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Hadamard Products of Stable Polynomials Are Stable

Jürgen Garloff*

Fachbereich Informatik, Fachhochschule Konstanz, Brauneggerstrasse 55, D-78462 Konstanz, Germany

and

David G. Wagner†

Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

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A real polynomial is (asymptotically) stable when all of its zeros lie in the open left half of the complex plane. We show that the Hadamard (coefficient-wise) product of two stable polynomials is again stable, improving upon some known results. Via the associated Hurwitz matrices we find another example of a class of totally nonnegative matrices which is closed under Hadamard multiplication.

1. INTRODUCTION

The *Hadamard product* of two polynomials

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$
$$q(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0$$

*Research partially supported by a sabbatical grant from the Ministry of Science and Research Baden-Württemberg/Germany and DFG/NSERC Bilateral Exchange Program. E-mail address: garloff@fh-konstanz.de.

†Research supported by NSERC Operating Grant OGP0105392. E-mail address: dgwagner@math.uwaterloo.ca.

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in $\mathbb{R}[x]$ is defined to be

$$(p*q)(x) = a_k b_k x^k + a_{k-1} b_{k-1} x^{k-1} + \cdots + a_1 b_2 x + a_0 b_0,$$

where $k = \min(n, m)$.

The problem of determining the locus of zeros of $p*q$ in terms of those if p and q has a long history. In 1895, Maló [13] proved that if the zeros of p are all real and the zeros of q are all real and of the same sign then all the zeros of $p*q$ are real as well. Weisner [19] generalized this result to the case in which the zeros of p are all real and negative, while those of q lie in a sector S_α with its vertex at the origin and with aperture $\alpha \leq \pi$; then all zeros of $p*q$ lie in S_α as well. More generally, from a theorem of de Bruijn [3] and using [16] it follows that if all the zeros of p are in a sector S_α of aperture α centered on the negative real axis, and all the zeros of q are in a similar sector S_β of aperture β, where $\alpha, \beta \leq \pi$, then all zeros of $p*q$ are in the sector $S_{\alpha+\beta}$.

Here we concentrate on classes of stable polynomials. A polynomial $p \in \mathbb{R}[x]$ is (*Hurwitz* or *asymptotically*) *stable* if every zero of p is in the open left half of the complex plane, and p is *quasi-stable* if every zero of p is in the closed left half of the complex plane. Also, $p \in \mathbb{R}[x]$ is *sinusoidal* if every zero of p is purely imaginary or 0, and p is *almost sinusoidal* if exactly one zero of p is not purely imaginary or 0, and is negative. This terminology is motivated by consideration of the long-term $(t \to \infty)$ qualitative behaviour of a general solution to the differential equation $p(d/dt)V(t) = 0$, see Section 3.1 of [2].

Our main result is the following. The multiplicity of ξ as a zero of p is denoted by $\operatorname{mult}(\xi, p)$.

THEOREM 1. *Let $F, P \in \mathbb{R}[x]$ be quasi-stable.*

(a) *Then $F*P$ is quasi-stable.*

(b) *If either F or P is sinusoidal then $F*P$ is sinusoidal. If both F and P are sinusoidal and* $\operatorname{mult}(0, F)$ *and* $\operatorname{mult}(0, P)$ *have different parities then* $F*P = 0$.

(c) *If both F and P are almost sinusoidal and* $\operatorname{mult}(0, F)$ *and* $\operatorname{mult}(0, P)$ *have the same parity, then $F*P$ is almost sinusoidal.*

(d) *If neither* (b) *nor* (c) *apply then $F*P$ has no purely imaginary zeros except possibly at the origin.*

(e) *As a special case of* (d), *if both F and P are stable then $F*P$ is also stable.*

As a simple consequence we have the following corollary.

COROLLARY 2. *For any* $P \in \mathbb{R}[x]$ *of degree* n, *if* P *is quasi-stable then* $(x+1)^n * P$ *is quasi-stable. If* P *is sinusoidal then* $(x+1)^n * P$ *is sinusoidal, and otherwise* $(x+1)^n * P$ *has no purely imaginary zeros except possibly at the origin.*

The key to the proof is the Hermite–Biehler Theorem (p. 228 of [4]) whereby the problem is recast as one involving polynomials with only real zeros. After proving Theorem 1 we apply it to a related problem regarding Hadamard products of totally nonnegative matrices.

It is interesting to note that a similar Hadamard product result does not hold for Schur stability. A polynomial $p \in \mathbb{R}[x]$ is *Schur stable* if every zero of p is in the open unit disc with center at 0. Clearly $p = (x + 0.9)(x + 0.8) = x^2 + 1.7x + 0.72$ is Schur stable but $p * p = x^2 + 2.89x + 0.5184$ has one zero less than $-2.89/2$ (in fact the zeros of $p * p$ are $-2.697\dots$ and $-0.1921\dots$).

A polynomial is *aperiodic* if all its zeros are simple and negative. In passing, we note that from Theorem 4(a), it follows that the Hadamard product of two aperiodic polynomials is aperiodic.

2. PROOFS

Suppose that $p, q \in \mathbb{R}[x]$ both have only real zeros, that those of p are $\xi_1 \leq \cdots \leq \xi_n$, and that those of q are $\theta_1 \leq \cdots \leq \theta_m$. We say that p *interlaces* q if $\deg q = 1 + \deg p$ and the zeros of p and q satisfy

$$\theta_1 \leq \xi_1 \leq \theta_2 \leq \cdots \leq \xi_n \leq \theta_{n+1}.$$

We also say that p *alternates left of* q if $\deg p = \deg q$ and the zeros of p and q satisfy

$$\xi_1 \leq \theta_1 \leq \xi_2 \leq \cdots \leq \xi_n \leq \theta_n.$$

We use the notations $p \dagger q$ for "p interlaces q," $p \ll q$ for "p alternates left of q," and $p \prec q$ for "either $p \dagger q$ or $p \ll q$." Of course, any polynomial which stands in one of these relations *a fortiori* has only real zeros. By convention, we say that for any polynomial p with only real zeros, all of $p \dagger 0$, $0 \dagger p$, $p \ll 0$, and $0 \ll p$ hold. A nonzero $p \in \mathbb{R}[x]$ is *standard* when its leading coefficient is positive. For brevity, we say that a polynomial *has only nonpositive zeros* to indicate that all of its zeros are real and nonpositive.

THEOREM 3 (Hermite-Biehler). *Let* $F(x) = f(x^2) + xg(x^2) \in \mathbb{R}[x]$ *be standard. Then*:

(a) *F is quasi-stable if and only if both f and g are standard, have only nonpositive zeros, and $g \prec f$.*

(b) *F is stable if and only if F is quasi-stable, $f(0) \neq 0$, and* $\gcd(f, g) = 1$.

(c) *F is sinusoidal if and only if F is quasi-stable and either $f = 0$ or $g = 0$.*

(d) *F is almost sinusoidal if and only if F is quasi-stable, $f \neq 0$ and $g \neq 0$, and either $f = cg$ or $f = cxg$ for some $c > 0$. In this case,* $\mathrm{mult}(0, F)$ *is even or odd according to whether $f = cg$ or $f = cxg$.*

The proof of the Hermite–Biehler Theorem in [4] covers only the case of stable polynomials, but the statement given here can be deduced from it easily by a limiting argument. Using Theorem 3 we reduce Theorem 1 to the following.

THEOREM 4. *Let $f, g, p, q \in \mathbb{R}[x]$ be standard with only nonpositive zeros.*

(a) *If $f * p \neq 0$ then $f * p$ has only nonpositive zeros which are simple except possibly at the origin.*

(b) *If $g \prec f$ and $q \prec p$ then $g * q \prec f * p$.*

(c) *If $g \prec f$ and $q \prec p$, and f, g, p, q are all nonzero, and neither $f = c_1 g$ and $p = c_2 q$ nor $f = c_1 xg$ and $p = c_2 xq$ for any $c_1 > 0$ and $c_2 > 0$, then* $\gcd(g * q, f * p) = x^r$, *where* $r := \max\{\mathrm{mult}(0, g), \mathrm{mult}(0, q)\}$.

Proof of Theorem 1. We may assume that F and P are standard. Since F and P are quasi-stable, we have $F(x) = f(x^2) + xg(x^2)$ and $P(x) = p(x^2) + xq(x^2)$, where f, g, p, q are standard, have only nonpositive zeros, and $g \prec f$ and $q \prec p$, by Theorem 3. Now $(F * P)(x) = (f * p)(x^2) + x(g * q)(x^2)$ and by Theorem 4, $f * p$ and $g * q$ are standard with only nonpositive zeros, and $g * q \prec f * p$. This proves (a). For (b), if either F or P is sinusoidal, then one of f, g, p, q is zero, so that one of $f * p$ or $g * q$ is zero, so that $F * P$ is sinusoidal. If both F and P are sinusoidal with multiplicities at 0 of opposite parity then either $f = 0 = q$ or $g = 0 = p$. In either case, $f * p = 0 = g * q$, so that $F * P = 0$. For (c), if F and P are almost sinusoidal with $\mathrm{mult}(0, F)$ and $\mathrm{mult}(0, P)$ of the same parity, then f, g, p, q are all nonzero, and either $f = c_1 g$ and $p = c_2 q$ or $f = c_1 xg$ and $p = c_2 xg$, for some $c_1 > 0$ and $c_2 > 0$. Thus, $f * p$ and $g * q$ are nonzero and either $f * p = c_1 c_2 (g * q)$ or $f * p = c_1 c_2 x(g * q)$, so that $F * P$ is almost sinusoidal. For part (d), case (c) of Theorem 4 applies, and since $\gcd(g * q, f * p) = x^r$, the only point of the imaginary axis which can be a zero of $F * P$ is the origin. Part (e) follows from (d). ∎

We now summarize the lemmas required for our proof of Theorem 4; proofs of Lemmas 5 and 6 are given in Section 3 of [18].

LEMMA 5. *Let* $p, q \in \mathbb{R}[x]$ *be standard with only real zeros.*

(a) *If* $q \ll p$ *then* $q \ll p + q$ *and* $p + q \ll p$.

(b) *If* $q \dagger p$ *then* $q \dagger p + q$ *and* $p + q \ll p$.

(c) *If* $q \dagger p$ *then* $q \dagger p - q$ *and* $p \ll p - q$.

(d) *Assume that* $q \ll p$, *and let the leading coefficients of* q *and* p *be* C_q *and* C_p, *respectively.*
Then

$$\begin{array}{llll} p - q \ll q & \text{and} & p - q \ll p & \text{if } C_q > C_p, \\ p - q \dagger q & \text{and} & p - q \dagger p & \text{if } C_q = C_p, \\ q \ll p - q & \text{and} & p \ll p - q & \text{if } C_q < C_p. \end{array}$$

LEMMA 6. *Let* $f, g_1, g_2 \in \mathbb{R}[x]$ *be standard with only real zeros.*

(a) *If* $g_1 \prec f$ *and* $g_2 \prec f$ *then* $g_1 + g_2 \prec f$.

(b) *If* $f \prec g_1$ *and* $f \prec g_2$ *then* $f \prec g_1 + g_2$.

(c) *Under either condition* (a) *or* (b), $\gcd(g_1 + g_2, f)$ *divides both* $\gcd(g_1, f)$ *and* $\gcd(g_2, f)$.

Lemma 7 is a useful characterization of the relation $g \prec f$, due essentially to Krein. Proposition 1.6 of [17] and results of Section 3 of [18] provide a proof.

LEMMA 7. *Let* $f \in \mathbb{R}[x]$ *be standard with only real zeros. Let the distinct zeros of* f *be* $\xi_1, \xi_2, \ldots, \xi_d$ *let* $\hat{f}_i := f/(x - \xi_i)$, *and let* $m_i = \mathrm{mult}(\xi_i, f)$ *for* $1 \leq i \leq d$.

(a) *Every* $g \in \mathbb{R}[x]$ *such that* $\deg(g) \leq \deg(f)$ *and* $m_i - 1 \leq \mathrm{mult}(\xi_i, g)$ *for each* $1 \leq i \leq d$ *may be written uniquely in the form*

$$g = c_0 f + \sum_{i=1}^{d} c_i \hat{f}_i,$$

where $c_i \in \mathbb{R}$ *for each* $0 \leq i \leq d$.

(b) *The polynomial* $g \in \mathbb{R}[x]$ *is standard, has only real zeros, and is such that* $g \prec f$ *if and only if* g *may be expanded as in* (a) *with* $c_i \geq 0$ *for each* $0 \leq i \leq d$.

(c) *If* g *is expanded as in* (a), *then* $\gcd(g, f) = \prod_{i=1}^{d} (x - \xi_i)^{m_i - \delta_i}$, *where* $\delta_i := 1$ *if* $c_i \neq 0$ *and* $\delta_i := 0$ *if* $c_i = 0$, *for each* $1 \leq i \leq d$.

We also need the following form of "Newton's Inequalities"; see Lemma 3 in Section 8.2 of [12], for example.

LEMMA 8. *Let* $f \in \mathbb{R}[x]$ *be standard, say* $f(x) = \sum_{i=0}^{n} a_i x^i$.

(a) *If f has only real zeros, then* $a_i^2 - a_{i-1}a_{i+1} > 0$ *for all* $\mathrm{mult}(0, f) + 1 \leq i \leq \deg(f)$.

(b) *If f has only nonpositive zeros, then* $a_i > 0$ *for all* $\mathrm{mult}(0, f) \leq i \leq \deg(f)$.

LEMMA 9. *Let* $f, g \in \mathbb{R}[x]$ *be standard, with only nonpositive zeros, and such that* $g \prec f$. *Say* $f(x) = \sum_{i=0}^{n} a_i x^i$ *and* $g(x) = \sum_{i=0}^{n} b_i x^i$ (*where* $b_n = 0$ *is possible*). *If there exists an index j such that* $a_j = b_j \neq 0$ *and* $a_{j-1} = b_{j-1} \neq 0$ *then* $f = g$.

Proof. From Lemma 5 it follows that $f - (1-\varepsilon)g$ has only real zeros, for all $\varepsilon \in \mathbb{R}$. Thus, by Lemma 8, we have

$$\left(a_i - (1-\varepsilon)b_i\right)^2 - \left(a_{i-1} - (1-\varepsilon)b_{i-1}\right)\left(a_{i+1} - (1-\varepsilon)b_{i+1}\right) > 0 \tag{1}$$

for all indices i such that $a_i - (1-\varepsilon)b_i \neq 0$. Now suppose that $f \neq g$ but that there exists an index j as in the hypothesis. Then there exists such an index j such that either $a_{j+1} \neq b_{j+1}$ or $a_{j-2} \neq b_{j-2}$; we assume the former case as the argument for the latter case is analogous. Substituting into (1) and rearranging, we obtain

$$\varepsilon^2\left(b_j^2 - b_{j-1}b_{j+1}\right) - \varepsilon b_{j-1}\left(a_{j+1} - b_{j+1}\right) > 0, \tag{2}$$

which is valid for all $0 \neq \varepsilon \in \mathbb{R}$. By Lemma 8 we have $b_j^2 - b_{j-1}b_{j+1} > 0$. Now take ε of the same sign as $b_{j-1}(a_{j+1} - b_{j+1})$, and with

$$0 < |\varepsilon| < \frac{\left|b_{j-1}\left(a_{j+1} - b_{j+1}\right)\right|}{b_j^2 - b_{j-1}b_{j+1}}.$$

This contradicts (2), completing the proof. ∎

To prove Theorem 4 we calculate with several additional products and linear transformations on $\mathbb{R}[x]$. Let $f = \sum_j a_j x^j$ and $g = \sum_j b_j x^j$ be polynomials, and define

$$f \odot g = \sum_j j! a_j b_j x^j$$

$$Lf = \sum_j a_j x^j / j!$$

$$Jf = \sum_j a_j x^{j+1} / (j+1)$$

$$Df = \sum_j j a_j x^{j-1}.$$

Notice that with the notation of Lemma 7 we have $Df = \sum_{i=1}^{d} m_i \hat{f}_i$. We also have the following useful computation rules; the proofs are simple calculations which are omitted.

LEMMA 10. *For any $f, g \in \mathbb{R}[x]$:*

(a) *The operations $*$ and $\odot$ are commutative, associative, and $\mathbb{R}$-bilinear.*

(b) *L, J, and D are $\mathbb{R}$-linear.*

(c) $DJf = f$.

(d) $JDf = f(x) - f(0)$.

(e) $Lxf = JLf$.

(f) $f \odot (xg) = x((Df) \odot g)$.

(g) $D(f \odot g) = (Df) \odot (Dg)$.

(h) $J(f \odot g) = (Jf) \odot (Jg)$.

(i) $L(f \odot g) = (Lf) \odot g = f \odot (Lg) = f * g$.

Theorem 11(a) sharpens a result of Laguerre; see [12, p. 341].

THEOREM 11. *Let $f, g \in \mathbb{R}[x]$ be standard with only real zeros.*

(a) *For each $k \in \mathbb{N}$, $J^k Lf$ has only real zeros.*

(b) *If f has only nonpositive zeros then for each $k \in \mathbb{N}$, $J^k Lf$ has only nonpositive zeros, which are simple except possibly for the origin.*

(c) *If $g \prec f$ then $J^k Lg \prec J^k Lf$ for each $k \in \mathbb{N}$.*

(d) *If f has only nonpositive zeros and $g \prec f$ and $g \neq cf$ for all $c \in \mathbb{R}$ then* $\gcd(J^k Lg, J^k Lf) = x^{k+m}$ *for each $k \in \mathbb{N}$, where $m :=$* $\operatorname{mult}(0, g)$.

Proof. For (a) and (b) we use induction on $\deg(f)$, the basis $\deg(f) \leq 1$ being clear. For the induction step we assume the results for f and prove them for $(x-\theta)f$, where $\theta \in \mathbb{R}$. By Lemma 10 we have

$$J^k L(x-\theta)f = J^k(J-\theta)Lf = (1-\theta D)J^{k+1}Lf.$$

By the induction hypothesis $J^{k+1}Lf$ has only real zeros, so that by Rolle's Theorem, $DJ^{k+1}Lf \dagger J^{k+1}Lf$. From Lemma 5 we deduce that $J^k L(x-\theta)f$ has only real zeros, and that

$$J^k Lf \dagger J^k L(x-\theta)f. \tag{3}$$

This proves (a). Furthermore, if f has only nonpositive zeros then we may assume that $\theta \le 0$, and by induction that $J^{k+1}Lf$ has only nonpositive zeros which are simple except possibly at the origin. By Lemma 7 we see that $(1-\theta D)J^{k+1}Lf \prec J^{k+1}Lf$ and that $\gcd((1-\theta D)J^{k+1}Lf, J^{k+1}Lf) = x^r$ for some $r \in \mathbb{N}$. Thus $J^k L(x-\theta)f = (1-\theta D)J^{k+1}Lf$ also has only nonpositive zeros which are simple except possibly at the origin. This proves (b).

For (c) and (d) we may write

$$g = c_0 f + \sum_{i=1}^{d} c_i \hat{f}_i$$

with each $c_i \ge 0$, by Lemma 7, with the notation explained there. Thus

$$J^k Lg = c_0 J^k Lf + \sum_{i=1}^{d} c_i J^k L\hat{f}_i.$$

By formula (3) we have $J^k L\hat{f}_i \dagger J^k Lf$ for each $1 \le i \le d$, so that $J^k Lg \prec J^k Lf$ by Lemmas 5 and 6, which proves (c).

For (d), first notice that the largest power of x which divides $\gcd(J^k Lg, J^k Lf)$ is x^{k+m}. Thus, it suffices to show that $\gcd(J^k Lg, J^k Lf)$ is a power of x. Consider any $q \in \mathbb{R}[x]$ with only nonpositive zeros. By part (b), and since $J^k Lq = DJ^{k+1}Lq$, we have $\gcd(J^k Lq, J^{k+1}Lq) = x^{k+r}$, where $r := \mathrm{mult}(0, q)$. Thus, for any $\theta \le 0$ we have $\gcd(J^k Lq, J^k L(x-\theta)q) = \gcd(J^k Lq, (1-\theta D)J^{k+1}Lq) = \gcd(J^k Lq, J^{k+1}Lq) = x^{k+r}$. Thus, for each $1 \le i \le d$ we have $\gcd(J^k L\hat{f}_i, J^k Lf) = x^{k+r_i}$, where $r_i := \mathrm{mult}(0, \hat{f}_i)$. Since $g \ne cf$ for all $c \in \mathbb{R}$, there is an index $1 \le i \le d$ with $c_i \ne 0$. By Lemma 6, $\gcd(J^k Lg, J^k Lf)$ divides $\gcd(J^k L\hat{f}_i, J^k Lf)$, and hence is a power of x. ∎

Theorem 12(a) is implied by a result of Schur [15].

THEOREM 12. *Let $f, g, p \in \mathbb{R}[x]$ be standard with only nonpositive zeros.*

(a) *If $f \odot p \neq 0$ then $f \odot p$ has only nonpositive zeros.*

(b) *If $g \prec f$ then $g \odot p \prec f \odot p$.*

(c) *If $g \prec f$ and $g \odot p = f \odot p \neq 0$ then either $f = g$ or $f \odot p = cx^r$ for some $c \in \mathbb{R}$ and $r \in \mathbb{N}$.*

Proof. We prove (a) and (b) together by induction of $d := \deg(f) + \deg(p)$, the basis $d \leq 1$ being clear. For (a) we assume the result for f and p and prove it for f and $(x - \theta)p$, where $\theta \leq 0$. By Lemma 10 we have

$$f \odot (x - \theta)p = x((Df) \odot p) - \theta(f \odot p).$$

Since $Df \dagger f$, the induction hypothesis implies that $(Df) \odot p \prec f \odot p$, and since $f \odot p$ has only nonpositive zeros, it follows that $f \odot p \prec x((Df) \odot p)$. Now by Lemma 5 it follows that $f \odot (x - \theta)p$ has only nonpositive zeros, proving part (a), and that

$$f \odot p \prec f \odot (x - \theta)p.$$

By symmetry, for each $\xi \leq 0$ we also have

$$f \odot p \prec (x - \xi) f \odot p. \tag{4}$$

To prove the induction step for (b), we show that if $f, h, p \in \mathbb{R}[x]$ have only nonpositive zeros and $f \prec h$ then $f \odot p \prec h \odot p$. By Lemma 7 we may write

$$f = c_0 h + \sum_{i=1}^{k} c_i \hat{h}_i,$$

where each $c_i \geq 0$, with the notation explained there. By (4) we have $\hat{h}_i \odot p \prec h \odot p$ for each $1 \leq i \leq k$. By Lemma 6 we conclude that $f \odot p \prec h \odot p$, as required.

For (c), since $f \odot p$ has only nonpositive zeros, Lemma 8 applies. Thus, either $f \odot p$ is a power of x, or it has two consecutive nonzero coefficients. In the latter case, since $g \odot p = f \odot p$, it follows that f and g have a pair of consecutive equal nonzero coefficients. By Lemma 9, since $g \prec f$ this implies that $f = g$. ∎

Proof of Theorem 4. Part (a) follows immediately from Theorems 11 and 12, since $f * p = L(f \odot p)$.

For part (b), from Theorems 11 and 12 we see that if $g \prec f$ then $g * p \prec f * p$, and (equivalently) that if $q \prec p$ then $f * q \prec f * p$. We now show that for $\xi \le 0$ and $\theta \le 0$, $g * q \prec (x-\xi)g * (x-\theta)q$. Notice that

$$\begin{aligned}(x-\xi)g*(x-\theta)q &= (x-\xi)g \odot L(x-\theta)q \\ &= (x-\xi)g \odot ((1-\theta D)JLq) \\ &= x(g \odot (1-\theta D)Lq) - \xi(g \odot (1-\theta D)JLq).\end{aligned}$$

By Theorem 11, JLq has only nonpositive zeros. For the first term on the right side we have $(1-\theta D)Lq \ll Lq$ by Rolle's Theorem and Lemma 5. Thus, by Theorem 11, $g \odot (1-\theta D)Lq \prec g \odot Lq$, and since these polynomials have only nonpositive zeros, it follows that $g \odot Lq \prec x(g \odot (1-\theta D)Lq)$. For the second term on the right side we have $Lq \dagger (1-\theta D)JLq$ by Rolle's Theorem and Lemma 5. It follows from Theorem 10 that $g \odot Lq \prec g \odot (1-\theta D)JLq$. From Lemma 6 we conclude that $g * q \prec (x-\xi)g*(x-\theta)q$.

To complete the proof of (b), we have by Lemma 7 that

$$g = c_0 f + \sum_{i=1}^{d} c_i \hat{f}_i$$

and

$$q = s_0 p + \sum_{j=1}^{e} s_j \hat{p}_j,$$

where each $c_i \ge 0$ and $s_j \ge 0$, with the notation explained there. Hence

$$g * q = \sum_{i=0}^{d} \sum_{j=0}^{e} c_i s_j \left(\hat{f}_i * \hat{p}_j\right), \tag{5}$$

where $\hat{f}_0 := f$ and $\hat{p}_0 := p$. By the previous paragraph, we have $\hat{f}_i * \hat{p}_j \prec f * p$ for each $0 \le i \le d$ and $0 \le j \le e$. By Lemma 6 it follows that $g * q \prec f * p$, as was to be shown.

For (c), notice that x^r is the largest power of x which divides $\gcd(g*q, f*p)$. Thus, it suffices to show that $\gcd(g*q, f*p)$ is a power of x. We may assume that f, g, p, q are all monic. If $q = p$ then $g \ne f$, and by Theorem 11 we see that $g \odot p \prec f \odot p$ and if $g \odot p = f \odot p$ then $f \odot p$ is a power of x. It then follows from Theorem 10 that $\gcd(g*p, f*p)$ is a power of x. Similarly, we are also done if $g = f$. In the remaining case $g \ne f$ and $q \ne p$, so that in the expansion (5) there is a term with $c_i > 0$ and $s_j > 0$, with $1 \le i \le d$ and $1 \le j \le e$. By Lemma 6, $\gcd(g*q, f*p)$ divides $\gcd(\hat{f}_i * \hat{p}_j, f*p)$, where $\hat{f}_i := f/(x-\xi_i)$ and $\hat{p}_j := p/(x-\theta_j)$. Since (by hypothesis) we do not have both $f = xg$ and $p = xq$, we may assume that either $\xi_i \ne 0$ or $\theta_j \ne 0$. Thus it suffices to show for $\xi \le 0$ and

$\theta \le 0$ with either $\xi \ne 0$ or $\theta \ne 0$ that $\gcd(g * q, (x - \xi)g * (x - \theta)q)$ is a power of x. By symmetry, we may assume that $\xi \ne 0$. By the above argument for (b),

$$(x - \xi)g * (x - \theta)q = x(g \odot (1 - \theta D)Lq) - \xi(g * (x - \theta)q),$$

and $g * q \prec x(g \odot (1 - \theta D)Lq)$ and $g * q \prec g * (x - \theta)q$. Thus, Lemma 6 implies that $\gcd(g * q, (x - \xi)g * (x - \theta)q)$ divides $\gcd(g * q, g * (x - \theta)q)$. By Theorem 12 we see that $g \odot q \prec g \odot (x - \theta)q$ and if $g \odot q = g \odot (x - \theta)q$ then $g \odot (x - \theta)q$ is a power of x. Now Theorem 11 implies that $\gcd(g * q, g * (x - \theta)q)$ is a power of x. This completes the proof. ∎

3. RELATED MATRIX RESULTS

We now turn to some matrix results related to stability and Hadamard products of polynomials. With a real polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ we associate the n-by-n *Hurwitz matrix* $H(p) = (h_{ij}(p))$, defined by $h_{ij}(p) = a_{2j-i}$ for each $1 \le i, j \le n$, where by convention $a_k = 0$ if $k < 0$ or $k > n$. That is,

$$H(p) = \begin{bmatrix} a_1 & a_3 & a_5 & \cdots & & 0 \\ a_0 & a_2 & a_4 & \cdots & & 0 \\ 0 & a_1 & a_3 & \cdots & & 0 \\ 0 & a_0 & a_2 & \cdots & & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & & \cdots & a_{n-3} & a_{n-1} & 0 \\ 0 & 0 & \cdots & & a_{n-2} & a_n \end{bmatrix}.$$

A real matrix M is *totally nonnegative* if every minor of M is nonnegative; in particular, each entry of M is nonnegative. This class of matrices has an interesting and applicable theory, as developed in [5, 10]. The relevant theorem for us is that the Hurwitz matrix $H(p)$ is nonsingular and totally nonnegative if and only if p is stable and $a_n > 0$, see [1, 11]. Also, if p is quasi-stable then $H(p)$ is totally nonnegative, but the converse does not hold [1, pp. 408, 411].

Given two n-by-n matrices $A = (a_{ij})$ and $B = (b_{ij})$ the *Hadamard product* of A and B is the n-by-n matrix $A * B$ defined by $A * B = (a_{ij}b_{ij})$. Comprehensive surveys of the Hadamard matrix product are found in [7, 8]. As shown in [9, 14], the Hadamard product of two totally nonnegative matrices need not be totally nonnegative. However, some subclasses of totally nonnegative matrices are known to be closed under Hadamard multiplication. These include:

(i) Generalized Vandermonde matrices $(x_i^{\alpha_j})$ with $1 \leq i, j \leq n$, where either the bases $0 < x_1 < x_2 < \cdots < x_n$ or the exponents $\alpha_1 < \cdots < \alpha_n$ are fixed (see p. 99 of [4]).

(ii) Tridiagonal totally nonnegative matrices [14].

(iii) Triangular totally nonnegative infinite Toeplitz matrices such that the value on the kth diagonal is a polynomial function of k [18].

(iv) Green's matrices (g_{ij}) which are totally nonnegative, where $g_{ij} = a_{\min(i,j)} b_{\max(i,j)}$ and $a_1, b_1, \ldots, a_n, b_n$ are positive real numbers (this fact follows from p. 91 of [5] and p. 111 of [10]). More precisely, the following is true: for any fixed number r, the set of totally nonnegative Green's matrices of rank at least r is closed under the Hadamard product, cf. p. 91 of [5].

(v) Finite moment matrices [6] of probability measures which are either symmetric around 0 or possess nonnegative support (Heiligers, private communication).

As a consequence of Theorem 1 and the above remarks, we may include another class of matrices in this list.

THEOREM 13. *If M and N are n-by-n nonsingular totally nonnegative Hurwitz matrices then $M * N$ is a nonsingular totally nonnegative Hurwitz matrix.*

The condition of nonsingularity can be weakened slightly by using Theorem 2 of [1].

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