Inclusion isotonicity of convex–concave extensions for polynomials based on Bernstein expansion

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Abstract

In this paper the expansion of a polynomial into Bernstein polynomials over an interval I is considered. The convex hull of the control points associated with the coefficients of this expansion encloses the graph of the polynomial over I. By a simple proof it is shown that this convex hull is inclusion isotonic, i.e. if one shrinks I then the convex hull of the control points on the smaller interval is contained in the convex hull of the control points on I. From this property it follows that the so-called Bernstein form is inclusion isotone, which was shown by a longish proof in 1995 in this journal by Hong and Stahl. Inclusion isotonicity also holds for multivariate polynomials on boxes. Examples are presented which document that two simpler enclosures based on only a few control points are in general not inclusion isotonic.

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1 Introduction

If we expand a polynomial p into Bernstein polynomials over an interval I then the coefficients of this expansion, the so-called Bernstein coefficients, enjoy the *convex hull property*, i.e. the convex hull of the control points associated with the Bernstein coefficients encloses the graph of p over I, e.g., [9]. From this fact it follows that the interval spanned by the minimum and maximum of the Bernstein coefficients, the so-called *Bernstein form*, encloses the range of p over I, cf. [1, 9]. This important property carries over to the expansion of multivariate polynomials into Bernstein polynomials over boxes and triangles, cf. [4]. Hong and Stahl [3] have shown that the Bernstein form is inclusion isotonic, i.e. that if we shrink the interval I to a smaller interval then the enclosure shrinks, too. Inclusion isotonicity is a fundamental property of interval computation, cf. [7]. In this note we extend the result by Hong and Stahl and show that the convex hull of the control points associated with the Bernstein coefficients is inclusion isotonic, too. Our proof is simple, thereby providing a shorter proof of the result of Hong and Stahl.

The organisation of this paper is as follows: In the next section we recall the Bernstein expansion and some of its properties. In Section 3 we present three so-called convex–concave extensions which provide for a given polynomial a convex lower bound and a concave upper bound function. In the literature, such bound functions are sometimes called convex underestimators and concave overestimators, respectively. All three convex–concave extensions are based on the control points of the polynomial with an increasing order of complexity. The first lower bound function is an affine function, the second one is comprised of two affine functions; similarly for the upper bound functions. Finally, the third convex–concave extension is provided by the convex hull of all control points. In Section 4 we state our main result that the convex–concave extension based on the convex hull of the control points is inclusion isotonic, whereas the other two convex–concave extensions are in general not.

For conciseness we concentrate here mostly on univariate polynomials. The extension of the inclusion isotonicity to multivariate polynomials is straightforward, see the remark after the proof of Theorem 1.

2 Bernstein expansion

The ith Bernstein polynomial of degree l over an interval $[\underline{a}, \overline{a}]$ is defined as

$$B_{l,i}(x) = \binom{l}{i} \frac{(x-\underline{a})^i(\overline{a}-x)^{l-i}}{(\overline{a}-\underline{a})^l}, \; i = 0, 1, \ldots, l.$$

A polynomial

$$p(x) = \sum_{i=0}^{l} a_i x^i \tag{1}$$

can be transformed over $[\underline{a}, \overline{a}]$ into its Bernstein form as

$$p(x) = \sum_{i=0}^{l} b_i B_{l,i}(x),$$

where the *Bernstein coefficients* b_i are given by

$$b_i = \sum_{j=0}^{i} \frac{\binom{i}{j}}{\binom{l}{j}} (\overline{a}-\underline{a})^j \sum_{k=j}^{l} \binom{k}{j} \underline{a}^{k-j} a_k, \; i = 0, 1, \ldots, l. \tag{2}$$

Without loss of generality we consider the unit interval $I = [0, 1]$ since any nonempty compact interval can be mapped affinely onto it. In this case, (2) reduces to

$$b_i = \sum_{j=0}^{i} \frac{\binom{i}{j}}{\binom{l}{j}} a_j, \; i = 0, 1, \ldots, l. \tag{3}$$

The extension to the Bernstein expansion on the unit box $I = [0, 1]^n$ is straightforward if we regard i as a multiindex, $i = (i_1, \ldots, i_n)$. Comparisons between multiindices are then understood componentwise, binomial coefficients $\binom{m}{j}$ as $\binom{m_1}{j_1} \ldots \binom{m_n}{j_n}$, and summation $\sum\limits_{j=0}^{i}$ as $\sum\limits_{j_1=0}^{i_1} \ldots \sum\limits_{j_n=0}^{i_n}$. If we define multipowers x^i as $x_1^{i_1} \ldots x_n^{i_n}$ we obtain the Bernstein coefficients b_i, $0 \leq i \leq l$, of a multivariate polynomial (1) in the same form (3). For the sake of simplicity we present our results only in the univariate case ($n = 1$) but outline extensions to the multivariate case.

Now we give two properties of the Bernstein expansion which we will later use. Let $b_0, \ldots, b_l$ be the Bernstein coefficients of a polynomial p (of degree l) over an interval $[\underline{a}, \overline{a}]$.

- **Convex hull property** (e.g. [9])

  The control points associated with the Bernstein coefficients of p over $[\underline{a}, \overline{a}]$ are indicated in bold, i.e. $\mathbf{b}_i = \binom{\underline{a} + \frac{i(\overline{a} - \underline{a})}{l}}{b_i}$ is the control point associated with b_i. The convex hull of p over $[\underline{a}, \overline{a}]$ is contained within the convex hull of the control points $\mathbf{b}_i$, i.e.

  $$conv \left\{ \begin{pmatrix} x \\ p(x) \end{pmatrix} \;\middle|\; x \in [\underline{a}, \overline{a}] \right\} \subseteq conv \{ \mathbf{b}_i \mid i = 0, \ldots, l \}. \tag{4}$$

  Note also that

  $$\begin{aligned} b_0 &= \sum_{i=0}^{l} a_i \underline{a}^i &= p(\underline{a}), \\ b_l &= \sum_{i=0}^{l} a_i \overline{a}^i &= p(\overline{a}). \end{aligned} \tag{5}$$

- **Subdivision** ([6])

  If we subdivide $[\underline{a}, \overline{a}]$ into $[\underline{a}, a_\lambda]$ and $[a_\lambda, \overline{a}]$ with $a_\lambda := (1 - \lambda)\underline{a} + \lambda \overline{a}$, where $\lambda \in (0, 1)$, then we can compute the Bernstein coefficients on both intervals by the de Casteljau algorithm:

  For $k = 1, \ldots, l$

  $$b_i^{(k)} := (1 - \lambda) b_{i-1}^{(k-1)} + \lambda b_i^{(k-1)}, \;\; i = k, \ldots, l. \tag{6}$$

The Bernstein coefficients on $[\underline{a}, a_\lambda]$, respectively $[a_\lambda, \overline{a}]$, can be found on the main diagonal of the resulting triangular scheme, respectively in its last column, however, in reverse order, i.e.,

$$\left.\begin{array}{l} b_i^{[\underline{a},a_\lambda]} = b_i^{(i)} \\ b_i^{[a_\lambda,\overline{a}]} = b_l^{(l-i)} \end{array}\right\} i = 0, \ldots, l. \tag{7}$$

The generalisation to the multivariate case is given in [10].

3 Convex–concave extensions

Convex–concave extensions were introduced and investigated in [5]. A *convex–concave extension* of a function $f : S \to \mathbf{R}$, $S \subseteq \mathbf{R}^n$, is a mapping $[\underline{f}, \overline{f}]$ which provides for each nonempty box $X \subseteq S$ a convex function $\underline{f}_X : X \to \mathbf{R}$ (lower bound function) and a concave function $\overline{f}_X : X \to \mathbf{R}$ (upper bound function) such that

$$\underline{f}_X(x) \leq f(x) \leq \overline{f}_X(x), \quad \forall x \in X. \tag{8}$$

These functions generalise the concept of interval extensions, e.g., p. 23 in [8]. We propose three different convex–concave extensions for a univariate polynomial p of degree l based upon the Bernstein coefficients and listed in increasing order of complexity. Examples are given in the following section. Let $b_0, \ldots, b_l$ be the Bernstein coefficients of p over $[\underline{a}, \overline{a}]$ and let $b_k = \min_{i=0,\ldots,l} b_i$. We will use the slope between $\mathbf{b}_k$ and $\mathbf{b}_i$ which is given by

$$\frac{l}{\overline{a} - \underline{a}} \frac{b_i - b_k}{i - k}, \quad i = 0, \ldots, k-1, k+1, \ldots, l. \tag{9}$$

3.1 One affine function

Let b_h be a Bernstein coefficient with

$$\left| \frac{b_h - b_k}{h - k} \right| = \min_{i=0,\ldots,k-1,k+1,\ldots,l} \left| \frac{b_i - b_k}{i - k} \right| \tag{10}$$

and let $d_k = l\frac{b_h - b_k}{h-k}$. Then for all $x \in [\underline{a}, \overline{a}]$, define an affine lower bound function $\underline{f}$ as

$$\underline{f}(x) = b_k + \frac{d_k}{\overline{a} - \underline{a}} \left(x - \frac{k}{l} \right). \tag{11}$$

An upper bound function $\overline{f}$ can be defined in a similar fashion, giving

$$\underline{f}(x) \leq p(x) \leq \overline{f}(x), \quad \text{for all} \quad x \in [\underline{a}, \overline{a}]. \tag{12}$$

In [2] we extend this affine lower bound function to multivariate polynomials. Then the coefficients of this bound function can be represented as the optimal solution of a linear programming problem.

3.2 Two affine functions

Let b_k be the minimum Bernstein coefficient, as above. Now define two slopes d_k^- and d_k^+, given by

$$d_k^- = l \max_{i=0,\ldots,k-1} \frac{b_i - b_k}{i-k}, \quad d_k^+ = l \min_{i=k+1,\ldots,l} \frac{b_i - b_k}{i-k}. \tag{13}$$

Then for all $x \in [\underline{a}, \overline{a}]$, define a lower bound function, comprised of two affine functions, $\underline{f}$ as

$$\underline{f}(x) = \begin{cases} b_k + \frac{d_k^-}{\overline{a}-\underline{a}} \left(x - \frac{k}{l}\right), & \text{if } x \le \frac{k}{l} \\ b_k + \frac{d_k^+}{\overline{a}-\underline{a}} \left(x - \frac{k}{l}\right), & \text{if } x > \frac{k}{l}. \end{cases} \tag{14}$$

An upper bound function $\overline{f}$ can be defined in a similar fashion, satisfying (12).

3.3 Convex hull

These convex–concave extensions have a natural extension to the use of (upto) l upper and lower affine functions, which are provided by the convex hull of the control points associated with the Bernstein coefficients. Starting from the left-most Bernstein coefficient $b_0 = \underline{f}(\underline{a}) = \overline{f}(\underline{a})$, we construct the lower bound function $\underline{f}$ by using the facets of the convex hull lying below (or on) the straight line connecting b_0 with the right-most Bernstein coefficient $b_l = \underline{f}(\overline{a}) = \overline{f}(\overline{a})$. Analogously, the upper bound function $\overline{f}$ is determined by the facets of the convex hull lying above (or on) this straight line. Examples are presented in Fig. 3.

4 Inclusion isotonicity

A convex–concave extension $[\underline{f}, \overline{f}]$ is *inclusion isotone* if, for all intervals X and Y with $X \subseteq Y$,

$$\underline{f}_Y(x) \le \underline{f}_X(x) \le \overline{f}_X(x) \le \overline{f}_Y(x), \quad \forall x \in X. \tag{15}$$

The following counterexample shows that the convex–concave extensions based upon either one or two affine upper and lower bound functions are in general not inclusion isotone. Let $p(x) = 32x^4 - 112x^3 + 118x^2 - 47x + 6$. Tab. 1 gives the convex–concave extensions (of each type) over the intervals $[0, 0.5]$, $[0, 0.6]$, $[0, 0.7]$ and $[0, 1]$. They are most easily specified by their upper and lower vertices (rounded to 3 decimal places). For example, the first entry for one affine function in Table 1 states that the affine upper bound function $\overline{f}$ has the values $\overline{f}(0) = 6$, $\overline{f}(0.5) = 0$, and the affine lower bound function $\underline{f}$ satisfies $\underline{f}(0) = -1.667$, $\underline{f}(0.5) = 0$.

The corresponding convex–concave extension is not inclusion isotone since $\underline{f}_{[0,0.5]}(0) = -1.667 < \underline{f}_{[0,0.6]}(0) = -1.08$, cf. Figure 1. A similar situation occurs on the same intervals for the lower bound function composed from two affine functions, cf. Figure 2.

X	**One affine function**	**Two affine functions**
$[0, 0.5]$	(0,6) (0.5,0) (0,-1.667) (0.5,0)	(0,6) (0.5,0) (0,1.083) (0.25,-0.833) (0.5,0)
$[0, 0.6]$	(0,6) (0.6,0.235) (0,-1.08) (0.6,-0.96)	(0,6) (0.6,0.235) (0,6) (0.15,-1.05) (0.6,-0.96)
$[0, 0.7]$	(0,6) (0.7,0.187) (0,-3.029) (0.7,0.187)	(0,6) (0.7,0.187) (0,6) (0.175,-2.225) (0.7,0.187)
$[0, 1]$	(0,6) (1,0.333) (0,-6.667) (1,-3)	(0,6) (1,0.333) (0,6) (0.25,-5.75) (1,-3)
	Convex hull	
$[0, 0.5]$	(0,6) (0.5,0) (0,6) (0.125,0.125) (0.25,-0.833) (0.5,0)	
$[0, 0.6]$	(0,6) (0.6,0.235) (0,6) (0.15,-1.05) (0.3,-1.02) (0.6,0.235)	
$[0, 0.7]$	(0,6) (0.7,0.187) (0,6) (0.175,-2.225) (0.7,0.187)	
$[0, 1]$	(0,6) (0.75,1.75) (1,-3) (0,6) (0.25,-5.75) (1,-3)	

Tab. 1. Vertices of the convex–concave extensions

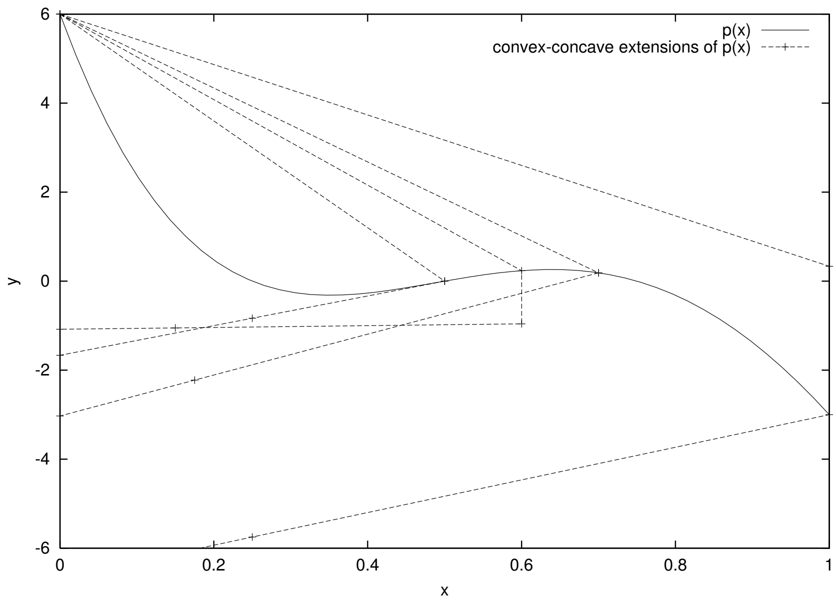

Fig. 1. Failure of inclusion isotonicity with one affine function

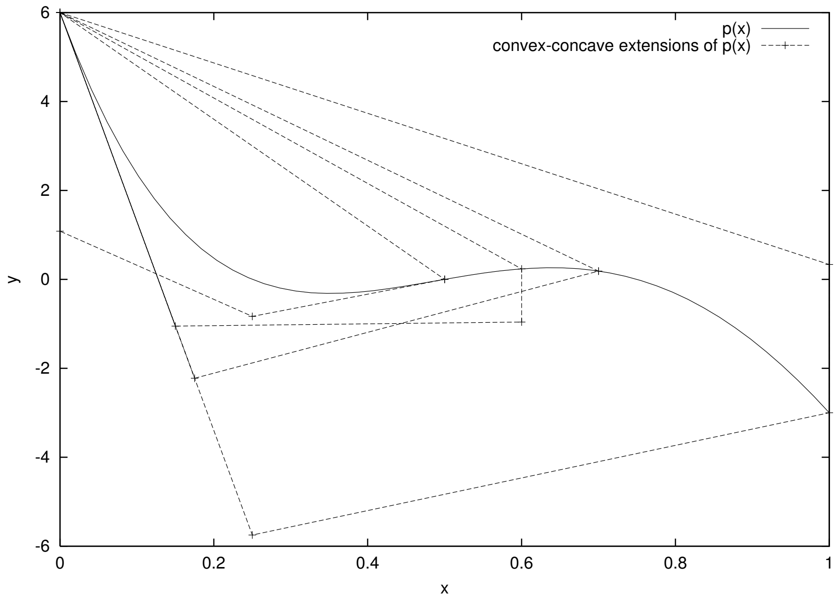

Fig. 2. Failure of inclusion isotonicity with two affine functions

We prove below, however, that the convex hull is inclusion isotone, yielding the inclusion isotonicity of the corresponding convex–concave extension. For simplicity we consider the case of the unit interval $I = [0, 1]$.

Theorem 1 *The convex hull of the control points associated with the Bernstein coefficients of a univariate polynomial is inclusion isotone.*

Proof: Let $p(x) = \sum_{i=0}^{l} a_i x^i$, with Bernstein coefficients $b_0, \ldots, b_l$ over the interval $[0, 1]$. It suffices to show that inclusion isotonicity holds true if we shrink only one of the two endpoints. Let $b_0^*, \ldots, b_l^*$ and $b_0^\dagger, \ldots, b_l^\dagger$ be the Bernstein coefficients of p over $[0, 1-\varepsilon]$ and $[\varepsilon, 1]$ respectively, where $\varepsilon \in (0, 1)$. We apply the algorithm (6) to compute the Bernstein coefficients on the intervals $[0, 1-\varepsilon]$ and $[\varepsilon, 1]$. Since we are forming convex combinations in (6), it follows from (7) that

$$\mathbf{b}_i^* \in conv\{\mathbf{b}_0, \ldots, \mathbf{b}_i\} \quad \text{and} \quad \mathbf{b}_i^\dagger \in conv\{\mathbf{b}_i, \ldots, \mathbf{b}_l\},\ i = 0, \ldots, l. \tag{16}$$

From (16) the statement of the theorem immediately follows. ■

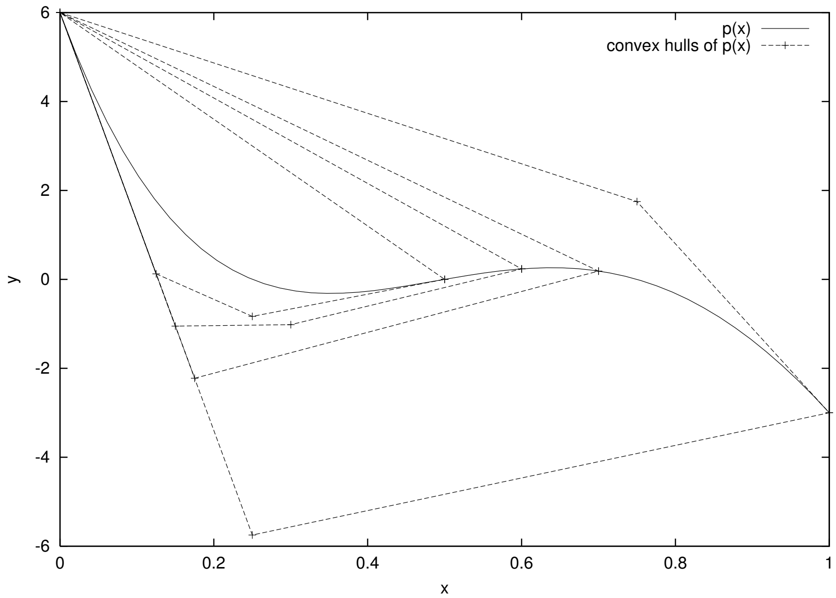

Fig. 3. The convex hull is inclusion isotone.

Figure 3 illustrates the property of inclusion isotonicity with the same polynomial and intervals as above.

Remark: In the n-variate case, the Bernstein polynomials $B_{l,i}$ are defined as

$$B_{l,i}(x) = B_{l_1,i_1}(x_1) \cdot \ldots \cdot B_{l_n,i_n}(x_n), \quad 0 \leq i \leq n. \tag{17}$$

The Bernstein coefficients now form an n-dimensional array. By shrinking the unit box in each dimension separately, we form convex combinations in only one direction. So the proof of the inclusion isotonicity of the convex hull proceeds in the multivariate case in a similar fashion.

The following Corollary is due to Hong and Stahl [3] and is a consequence of Theorem 1.

Corollary 1 *The enclosure for the range of a univariate polynomial p over an interval X provided by its Bernstein coefficients $b_0, \ldots, b_l$,*

$$\min_{i=0,\ldots,l} b_i \;\leq\; p(x) \;\leq\; \max_{i=0,\ldots,l} b_i, \quad x \in X, \tag{18}$$

is inclusion isotone.

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