the inequalities $|\lambda_1| \geq |\lambda_2| > |\lambda_3| \geq |\lambda_4| > |\lambda_5| > \dots$.*
*(ii) For every odd $k$ in the range $\{1, \dots, n-1\}$, the inequality $\epsilon_{k-1}\epsilon_{k+1}\lambda_k\lambda_{k+1} > 0$ holds.*
*(iii) If $n$ is odd then $\lambda_n$ is real, and $\epsilon_{n-1}\det(A)\lambda_n > 0$.*
*(iv) For any $i$, the eigenvectors have the following properties: if $v^{2i}$ is real then $s^+(v^{2i}) \leq 2i-1$, and if $v^{2i-1}, v^{2i}$ is a complex conjugate pair then for any matching $c_1, c_2 \in \mathbb{C}$*

$$s^+(c_1 v^{2i-1} + c_2 v^{2i}) \leq 2i - 1.$$

*Also, if $v^{2i-1}$ is real then $s^-(v^{2i-1}) \geq 2i-2$, and if $v^{2i}, v^{2i+1}$ is a complex conjugate pair then for any matching $c_1, c_2 \in \mathbb{C}$*

$$s^-(c_1 v^{2i-1} + c_2 v^{2i}) \geq 2i - 2.$$

*Furthermore, the vectors $u^1, \dots, u^p$, with $p$ the largest even number satisfying $p \leq n$, are linearly independent.*

We now apply Theorem 1 to show that the matrices which are $SSR_k$ for all even $k$ also possess a certain VDP.

**Definition 1.** Let $x \in \mathbb{R}^n$. Define

$$s_o^+(x) := \begin{cases} s^+(x), & \text{if } s^+(x) \text{ is odd,} \\ s^+(x) + 1, & \text{if } s^+(x) \text{ is even.} \end{cases}$$

Note that $s_o^+(x)$ is the smallest odd number greater than or equal to $s^+(x)$. We define analogously $s_o^-(x)$ by replacing the superscript $+$ by $-$.

**Definition 2.** A matrix $A \in \mathbb{R}^{m \times n}$ is said to have the *odd VDP* if

$$s_o^+(Ax) \leq s_o^-(x) \text{ for all } x \in \mathbb{R}^n \setminus \{0\}. \tag{27}$$

**Theorem 4.** *Let $A \in \mathbb{R}^{n \times n}$ be a nonsingular matrix. The following two statements are equivalent:*

*(i) A has the odd VDP.*
*(ii) The matrix $A$ is $SSR_k$ for all even $k$ in the range $\{2, \dots, n-1\}$.*

**Proof.** Assume that condition (i) holds. Pick an *odd* number $k \in \{1, \dots, n-1\}$ and pick $x \in \mathbb{R}^n \setminus \{0\}$ such that $s_o^-(x) \leq k$. Then $s^-(x) \leq k$. Condition (i) yields $s_o^+(Ax) \leq k$. Theorem 1 implies that $A$ is $SSR_{k+1}$. Since $k$ is an arbitrary odd number in $\{1, \dots, n-1\}$, we conclude that condition (ii) holds.

To prove the converse implication, suppose that condition (ii) holds. Pick $x \in \mathbb{R}^n \setminus \{0\}$. Let $p$ be such that $s_o^-(x) = 2p-1$ and thus $s^-(x) \leq 2p-1$. If $2p-1 = n$ then clearly (27) holds; so we may assume that $2p-1 \leq n-1$. Condition (ii) implies in particular that $A$ is $SSR_{2p}$ and application of Theorem 1 yields $s^+(Ax) \leq 2p-1$, hence $s_o^+(Ax) \leq 2p-1 = s_o^-(x)$. $\square$

The next section describes an application of the properties analyzed above to dynamical systems.

## 4. Applications to discrete-time dynamical systems

### 4.1. Linear systems

Consider the linear time-varying system:

$$x(i+1) = A(i)x(i), \quad x(0) = x_0, \tag{28}$$

with $A(i) \in \mathbb{R}^{n\times n}$ for all $i \geq 0$, and $x_0 \in \mathbb{R}^n$.

Our goal is to develop for the discrete-time system (28) an analogue of the important notion of a TPDS, derived by Schwarz [15] for continuous-time systems. The main idea is to require every $A(i)$ to satisfy a VDP.

**Lemma 1.** *Suppose that there exists $k \in \{1, \dots, n-1\}$ such that for every $i \geq 0$ the matrix $A(i)$ is nonsingular and $SSR_k$. Pick $x_0 \in \mathbb{R}^n \setminus \{0\}$ such that $s^-(x_0) \leq k-1$. Then the solution of (28) satisfies*

$$s^+(x(i)) \leq k-1 \text{ for all } i \geq 1.$$

**Proof.** Fix $i \geq 1$. Then $x(i) = A(i-1)\dots A(1)A(0)x_0$. The matrix $A(i-1)\dots A(1)A(0)$ is nonsingular and $SSR_k$, as it is the product of nonsingular and $SSR_k$ matrices. Theorem 1 implies that $s^+(x(i)) \leq k-1$. □

Let $\mathcal{V} := \{y \in \mathbb{R}^n : s^-(y) = s^+(y)\}$. For example, for $n = 3$ the vector $y = [1 \quad 0 \quad -1]' \in \mathcal{V}$, as $s^-(y) = s^+(y) = 2$. It is not difficult to show that

$$\mathcal{V} = \{y \in \mathbb{R}^n : y_1 \neq 0, y_n \neq 0, \text{ and if } y_i = 0 \text{ for some } i \in \{2, \dots, n-1\} \text{ then } y_{i-1}y_{i+1} < 0\}.$$

**Theorem 5.** *Suppose that for every $i \geq 0$ the matrix $A(i)$ is SSR. Then*

*(i) For any $x_0 \in \mathbb{R}^n \setminus \{0\}$ the solution of (28) satisfies*

$$s^+(x(i+1)) \leq s^-(x(i)) \text{ for all } i \geq 0, \tag{29}$$

*and $x(i) \in \mathcal{V}$ for all $i \geq 0$ except perhaps for up to $n-1$ values of $i$.*

*(ii) Pick two different initial conditions $x_0, \bar{x}_0 \in \mathbb{R}^n$, and let $x(q), \bar{x}(q)$ denote the corresponding solutions of (28) at time $q$. Then there exists $m \geq 0$ such that*

$$x_1(q) \neq \bar{x}_1(q) \text{ for all } q \geq m. \tag{30}$$

**Proof.** Pick $x_0 \in \mathbb{R}^n \setminus \{0\}$. If $s^-(x_0) = n-1$ then clearly (29) holds, so we may assume that $s^-(x_0) < n-1$. Let $q \in \{1, \dots, n-1\}$ be such that $s^-(x_0) = q-1$. Since $A(0)$ is nonsingular and $SSR_q$, Lemma 1 implies that $s^-(x(1)) \leq s^+(x(1)) \leq q-1 = s^-(x_0)$. Let $p \leq q$ be such that $s^-(x(1)) = p-1$. Since $A(1)$ is nonsingular and $SSR_p$, Lemma 1 implies that $s^-(x(2)) \leq s^+(x(2)) \leq p-1 = s^-(x(1))$. Proceeding in this fashion proves (29).

The analysis above shows that the mappings $i \to s^+(x(i))$ and $i \to s^-(x(i))$ are nonincreasing, and if $x(i) \notin \mathcal{V}$, that is, $s^-(x(i)) < s^+(x(i))$ then $s^+(x(i+1)) < s^+(x(i))$. Since $s^+$ takes values in $\{0, \dots, n-1\}$, this proves (i).

To prove (ii), let $z(i) := x(i) - \bar{x}(i)$. Then $z(0) \neq 0$, $z(i+1) = A(i)z(i)$, and (i) implies that there exists $m \geq 0$ such that $z(p) \in \mathcal{V}$ for all $p \geq m$. In particular, $z_1(p) \neq 0$ for all $p \geq m$, and this completes the proof. □

**Remark 2.** Suppose that $A(i) = I$ for all $i \geq 0$. Pick $x_0 \in \mathbb{R}^n \setminus \{0\}$ such that $x_0 \notin \mathcal{V}$. Then the solution of (28) satisfies $x(i) \notin \mathcal{V}$ for all $i \geq 0$. Thus, Theorem 5 does not hold if we weaken the hypothesis of the theorem to: "$A(i)$ is nonsingular and $SR$ for all $i \geq 0$".

For the analysis of periodic discrete-time systems below, we need to strengthen the result in (30). To do this, we use a result from [4]. Recall that two vectors $v, w \in \mathbb{R}^n$ are said to *oscillate in the same way* if $s^-(v) = s^-(w)$ and the first non-zero entry in $v$ and in $w$ has the same sign. For example, $v = [0 \quad 0 \quad 1 \quad 0 \quad -2]'$ and $w = [5 \quad 0 \quad -2 \quad -3 \quad -3]'$ oscillate in the same way. Theorem 5 in [4, p. 254] implies that $A \in \mathbb{R}^{n\times n}$ is TN if and only if for any $x \in \mathbb{R}^n$ the vector $y := Ax$ satisfies $s^-(y) \leq s^-(x)$ and if $s^-(y) = s^-(x)$ then $x$ and $y$ oscillate in the same way.

**Definition 3.** We call (28) a *totally positive discrete-time system* (TPDTS) if $A(i)$ is TP for all $i \geq 0$.

This is the analogue of the notion of a TPDTS defined by Schwarz [15] for continuous-time systems. Note that if the system is TPDTS then, in particular, every entry of $A(i)$ is positive for all $i$, so the system is cooperative [18].

**Theorem 6.** *Suppose that* (28) *is TPDTS. Then*

*(i) For any $x_0 \in \mathbb{R}^n \setminus \{0\}$ the solution of* (28) *satisfies*

$$s^+(x(i+1)) \leq s^-(x(i)) \text{ for all } i \geq 0, \tag{31}$$

*and $x(i) \in \mathcal{V}$ for all $i \geq 0$ except perhaps for up to $n-1$ values of $i$.*

*(ii) Pick two different initial conditions $x_0, \bar{x}_0 \in \mathbb{R}^n$, and let $x(q), \bar{x}(q)$ denote the corresponding solutions of* (28) *at time $q$. Then there exists $r \geq 0$ such that either*

$$x_1(q) > \bar{x}_1(q) \text{ for all } q \geq r$$

*or*

$$x_1(q) < \bar{x}_1(q) \text{ for all } q \geq r.$$

**Proof.** Since a TP matrix is SSR, statement (i) follows immediately from Theorem 5. To prove (ii), recall that $z(i) := x(i) - \bar{x}(i)$ satisfies $z(i) = A(i)z(i)$, and that there exists $m \geq 0$ such that $z_1(p) \neq 0$ for all $p \geq m$. By the analysis above, there exists $\ell \geq 0$ such that

$$s^-(z(i)) = s^+(z(i)) = s^-(z(i+1)) = s^+(z(i+1)) \text{ for all } i \geq \ell.$$

In particular, we obtain

$$s^-(z(i)) = s^-(A(i)z(i)) \text{ for all } i \geq \ell. \tag{32}$$

Let $r := \max(m, \ell)$. Pick $i \geq r$. Then the first non-zero entry in $z(i)$ and in $z(i+1) = A(i)z(i)$ is the first entry. Since $A(i)$ is TP (and thus TN), (32) implies that $z_1(i)$ and $z_1(i+1)$ have the same sign. Since $i \geq r$ is arbitrary, this proves (ii). □

**Remark 3.** Consider the special case, where (28) is time-invariant, that is, $A(i) = A$ for all $i \geq 0$, with $A$ a TP matrix. In this case, it is possible to give a simpler proof for the "eventual monotonicity" result in (ii).

Indeed, let $\lambda_1 > \cdots > \lambda_n > 0$ be the eigenvalues of $A$ with corresponding eigenvectors $v^i \in \mathbb{R}^n$, $i = 1, \ldots, n$. Write $z_0 = x_0 - \bar{x}_0$ as $z_0 = \sum_{\ell=1}^n c_\ell v^\ell$. Since $z_0 \neq 0$, there exists a minimal index $p$ such that $c_p \neq 0$, that is, $z_0 = \sum_{\ell=p}^n c_\ell v^\ell$ and thus

$$z(i) = A^i z_0 = \sum_{\ell=p}^{n} c_\ell (\lambda_\ell)^i v^\ell. \tag{33}$$

Recall that every eigenvector $v^i$ of an $SSR$ (and in particular TP) matrix satisfies $v^i \in \mathcal{V}$, and thus the first entry of $v^p$ is not zero. Since $\lambda_p > \lambda_{p+1} > \cdots > \lambda_n$, it follows from (33) that for any $i$ sufficiently large, the first entry of $z(i)$ is not zero and has the same sign as the first entry of $c_p v^p$.

Our next goal is to use the notion of a TPDTS to analyze nonlinear systems.

### *4.2. Nonlinear systems*

Consider the time-varying nonlinear discrete-time dynamical system

$$x(i+1) = f(i, x(i)), \tag{34}$$

with $f$ continuously differentiable with respect to $x$. Let $J(i,x) := \frac{\partial}{\partial x} f(i,x)$ denote its Jacobian with respect to $x$. We assume throughout that the trajectories of (34) evolve on a convex and compact set $\Omega \subset \mathbb{R}^n$.

Pick two different initial conditions $x_0, \bar{x}_0 \in \Omega$ and let $z(i) := x(i) - \bar{x}(i)$. Then

$$\begin{aligned} z(i+1) &= f(i, x(i)) - f(i, \bar{x}(i)) \\ &= \int_0^1 \frac{d}{dr} f(i, rx(i) + (1-r)\bar{x}(i))\, \mathrm{d}r \\ &= \int_0^1 \frac{\partial}{\partial x} f(i, rx(i) + (1-r)\bar{x}(i)) \frac{\partial}{\partial r}(rx(i) + (1-r)\bar{x}(i))\, \mathrm{d}r \\ &= \left( \int_0^1 J(i, rx(i) + (1-r)\bar{x}(i))\, \mathrm{d}r \right) z(i). \end{aligned} \tag{35}$$

This is a discrete-time variational system, as it describes how a variation in the initial condition propagates with time.

The next assumption guarantees that (35) is TPDTS.

**Assumption 1.** The matrix $\int_0^1 J(i, ra + (1-r)b)\, \mathrm{d}r$ is TP for any $a, b \in \Omega$ and any $i \geq 0$.

To illustrate an application of Theorem 6, we assume that there exists an integer $T > 0$ such that the nonlinear system (34) is $T$-periodic.

**Assumption 2.** The vector field in (34) satisfies $f(i+T, a) = f(i, a)$ for all $a \in \Omega$ and all $i \geq 0$.

We now state the main result in this section. We say that a solution $x(i)$ of (34) is a *$T$-periodic solution* if $x(i+T) = x(i)$ for all $i \geq 0$.

**Theorem 7.** *If Assumptions 1 and 2 hold then any solution of (34) converges to a $T$-periodic solution as $i \to \infty$.*

If we view the $T$-periodic vector field in (34) as representing a periodic excitation then Theorem 7 implies that the system *entrains* to the excitation, as every solution converges to a periodic solution with the same period as the excitation. Entrainment is important in many natural and artificial systems. Proper functioning of various organisms requires internal processes to entrain to the 24h solar day. Synchronous generators must entrain to the frequency of the grid. For more details, see, e.g., [8,9,13].

Assumption 1 implies in particular that every minor of $J(i, x)$ is positive for all $i \geq 0$ and all $x \in \Omega$. In particular, the first-order minors, i.e. the entries of $J(i, x)$ are positive, so the nonlinear system is strongly cooperative [18]. Our conditions here require more than cooperativity and as a consequence yield more powerful results on the asymptotic behavior of the system (see, e.g. [19,5]).

**Proof of Theorem 7.** Pick $a \in \Omega$. Let $x(i, a)$ denote the solution of (34) at time $i$ with $x_0 = a$. If $x(i, a)$ is $T$-periodic then there is nothing to prove. Thus, we may assume that $y(i) := x(i, a)$ and $w(i) := x(i+T, a)$ are not identical and the $T$-periodicity of the vector field implies that both $y, w$ are solutions of the dynamical system.

Theorem 6 implies that there exists $m \geq 0$ such that either $w_1(i) > y_1(i)$ or $w_1(i) < y_1(i)$ for all $i \geq m$. Without loss of generality, we may assume that the first of these two inequalities holds and this yields

$$x_1((k+1)T, a) > x_1(kT, a) \text{ for all } k \geq m. \tag{36}$$

Define

$$\omega_T(a) := \{v \in \Omega : \lim_{k\to\infty} x(n_k T, a) = v \text{ for some sequence } n_1 < n_2 < \dots\}.$$

Since the solutions remain in the compact set $\Omega$, the set $\omega_T(a)$ is not empty. If $v \in \omega_T(a)$ then $x(T, v) = v$, so the solution emanating from $v$ is $T$-periodic. Thus, to complete the proof we need to show that $\omega_T(a)$ is a singleton. We assume on the contrary that there exist $p, q \in \Omega$, with $p \neq q$, such that $p, q \in \omega_T(a)$. Then there exist sequences $\{n_k\}_{k=1}^{\infty}$ and $\{m_k\}_{k=1}^{\infty}$ such that

$$p = \lim_{k\to\infty} x(n_k T, a) \text{ and } q = \lim_{k\to\infty} x(m_k T, a).$$

Passing to subsequences, we may assume that $n_k < m_k < n_{k+1}$ for all $k$. Now (36) yields $x_1(n_k T, a) < x_1(m_k T, a) < x_1(n_{k+1} T, a)$ for all $k$ sufficiently large and taking $k \to \infty$ yields $p_1 = q_1$. We conclude that any two points in $\omega_T(a)$ have the same first coordinate.

Consider the trajectories emanating from $p$ and $q$, that is, $x(i, p)$ and $x(i, q)$. Since $\omega_T(a)$ is an invariant set, $x(kT, p), x(kT, q) \in \omega_T(a)$ for all $k$, so

$$x_1(kT, p) = x_1(kT, q) \text{ for all } k.$$

However, since $p \neq q$ this contradicts the eventual monotonicity property described in Theorem 6. This contradiction proves that $\omega_T(a)$ is a singleton which completes the proof. □

**Example 3.** Consider the nonlinear system

$$\begin{aligned} x_1(i+1) &= \frac{c_{11}(i)x_1(i)}{1 + x_1(i)} + \frac{c_{12}(i)x_2(i)}{1 + x_2(i)}, \\ x_2(i+1) &= \frac{c_{21}(i)x_1(i)}{1 + x_1(i)} + \frac{c_{22}(i)x_2(i)}{1 + x_2(i)}. \end{aligned} \tag{37}$$

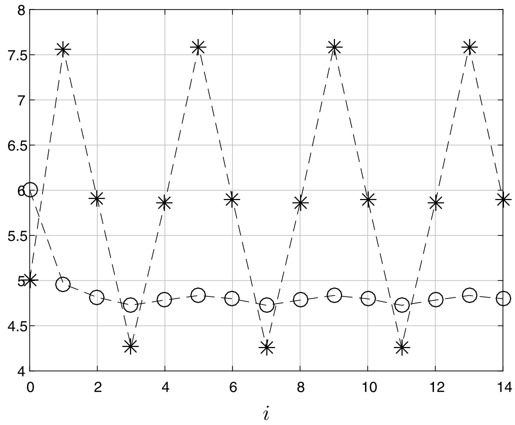


**Fig. 1.** Solution $x_1(i)$ (marked by asterisks) and $x_2(i)$ (marked by circles) as a function of $i$ for the system in Example 3.

Its Jacobian is

$$J(i,x(i)) = \begin{bmatrix} c_{11}(i)(1+x_1(i))^{-2} & c_{12}(i)(1+x_2(i))^{-2} \\ c_{21}(i)(1+x_1(i))^{-2} & c_{22}(i)(1+x_2(i))^{-2} \end{bmatrix}.$$

Calculation of the integral in Assumption 1 yields

$$\int_0^1 J(i, ra+(1-r)b)\,\mathrm{d}r = \begin{bmatrix} c_{11}(i)(1+a_1+b_1+a_1b_1)^{-1} & c_{12}(i)(1+a_2+b_2+a_2b_2)^{-1} \\ c_{21}(i)(1+a_1+b_1+a_1b_1)^{-1} & c_{22}(i)(1+a_2+b_2+a_2b_2)^{-1} \end{bmatrix}.$$

We assume that there exist $\alpha, \beta, \gamma > 0$ such that

$$\alpha < c_{pq}(i) < \beta \text{ and } c_{11}(i)c_{22}(i) - c_{12}(i)c_{21}(i) > \gamma$$

for all $1 \le p, q \le 2$ and all $i \ge 0$. Then it is clear that $\Omega := [0,v] \times [0,v]$ is an invariant set of (37) for any $v > 0$ sufficiently large, and that Assumption 1 holds. We also assume that the $c_{pq}(i)$'s are all periodic with a common period $T$. Then Theorem 7 implies that every solution converges to a $T$-periodic trajectory.

Fig. 1 depicts the solution of (37) for

$$\begin{aligned} c_{11}(i) &= 5 + \sin(\frac{i\pi}{2} + 0.2), \\ c_{12}(i) &= 2 + \sin(\frac{i\pi}{2}), \\ c_{21}(i) &= 3/2, \\ c_{22}(i) &= 4.2432, \end{aligned}$$

and the initial condition $x_0 = [5 \quad 6]'$. Note that for these values the system is $T$-periodic for $T = 4$. It may be seen that the trajectory $x(i)$ converges to a periodic pattern, with the same period $T$. $\square$

Consider now the time-invariant nonlinear system

$$x(i+1) = f(x(i)) \tag{38}$$

whose trajectories evolve on a compact and convex set $\Omega \subset \mathbb{R}^n$. The associated variational system is

$$z(i+1)=\left(\int_0^1 J(rx(i)+(1-r)\bar{x}(i))\,\mathrm{d}r\right)z(i). \tag{39}$$

The next assumption guarantees that (39) is TPDTS.

**Assumption 3.** The matrix $\int_0^1 J(ra+(1-r)b)\,\mathrm{d}r$ is TP for any $a,b\in\Omega$.

Pick an arbitrary integer $k\geq 0$. The time-invariant vector field is $T$-periodic for $T=k$ and thus entrainment implies that every solution converges to a $k$-periodic trajectory. Since $k$ is arbitrary this yields the following result.

**Corollary 4.** *If Assumption 3 holds then any solution of* (39) *converges to an equilibrium point.*

This result may be regarded as the analogue of a result of Smillie [16] on convergence to an equilibrium in a continuous-time nonlinear cooperative systems with a special Jacobian.

## 5. Conclusion

$SSR$ matrices appear in many fields. The most prominent example are $TP$ matrices. Here we have studied firstly the spectral properties of matrices that are $SSR_k$ for a single value $k$. An important property of such matrices is that some eigenvalues can be complex (unlike in the $SSR$ case, where all eigenvalues are real). We then showed that the investigation of matrices that are $SSR_k$ for a single value $k$ can be used as a basic building block for studying matrices that are $SSR_k$ for several values of $k$. e.g., $SSR$ matrices or matrices that are $SSR_k$ for all odd $k$. As an application, we derived an analogue of the notion of TPDS for the discrete-time case and analyzed its asymptotic behavior.

As explained in the recent paper [10], VDPs satisfied by the solutions of linear time-varying systems can be used to prove the stability of certain nonlinear dynamical systems. In this context, it may be of interest to study the following problem. Consider the system

$$\dot{x}(s)=A(s)x(s), \tag{40}$$

where $x(s)\in\mathbb{R}^n$ and $A(s)$ is a continuous matrix function of $s$. For any pair $t_0\leq t$, the solution of (40) satisfies $x(t)=\Phi(t,t_0)x(t_0)$, where $\Phi(t,t_0)$ is the transition matrix from time $t_0$ to time $t$, that is, the solution at time $t$ of the matrix differential equation

$$\dot{\Phi}(s)=A(s)\Phi(s),\ \text{with}\ \Phi(t_0)=I.$$

An interesting question is: given a value $k\in\{1,\ldots,n\}$, when will $\Phi(t,t_0)$ be $SSR_k$ for all pairs $t_0,t$ with $t>t_0$, and what will be the implications of this for the solution of (40)?

It is well-known that the sum of two TP matrices is not necessarily a TP matrix. This means that establishing that Assumption 1 indeed holds, that is, that the matrix $\int_0^1 J(i,ra+(1-r)b)\,\mathrm{d}r$ is TP for any $a,b\in\Omega$ and any $i\geq 0$ is not trivial. An interesting direction for further research is to find simple conditions guaranteeing that this indeed holds.

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