Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 40, pp. 574-584, August 2024.

A STUDY OF THE VALIDITY OF OPPENHEIM'S INEQUALITY FOR HURWITZ MATRICES ASSOCIATED WITH HURWITZ POLYNOMIALS*

FATIMAH ALSAAFIN†, DOAA AL-SAAFIN†, AND JÜRGEN GARLOFF†‡

Abstract. In this paper, Hurwitz polynomials, i.e., real polynomials whose roots are located in the open left half of the complex plane, and their associated Hurwitz matrices are considered. New formulae for the principal minors of Hurwitz matrices are presented which lead to (i) a new criteria for deciding whether a polynomial is Hurwitz, (ii) an inequality of a type of Oppenheim's inequality for the Hurwitz matrices up to order 6, and (iii) a necessary and sufficient condition for the Hadamard square root of Hurwitz polynomials of degree five to be Hurwitz.

Key words. Hurwitz polynomial, Hurwitz matrix, Hadamard product, Oppenheim's inequality.

AMS subject classifications. 15A45, 15A48.

1. Introduction.

A real symmetric $n \times n$ matrix A is said to be *positive semidefinite* if $\mathbf{x}^T A\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$; A is *positive definite* if $\mathbf{x}^T A\mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{R}^n,\ \mathbf{x} \neq 0$. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be real $n \times n$ matrices. Their *Hadamard product* (also called Schur product) $A \circ B$ is defined as the entrywise product of A and B, $A \circ B = [a_{ij}b_{ij}]$. Let A and B are symmetric. The Loewner partial order $A \succeq B$ denotes that $A - B$ is positive semidefinite, and $A \succ B$ that $A - B$ is positive definite.

We consider polynomials with positive coefficients, i.e., polynomials of the form

$$p(x) = \sum_{k=0}^{n} a_k x^k, \tag{1.1}$$

where a_i , $i = 0, \ldots, n$, are positive numbers. The polynomial p is said to be *Hurwitz* or *stable* if all the roots of p lie in the open left half of the complex plane. By $\boldsymbol{P}_n$ we denote the family of all polynomials of degree n with positive coefficients and by $\boldsymbol{H}_n$ the family of all Hurwitz polynomials in $\boldsymbol{P}_n$.
Let p, q are two polynomials of equal degree n,

$$p(x) = \sum_{k=0}^{n} a_k x^k\ ,\ \ q(x) = \sum_{k=0}^{n} b_k x^k.$$

Then, the *Hadamard product* $p \circ q$ of the two polynomials is defined by

$$(p \circ q)(x) := \sum_{k=0}^{n} a_k b_k x^k.$$

If $p \in \boldsymbol{P}_n$ and $t \in \mathbb{R},\ t > 0$, the t-th *Hadamard power* of p is the polynomial $p^{\circ t}(x) := \sum_{k=0}^{n} a_k^t x^k$. In 1996, Garloff and Wagner proved in [7] that $p \circ q \in \boldsymbol{H}_n$, if $p, q \in \boldsymbol{H}_n$. In particular, $f \in \boldsymbol{H}_n$ implies $f^{\circ t} \in \boldsymbol{H}_n$ for

*Received by the editors on December 16, 2023. Accepted for publication on July 31, 2024. Handling Editor: Froilán Dopico. Corresponding Author: Jürgen Garloff.

†Department of Mathematics and Statistics, University of Konstanz, Germany (fatimah.alsaafin@uni-konstanz.de, doaa.al-saafin@uni-konstanz.de).

‡HTWG Konstanz – University of Applied Sciences, Faculty for Computer Science, Konstanz, Germany (juergen.garloff@htwg-konstanz.de).

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 40, pp. 574-584, August 2024.

all $t \in \{1, 2, 3, \dots\}$. In general, if $p \in \boldsymbol{H}_n$ then $p^{\circ t}$ needs not be a Hurwitz polynomial for every $t > 1$. Let $p \in \boldsymbol{H}_n$ and $t > 1$. Then, $p^{\circ t}$ is Hurwitz for $n \leq 5$, while $p^{\circ t}$ needs not be Hurwitz for $n \geq 6$, see [3].

2. Background and key lemmata. We collect here some key facts needed for our main results. The following lemmata are well-known.

LEMMA 2.1. *(Schur Product Theorem): Suppose $A, B \succeq (\succ)\ 0$ are of the same order. Then $A \circ B \succeq (\succ) 0$.*

LEMMA 2.2. *([15, Theorem 7.7]): Suppose $A, B \succeq 0$ have the same order (> 1). Then*

$$\det(A + B) \geq \det A + \det B$$

with equality if and only if $A + B$ is singular or $A = 0$ or $B = 0$.

For a polynomial p given by (1.1), the *Hurwitz matrix $H(p)$ associated with p* is given by

$$H(p) = \begin{bmatrix} a_{n-1} & a_{n-3} & a_{n-5} & \dots & 0 \\ a_n & a_{n-2} & a_{n-4} & \dots & 0 \\ 0 & a_{n-1} & a_{n-3} & \dots & 0 \\ 0 & a_n & a_{n-2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_0 \end{bmatrix}. \tag{2.1}$$

If we consider a Hurwitz matrix without reference to a certain polynomial, we suppress the reference to a polynomial and write only H. The leading principal minors of the matrix (2.1) are given by the following determinants, called *Hurwitz determinants*,

$$\Delta_1 := a_{n-1},\ \Delta_2 := \begin{vmatrix} a_{n-1} & a_{n-3} \\ a_n & a_{n-2} \end{vmatrix},\ \Delta_3 := \begin{vmatrix} a_{n-1} & a_{n-3} & a_{n-5} \\ a_n & a_{n-2} & a_{n-4} \\ 0 & a_{n-1} & a_{n-3} \end{vmatrix}, \dots, \Delta_n := \det(H(p)).$$

LEMMA 2.3. *(Routh-Hurwitz Criterion): The polynomial p in (1.1) is a Hurwitz polynomial if and only if all leading principal minors $\Delta_1, \Delta_2, \dots, \Delta_n$ of $H(p)$ are positive.*

For the polynomial p in (1.1), and the leading principal minors of the matrix (2.1), define the sequences Q_1, Q_2 as

$$Q_1 := (\Delta_1, \Delta_3, \Delta_5, \dots),$$
$$Q_2 := (\Delta_2, \Delta_4, \Delta_6, \dots).$$

LEMMA 2.4. *([11, Liénard-Chipart Criterion]): The polynomial p in (1.1) is a Hurwitz polynomial if and only if one of the sequences Q_1, Q_2 has all its members positive.*

LEMMA 2.5. *([2, Lemma 1.4]; [9, Lemma 1]) For $p \in \boldsymbol{P}_n$ given by (1.1), define the polynomial*

$$q(x) := a_{n-1}x^{n-1} + (a_{n-2} - \mu a_{n-3})\, x^{n-2} + a_{n-3}x^{n-3} + (a_{n-4} - \mu a_{n-5})\, x^{n-4} + \dots,$$

where $\mu = \frac{a_n}{a_{n-1}}$. Then, $p \in \boldsymbol{H}_n$ if and only if $q \in \boldsymbol{H}_{n-1}$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 40, pp. 574-584, August 2024.

THEOREM 2.6. *(Oppenheim's Inequality [14]): Suppose $A = [a_{ij}]$, $B = [b_{ij}]$ are positive semidefinite matrices of order n. Then,*

$$\det(A \circ B) \geq \det A \cdot \det B. \tag{2.2}$$

Over the years, various generalizations for Oppenheim's inequality (2.2) have been obtained in the literature, e.g., generalizations of Oppenheim's inequality for positive definite block matrices [12], H-matrices [10], and M-matrices [13].

An interesting question is for which totally nonnegative matrices (2.2) is valid. A real matrix is called *totally nonnegative* (abbreviated TN) if all its minors are nonnegative. Such matrices arise in a great variety in mathematics and its applications, see, e.g., [6]. If A and B are TN, then $A \circ B$ needs not be TN such that $\det(A \circ B)$ may be negative. So there is no hope that Oppenheim's inequality holds for arbitrary TN matrices. However, it turns out that (2.2) is valid for a very restricted class of TN matrices, see Subsection 3.2.

The Hurwitz matrix (2.1) associated with a Hurwitz polynomial p is known to be TN [1, 9]. This means that all minors of $H(p)$ are nonnegative. But much more can be said: Not only each leading principal minor of $H(p)$ is positive, cf. Lemma 2.3, but also the determinant of each submatrix which does not contain a zero entry on its main diagonal [8, 9].

In this paper, we study the problem whether Oppenheim's inequality holds for Hurwitz matrices. To exploit the validity of (2.2) for positive definite matrices, we associate in Subsection 3.1 with $H(p)$ a positive definite matrix of roughly the half order. In the proofs, we will make use of the positivity of the minors with nonvanishing diagonal entries without any further reference. In Subsection 3.2, we report on a different approach, viz. the use of the so-called Hadamard core. As an application, we give in Subsection 3.3 a result for the Hadamard square root of a Hurwitz polynomial of degree 5.

3. Main results.

3.1. Reduction to a positive definite matrix.

THEOREM 3.1. *Let $A = H(p)$ be the Hurwitz matrix associated with the Hurwitz polynomial p given by (1.1). Then if n is even, $\det A = \det C_e(A)$, where the $n/2 \times n/2$ symmetric matrix $C_e(A)$ is defined by*

(3.1)

$$C_e(A) := \begin{bmatrix} \begin{vmatrix} a_{n-1} & a_{n-3} \\ a_n & a_{n-2} \end{vmatrix} & \begin{vmatrix} a_{n-1} & a_{n-5} \\ a_n & a_{n-4} \end{vmatrix} & \begin{vmatrix} a_{n-1} & a_{n-7} \\ a_n & a_{n-6} \end{vmatrix} & \cdots \\ \begin{vmatrix} a_{n-1} & a_{n-5} \\ a_n & a_{n-4} \end{vmatrix} & \begin{vmatrix} a_{n-1} & a_{n-7} \\ a_n & a_{n-6} \end{vmatrix} + \begin{vmatrix} a_{n-3} & a_{n-5} \\ a_{n-2} & a_{n-4} \end{vmatrix} & \begin{vmatrix} a_{n-1} & a_{n-9} \\ a_n & a_{n-8} \end{vmatrix} + \begin{vmatrix} a_{n-3} & a_{n-7} \\ a_{n-2} & a_{n-6} \end{vmatrix} & \cdots \\ \begin{vmatrix} a_{n-1} & a_{n-7} \\ a_n & a_{n-6} \end{vmatrix} & \begin{vmatrix} a_{n-1} & a_{n-9} \\ a_n & a_{n-8} \end{vmatrix} + \begin{vmatrix} a_{n-3} & a_{n-7} \\ a_{n-2} & a_{n-6} \end{vmatrix} & \begin{vmatrix} a_{n-1} & a_{n-11} \\ a_n & a_{n-10} \end{vmatrix} + \begin{vmatrix} a_{n-3} & a_{n-9} \\ a_{n-2} & a_{n-8} \end{vmatrix} + \begin{vmatrix} a_{n-5} & a_{n-7} \\ a_{n-4} & a_{n-6} \end{vmatrix} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 40, pp. 574-584, August 2024.

and if n is odd, $\det A = \det C_o(A)$*, where the* $(n+1)/2 \times (n+1)/2$ *matrix* $C_o(A)$ *is defined by*

$$
C_o(A) := \begin{bmatrix}
a_{n-1} & a_{n-3} & a_{n-5} & \dots \\
a_n a_{n-3} & a_n a_{n-5} + \begin{vmatrix} a_{n-2} & a_{n-4} \\ a_{n-1} & a_{n-3} \end{vmatrix} & a_n a_{n-7} + \begin{vmatrix} a_{n-2} & a_{n-6} \\ a_{n-1} & a_{n-5} \end{vmatrix} & \dots \\
a_n a_{n-5} & a_n a_{n-7} + \begin{vmatrix} a_{n-2} & a_{n-6} \\ a_{n-1} & a_{n-5} \end{vmatrix} & a_n a_{n-9} + \begin{vmatrix} a_{n-2} & a_{n-8} \\ a_{n-1} & a_{n-7} \end{vmatrix} + \begin{vmatrix} a_{n-4} & a_{n-6} \\ a_{n-3} & a_{n-5} \end{vmatrix} & \dots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}, \tag{3.2}
$$

with the convention that $a_{n-i} = 0$*, for all* $i > n$*.*

Proof. Case 1: n is even (for an illustration of $n = 8$, see Example 3.2):
First, define the $n \times n$ matrix E_A and the $n/2 \times n/2$ matrices $E_A^{(1)}$, $E_A^{(2)}$, all having entries from A, as follows:

$$
E_A := \begin{bmatrix}
1 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 1 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & \dots & a_{n-1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & -a_n & a_{n-1} & \dots & a_3 \\
-a_n & a_{n-1} & -a_{n-2} & a_{n-3} & \dots & a_1
\end{bmatrix},
$$

$$
E_A^{(1)} := \begin{bmatrix}
a_{n-1} & a_{n-3} & a_{n-5} & \dots & a_1 \\
0 & a_{n-1} & a_{n-3} & \dots & a_3 \\
0 & 0 & a_{n-1} & \dots & a_5 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \dots & a_{n-1}
\end{bmatrix}, \quad
E_A^{(2)} := \begin{bmatrix}
0 & 0 & \dots & 0 & 0 \\
a_1 & 0 & \dots & 0 & 0 \\
a_3 & a_1 & \dots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{n-3} & a_{n-5} & \dots & a_1 & 0
\end{bmatrix}.
$$

Then, $\det E_A = a_{n-1}^{n/2}$ and $E_A A = \left(\begin{array}{c|c} E_A^{(1)} & E_A^{(2)} \\ \hline 0 & C_e(A) \end{array}\right)$.

Thus, $\det(E_A A) = a_{n-1}^{n/2} \det C_e(A)$, and so (3.1) is shown.
Case 2: n is odd:
First, define the $n \times n$ matrix O_A and the $(n-1)/2 \times (n-1)/2$, $(n-1)/2 \times (n+1)/2$ matrices $O_A^{(1)}$, $O_A^{(2)}$, respectively, as follows:

$$
O_A := \begin{bmatrix}
1 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 1 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \dots & 1 \\
0 & 0 & 0 & 0 & \dots & a_{n-2} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & a_n & -a_{n-1} & \dots & a_3 \\
a_n & -a_{n-1} & a_{n-2} & -a_{n-3} & \dots & a_1
\end{bmatrix},
$$

$$
O_A^{(1)} := \begin{bmatrix}
a_{n-1} & a_{n-3} & a_{n-5} & \dots & a_0 \\
0 & a_{n-1} & a_{n-3} & \dots & a_2 \\
0 & 0 & a_{n-1} & \dots & a_4 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \dots & a_{n-1}
\end{bmatrix}, \quad
O_A^{(2)} := \begin{bmatrix}
0 & 0 & \dots & 0 & 0 \\
a_0 & 0 & \dots & 0 & 0 \\
a_2 & a_0 & \dots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{n-3} & a_{n-5} & \dots & a_0 & 0
\end{bmatrix}.
$$

Thus, $\det(O_A A) = a_{n-1}^{(n-1)/2} \det C_o(A)$, and so (3.2) is shown. ∎

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 40, pp. 574-584, August 2024.

EXAMPLE 3.2. *Let $n = 8$. We get:*

$$E_A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -a_8 & a_7 \\ 0 & 0 & 0 & 0 & -a_8 & a_7 & -a_6 & a_5 \\ 0 & 0 & -a_8 & a_7 & -a_6 & a_5 & -a_4 & a_3 \\ -a_8 & a_7 & -a_6 & a_5 & -a_4 & a_3 & -a_2 & a_1 \end{bmatrix},$$

and so,

$$E_A A = \left(\begin{array}{c|c} E_A^{(1)} & E_A^{(2)} \\ \hline 0 & C_e(A) \end{array} \right),$$

where

$$E_A^{(1)} = \begin{bmatrix} a_7 & a_5 & a_3 & a_1 \\ 0 & a_7 & a_5 & a_3 \\ 0 & 0 & a_7 & a_5 \\ 0 & 0 & 0 & a_7 \end{bmatrix}, \; E_A^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 \\ a_1 & a_2 & 0 & 0 \\ a_1 & a_2 & a_3 & 0 \end{bmatrix},$$

$$C_e(A) = \begin{bmatrix} \begin{vmatrix} a_7 & a_5 \\ a_8 & a_6 \end{vmatrix} & \begin{vmatrix} a_7 & a_3 \\ a_8 & a_4 \end{vmatrix} & \begin{vmatrix} a_7 & a_1 \\ a_8 & a_2 \end{vmatrix} & a_7a_0 \\ \begin{vmatrix} a_7 & a_3 \\ a_8 & a_4 \end{vmatrix} & \begin{vmatrix} a_7 & a_1 \\ a_8 & a_2 \end{vmatrix} + \begin{vmatrix} a_5 & a_3 \\ a_6 & a_4 \end{vmatrix} & a_7a_0 + \begin{vmatrix} a_5 & a_1 \\ a_6 & a_2 \end{vmatrix} & a_5a_0 \\ \begin{vmatrix} a_7 & a_1 \\ a_8 & a_2 \end{vmatrix} & a_7a_0 + \begin{vmatrix} a_5 & a_1 \\ a_6 & a_2 \end{vmatrix} & a_5a_0 + \begin{vmatrix} a_3 & a_1 \\ a_4 & a_2 \end{vmatrix} & a_3a_0 \\ a_7a_0 & a_5a_0 & a_3a_0 & a_1a_0 \end{bmatrix}.$$

COROLLARY 3.3. *Let $p \in \boldsymbol{P}_n$ given by (1.1) and A be the Hurwitz matrix associated with p . Then if $n = 3$,*

$$\det A = \begin{vmatrix} a_2 & a_0 \\ a_0a_3 & a_1a_0 \end{vmatrix},$$

if $n = 4$,

$$\det A = \begin{vmatrix} \begin{vmatrix} a_3 & a_1 \\ a_4 & a_2 \end{vmatrix} & a_3a_0 \\ a_3a_0 & a_1a_0 \end{vmatrix},$$

if $n = 5$,

$$\det A = \begin{vmatrix} a_4 & a_2 & a_0 \\ a_5a_2 & a_5a_0 + \begin{vmatrix} a_3 & a_1 \\ a_4 & a_2 \end{vmatrix} & a_3a_0 \\ a_5a_0 & a_3a_0 & a_1a_0 \end{vmatrix},$$

if $n = 6$,

$$\det A = \begin{vmatrix} \begin{vmatrix} a_5 & a_3 \\ a_6 & a_4 \end{vmatrix} & \begin{vmatrix} a_5 & a_1 \\ a_6 & a_2 \end{vmatrix} & a_5a_0 \\ \begin{vmatrix} a_5 & a_1 \\ a_6 & a_2 \end{vmatrix} & a_5a_0 + \begin{vmatrix} a_3 & a_1 \\ a_4 & a_2 \end{vmatrix} & a_3a_0 \\ a_5a_0 & a_3a_0 & a_1a_0 \end{vmatrix}.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 40, pp. 574-584, August 2024.

We denote by $A[\kappa|\mu]$ the submatrix of A lying in the rows indexed by the sequence κ and columns indexed by the sequence μ. When $\kappa = \mu$, the principal submatrix $A[\kappa|\kappa]$ is abbreviated to $A[\kappa]$. The set theoretic symbol $\cup$ denotes the union of sequences, where we always assume that the resulting sequences are ordered increasingly.

THEOREM 3.4. *Let $p \in \boldsymbol{P}_n$ given by (1.1) and A be the Hurwitz matrix associated with p. Then, the following relations hold for the Hurwitz determinants.*

(i) If n is even, then for $i = 1, \dots, \frac{n}{2}$,

$$\Delta_{2i} = \det A[1, \dots, 2i] = \det C_e(A)[1, \dots, i].$$

(ii) If n is odd, then for $i = 1, \dots, \frac{n+1}{2}$,

$$\Delta_{2i-1} = \det A[1, \dots, 2i-1] = \det C_o(A)[1, \dots, i].$$

Proof. Let $\Delta_1, \Delta_2, \dots, \Delta_n$ be the Hurwitz determinants of A and n be even. Define the following three index sets

$$\begin{aligned}
\alpha &= \left(\alpha_1, \alpha_2, \dots, \alpha_{\frac{n}{2}}\right) := \left(\frac{n}{2} + 1, \frac{n}{2} + 2, \dots, n\right), \\
\beta &= \left(\beta_1, \beta_2, \dots, \beta_{\frac{n}{2}}\right) := \left(\frac{n}{2}, \frac{n}{2} - 1, \dots, 1\right), \\
\zeta &= (\zeta_1, \zeta_2, \dots, \zeta_n) := (n, n-1, \dots, 1).
\end{aligned}$$

For $i = 1, \dots, \frac{n}{2}$, we have

$$\det\left(E_A\left[(\alpha_1, \alpha_2, \dots, \alpha_i) \cup (\beta_1, \beta_2, \dots, \beta_i) \mid \zeta_1, \zeta_2, \dots, \zeta_{2i}\right]\right) = a_{n-1}^i,$$

and

$$\det\left(E_A\left[(\alpha_1, \alpha_2, \dots, \alpha_i) \cup (\beta_1, \beta_2, \dots, \beta_i) \mid \zeta_1, \zeta_2, \dots, \zeta_{2i}\right] A[1, \dots, 2i]\right)$$

$$= \det\left(\begin{array}{c|c} E_A^{(1)}[\beta_1, \dots, \beta_i] & \star \\ \hline 0 & C_e(A)[1, \dots, i] \end{array}\right)$$

$$= \det E_A^{(1)}[\beta_1, \dots, \beta_i] \det C_e(A)[1, \dots, i] = a_{n-1}^i \det C_e(A)[1, \dots, i].$$

Thus,

$$\Delta_{2i} = \det A[1, \dots, 2i] = \det C_e(A)[1, \dots, i].$$

The proof of the odd case is similar. ∎

EXAMPLE 3.5. *Let $n = 8$. Then, the fourth Hurwitz determinant is*

$$\Delta_4 = \det A[1, \dots, 4] = \det \begin{bmatrix} a_7 & a_5 & a_3 & a_1 \\ a_8 & a_6 & a_4 & a_2 \\ 0 & a_7 & a_5 & a_3 \\ 0 & a_8 & a_6 & a_4 \end{bmatrix}.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 40, pp. 574-584, August 2024.

We have

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -a_8 & a_7 \\ -a_8 & a_7 & -a_6 & a_5 \end{bmatrix} \begin{bmatrix} a_7 & a_5 & a_3 & a_1 \\ a_8 & a_6 & a_4 & a_2 \\ 0 & a_7 & a_5 & a_3 \\ 0 & a_8 & a_6 & a_4 \end{bmatrix} = \begin{bmatrix} a_7 & a_5 & a_3 & a_1 \\ 0 & a_7 & a_5 & a_3 \\ 0 & 0 & a_7a_6 - a_8a_5 & a_7a_4 - a_8a_3 \\ 0 & 0 & a_7a_4 - a_8a_3 & a_7a_2 - a_8a_1 + a_5a_4 - a_6a_3 \end{bmatrix}.$$

Thus,

$$\det\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -a_8 & a_7 \\ -a_8 & a_7 & -a_6 & a_5 \end{bmatrix} \begin{bmatrix} a_7 & a_5 & a_3 & a_1 \\ a_8 & a_6 & a_4 & a_2 \\ 0 & a_7 & a_5 & a_3 \\ 0 & a_8 & a_6 & a_4 \end{bmatrix} \right) = a_7^2 \, \det C_e(A)[1,2],$$

and therefore,

$$\Delta_4 = \det\, C_e(A)[1,2].$$

By application of Lemma 2.4, we obtain a new necessary and sufficient condition for a polynomial to be Hurwitz.

Theorem 3.6. *Let $p \in \boldsymbol{P}_n$ given by (1.1), with $a_n = 1$ if n is odd*[1]*, and A be the Hurwitz matrix associated with p. Then, $p \in \boldsymbol{H}_n$ if and only if $C_e(A) \succ 0$ ($C_o(A) \succ 0$).*

Remark 3.7. *If one checks a symmetric matrix for positive definiteness by the positivity of its leading principal minors, then Theorem 3.6 requires about the same number of minors to be checked as Lemma 2.4. However, the order of minors in Theorem 3.6 is roughly half the order of the respective minors in the Liénard-Chipart Criterion.*

The next theorem presents inequalities of type (2.2) for Hurwitz matrices of order $n \leq 6$. Taking into account that for positive definite matrices, the equality case in inequality (2.2) can occur only in very restricted cases, see, e.g., [16], it is not surprising that the following inequalities are strict.

Theorem 3.8. *Let $f(x) = \sum_{k=0}^{n} a_k x^k$, $g(x) = \sum_{k=0}^{n} b_k x^k$ be Hurwitz polynomials and let A and B be the Hurwitz matrices associated with f and g, respectively. Then the following statements hold.*

(i) If $n = 3, 4, 5$, then $\det(A \circ B) > \det A \cdot \det B$.
(ii) If $n = 6$, then $\det(A \circ B) + (a_3a_2b_5b_0 + a_5a_0b_3b_2)\, \det(C_e(A \circ B)[1,3]) > \det A \cdot \det B$.

Proof. In each part, we use the determinant forms in Corollary 3.3.

(i) If $n = 3$, then the result is trivial.
If $n = 4$, then

[1] In this case, the matrix $C_o(A)$ in (3.2) is symmetric.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 40, pp. 574-584, August 2024.

$$\det(A\circ B) = \begin{vmatrix} \begin{vmatrix} a_3b_3 & a_1b_1 \\ a_4b_4 & a_2b_2 \end{vmatrix} & a_3a_0b_3b_0 \\ a_3a_0b_3b_0 & a_1a_0b_1b_0 \end{vmatrix}$$

$$= \begin{vmatrix} \begin{vmatrix} a_3 & a_1 \\ a_4 & a_2 \end{vmatrix} \begin{vmatrix} b_3 & b_1 \\ b_4 & b_2 \end{vmatrix} + a_4a_1 \begin{vmatrix} b_3 & b_1 \\ b_4 & b_2 \end{vmatrix} + b_4b_1 \begin{vmatrix} a_3 & a_1 \\ a_4 & a_2 \end{vmatrix} & a_3a_0b_3b_0 \\ a_3a_0b_3b_0 & a_1a_0b_1b_0 \end{vmatrix}.$$

We make use of Theorem 3.6 and Lemma 2.1 to conclude

$$\det(A\circ B) = \det\left(\begin{pmatrix} \begin{vmatrix} a_3 & a_1 \\ a_4 & a_2 \end{vmatrix} & a_3a_0 \\ a_3a_0 & a_1a_0 \end{pmatrix} \circ \begin{pmatrix} \begin{vmatrix} b_3 & b_1 \\ b_4 & b_2 \end{vmatrix} & b_3b_0 \\ b_3b_0 & b_1b_0 \end{pmatrix} + \operatorname{diag}\left(a_1a_4 \begin{vmatrix} b_3 & b_1 \\ b_4 & b_2 \end{vmatrix} + b_1b_4 \begin{vmatrix} a_3 & a_1 \\ a_4 & a_2 \end{vmatrix}, 0\right)\right)$$

$$> \det C_e(A)\cdot \det C_e(B) \text{ (by Lemma 2.2 with the exclusion of the equality case and Theorem 2.6)}$$

$$= \det A \cdot \det B.$$

Let $n = 5$. Without loss of generality, we may assume that $a_5 = b_5 = 1$. Then, the matrices A and B are symmetric and

$$\det(A\circ B) = \begin{vmatrix} a_4b_4 & a_2b_2 & a_0b_0 \\ a_2b_2 & a_0b_0 + \begin{vmatrix} a_3b_3 & a_1b_1 \\ a_4b_4 & a_2b_2 \end{vmatrix} & a_3a_0b_3b_0 \\ a_0b_0 & a_3a_0b_3b_0 & a_1a_0b_1b_0 \end{vmatrix}.$$

Since

$$a_0b_0 + \begin{vmatrix} a_3b_3 & a_1b_1 \\ a_4b_4 & a_2b_2 \end{vmatrix} = \left(a_0 + \begin{vmatrix} a_3 & a_1 \\ a_4 & a_2 \end{vmatrix}\right)\left(b_0 + \begin{vmatrix} b_3 & b_1 \\ b_4 & b_2 \end{vmatrix}\right) + \begin{vmatrix} a_1 & a_0 \\ 1 & a_4 \end{vmatrix}\begin{vmatrix} b_3 & b_1 \\ b_4 & b_2 \end{vmatrix} + \begin{vmatrix} b_1 & b_0 \\ 1 & b_4 \end{vmatrix}\begin{vmatrix} a_3 & a_1 \\ a_4 & a_2 \end{vmatrix},$$

we obtain similarly as for $n = 4$

$$\det(A\circ B) \geq \det C_o(A)\cdot \det C_o(B) = \det A\cdot \det B.$$

(ii) If $n = 6$, then

$$C_e(A\circ B) = \begin{bmatrix} \begin{vmatrix} a_5 & a_6 \\ a_3 & a_4 \end{vmatrix} & \begin{vmatrix} a_5 & a_1 \\ a_6 & a_2 \end{vmatrix} & a_5a_0 \\ \begin{vmatrix} a_5 & a_1 \\ a_6 & a_2 \end{vmatrix} & a_5a_0 + \begin{vmatrix} a_3 & a_1 \\ a_4 & a_2 \end{vmatrix} & a_3a_0 \\ a_5a_0 & a_3a_0 & a_1a_0 \end{bmatrix} \circ \begin{bmatrix} \begin{vmatrix} b_5 & b_6 \\ b_3 & b_4 \end{vmatrix} & \begin{vmatrix} b_5 & b_1 \\ b_6 & b_2 \end{vmatrix} & b_5b_0 \\ \begin{vmatrix} b_5 & b_1 \\ b_6 & b_2 \end{vmatrix} & b_5b_0 + \begin{vmatrix} b_3 & b_1 \\ b_4 & b_2 \end{vmatrix} & b_3b_0 \\ b_5b_0 & b_3b_0 & b_1b_0 \end{bmatrix}$$

$$+ \begin{bmatrix} b_6b_3\begin{vmatrix} a_5 & a_3 \\ a_6 & a_4 \end{vmatrix} & b_6b_1\begin{vmatrix} a_5 & a_1 \\ a_6 & a_2 \end{vmatrix} & 0 \\ b_6b_1\begin{vmatrix} a_5 & a_1 \\ a_6 & a_2 \end{vmatrix} & b_4b_1\left(a_5a_0 + \begin{vmatrix} a_3 & a_1 \\ a_4 & a_2 \end{vmatrix}\right) & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} a_6a_3\begin{vmatrix} b_5 & b_3 \\ b_6 & b_4 \end{vmatrix} & a_6a_1\begin{vmatrix} b_5 & b_1 \\ b_6 & b_2 \end{vmatrix} & 0 \\ a_6a_1\begin{vmatrix} b_5 & b_1 \\ b_6 & b_2 \end{vmatrix} & a_4a_1\left(b_5b_0 + \begin{vmatrix} b_3 & b_1 \\ b_4 & b_2 \end{vmatrix}\right) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$- \operatorname{diag}(0, c, 0), \text{ where } c := a_3a_2b_5b_0 + a_5a_0b_3b_2.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 40, pp. 574-584, August 2024.

The Hadamard product on the right-hand side is positive definite because it is the Hadamard product of the two positive definite matrices $C_e(A)$ and $C_e(B)$. Also, the matrix

$$X := \begin{bmatrix} b_3b_6\begin{vmatrix} a_5 & a_3 \\ a_6 & a_4 \end{vmatrix} & b_6b_1\begin{vmatrix} a_5 & a_1 \\ a_6 & a_2 \end{vmatrix} & 0 \\ b_6b_1\begin{vmatrix} a_5 & a_1 \\ a_6 & a_2 \end{vmatrix} & b_4b_1\left(a_5a_0 + \begin{vmatrix} a_3 & a_1 \\ a_4 & a_2 \end{vmatrix}\right) & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} b_6b_3 & b_6b_1 & 0 \\ b_6b_1 & b_4b_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \circ \begin{bmatrix} \begin{vmatrix} a_5 & a_3 \\ a_6 & a_4 \end{vmatrix} & \begin{vmatrix} a_5 & a_1 \\ a_6 & a_2 \end{vmatrix} & 0 \\ \begin{vmatrix} a_5 & a_1 \\ a_6 & a_2 \end{vmatrix} & a_5a_0 + \begin{vmatrix} a_3 & a_1 \\ a_4 & a_2 \end{vmatrix} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

is the Hadamard product of two positive semidefinite matrices by

$$\det \begin{bmatrix} b_6b_3 & b_6b_1 \\ b_6b_1 & b_4b_1 \end{bmatrix} = b_6b_1(b_4b_3 - b_6b_1) > 0,$$

and so it is positive semidefinite with a similar conclusion for the matrix

$$Y := \begin{bmatrix} a_6a_3\begin{vmatrix} b_5 & b_3 \\ b_6 & b_4 \end{vmatrix} & a_6a_1\begin{vmatrix} b_5 & b_1 \\ b_6 & b_2 \end{vmatrix} & 0 \\ a_6a_1\begin{vmatrix} b_5 & b_1 \\ b_6 & b_2 \end{vmatrix} & a_4a_1\left(b_5b_0 + \begin{vmatrix} b_3 & b_1 \\ b_4 & b_2 \end{vmatrix}\right) & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

After rearranging terms, we obtain

$$\begin{aligned} \det(C_e(A \circ B) + \operatorname{diag}(0, c, 0)) &= \det(C_e(A \circ B)) + c \det(C_e(A \circ B)[1, 3]) \\ &= \det(A \circ B) + c \det C_e(A \circ B)[1, 3]. \end{aligned}$$

On the other side, application of Lemma 2.2 with exclusion of the equality case and Theorem 2.6 yields

$$\begin{aligned} \det(C_e(A \circ B) + \operatorname{diag}(0, c, 0)) &\geq \det(C_e(A) \circ C_e(B) + X + Y) \\ &> \det(C_e(A) \circ C_e(B)) \\ &\geq \det A \cdot \det B, \end{aligned}$$

from which the statement follows. □

3.2. Use of the Hadamard core. Another approach is to use the concept of the Hadamard core of TN matrices [4], see also Section 8.2 in [6]. The *Hadamard core (for TN matrices) of order n* is defined as

$$\{A \in \mathbb{R}^{n,n} \mid B \in \mathbb{R}^{n,n} \text{ is } TN \ \rightarrow A \circ B \text{ is } TN\}.$$

By choosing B as the matrix which contains only 1's as entries, we see that all members of the Hadamard core are TN. By [4, Corollary 5.2], [6, Corollary 8.3.2], if A is in the Hadamard core, then (2.2) is fulfilled for any TN matrix B. Since a tridiagonal TN matrix of any order is in the Hadamard core [4, Theorem 2.6], [6, Theorem 8.2.5], inequality (2.2) is valid for Hurwitz matrices of order 3 (which are tridiagonal). Obviously, for any polynomial $p \in \boldsymbol{P}_n$ with coefficients a_i, it holds that $\det H(p) = a_0 \det H(p)[1, \ldots, n-1]$. Therefore, Oppenheim's inequality holds for Hurwitz matrices of order n, if it is valid for their leading principal submatrices of order $n-1$. Since for a Hurwitz matrix H of order 4, the submatrix $H[1, 2, 3]$ is tridiagonal, it follows that (2.2) is valid for all Hurwitz matrices of order 4. Conditions for a matrix of order 4 to be in the Hadamard core are presented in [5]. Unfortunately, the results therein are not applicable,

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 40, pp. 574-584, August 2024.

because they would require that for a Hurwitz matrix H of order 5, the submatrix $H[1,2,3,4]$ contains a zero entry within its tridiagonal part which is formed by the main diagonal, the superdiagonal, and the subdiagonal. However, we have not yet found an example in which $H[1,2,3,4]$ is not in the Hadamard core.

3.3. Application to the Hadamard square root of a Hurwitz polynomial. As an application, we consider now the Hadamard square root of a Hurwitz polynomial of degree 5. In general, if a polynomial f of degree 5 is Hurwitz, then $f^{\circ\frac{1}{2}}$ does not need to be Hurwitz.

Example 3.9. *The polynomial*

$$f(x) = 0.1x^5 + 1.5x^4 + x^3 + 3x^2 + x + 1,$$

is Hurwitz, but $f^{\circ\frac{1}{2}}(x) \notin \boldsymbol{H}_5$.

We give a necessary and sufficient condition for the Hadamard square root of a Hurwitz polynomial of degree 5 to be Hurwitz.

Theorem 3.10. *Let* $f(x) = \sum_{k=0}^{5} a_k x^k \in \boldsymbol{H}_5$, $\omega := \frac{\sqrt{a_4a_3}-\sqrt{a_5a_2}}{\sqrt{a_4a_1}-\sqrt{a_5a_0}}$. *Then* $f^{\circ\frac{1}{2}}(x) = \sum_{k=0}^{5} \sqrt{a_k}x^k$ *is Hurwitz if and only if*

$$\omega^2 > \frac{a_4}{a_2}\ , \ \textit{and} \quad \sqrt{a_0}\,\omega^2 - \sqrt{a_2}\,\omega + \sqrt{a_4} < 0.$$

Proof. By Lemma 2.5, it is enough to show that k is Hurwitz, where

$$k(x) := a_4x^4 + (\sqrt{a_4a_3} - \sqrt{a_5a_2})\,x^3 + \sqrt{a_4a_2}x^2 + (\sqrt{a_4a_1} - \sqrt{a_5a_0})\,x + \sqrt{a_4a_0}.$$

Let K be the Hurwitz matrix associated with the polynomial k. Theorem 3.6 implies that k is Hurwitz if and only if

$$C_e(K) = \sqrt{a_0a_4}\begin{bmatrix} \begin{vmatrix} \sqrt{a_4a_3} - \sqrt{a_5a_2} & \sqrt{a_4a_1} - \sqrt{a_5a_0} \\ a_4 & \sqrt{a_4a_2} \end{vmatrix} & \sqrt{a_0a_4}\left(\sqrt{a_4a_3} - \sqrt{a_5a_2}\right) \\ \sqrt{a_4a_3} - \sqrt{a_5a_2} & \sqrt{a_4a_1} - \sqrt{a_5a_0} \end{bmatrix} \succ 0.$$

The two conditions imply that the two leading principal minors in $C_e(K)$ are positive. □

Conclusions. In this paper, we have presented a new necessary and sufficient condition for a polynomial to be Hurwitz. Based on this, we have shown that the Hurwitz matrices associated with a Hurwitz polynomial up to degree six satisfy an inequality of Oppenheim's type. We have tested a huge number of polynomials of degree six but did not find one for which the inequality in Theorem 3.8 (ii) is not valid without the extra term on the left-hand side. Also, we did not find a Hurwitz polynomial of degree greater than six which does not satisfy the inequality (2.2), even more, in all cases we have tried the left-hand side was much greater than the right-hand side.

One may ask whether Theorem 3.8 holds for *quasi-Hurwitz* polynomials, i.e., polynomials having all their roots inside the closed left half of the complex plane. By [1], the Hurwitz matrix associated with a quasi-Hurwitz polynomial is *TN*, and by [7], the Hadamard product of two quasi-Hurwitz polynomials is again quasi-Hurwitz. By the continuous dependency of the coefficients of a polynomial from its roots, Theorem 3.8 remains in force for quasi-Hurwitz polynomials, however, with the non-strict inequality. Equality is possible

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 40, pp. 574-584, August 2024.

as it can be seen from the following scenario: If one of both polynomials (of arbitrary degree) has only purely imaginary roots, it is a polynomial in $x^{2,}$ and therefore, its associated Hurwitz matrix has a null row as its first row. As a consequence, both sides of (2.2) are zero.

Acknowledgments. The third author thanks Stanisław Białas for having brought the problem of this paper to his attention. The authors thank one of the reviewers for the comments which have improved the presentation of their results and Florentien Bach for performing the numerical experiments.

REFERENCES

[1] B.A. Asner, Jr. On the total nonnegativity of the Hurwitz matrix. *SIAM J. Appl. Math.*, 18(2):407–414, 1970.
[2] S.P. Bhattacharyya, H. Chapellat, and L.H. Keel. *Robust Control: The Parametric Approach.* Prentice Hall, Upper Saddle River, NJ, 1995.
[3] S. Białas, L. Białas-Cież, and M. Kudra. On the Hurwitz stability of noninteger Hadamard powers of stable polynomials. *Linear Algebra Appl.*, 683:111–124, 2014.
[4] A.S. Crans, S.M. Fallat, and C.R. Johnson. The Hadamard core of the totally nonnegative matrices. *Linear Algebra Appl.*, 328(1–3):203–222, 2001.
[5] K.B. Dwelle. *Some Results on Hadamard Closure and Variation Diminishing Properties of Totally Nonnegative Matrices*. PhD thesis, Purdue University, 2007.
[6] S.M. Fallat and C.R. Johnson. *Totally Nonnegative Matrices.* Princeton Series in Applied Mathematics. Princeton University Press, Princeton and Oxford, 2011.
[7] J. Garloff and D.G. Wagner. Hadamard products of stable polynomials are stable. *J. Math. Anal. Appl.*, 202(3):797–809, 1996.
[8] M. Gasca, C.A. Micchelli, and J.M. Peña. Almost strictly totally positive matrices. *Numer. Algorithms*, 2:225–236, 1992.
[9] J.H.B. Kemperman. A Hurwitz matrix is totally positive. *SIAM J. Math. Anal.*, 13(2):331–341, 1982.
[10] Y.-T. Li and J.-C. Li. On the estimations of bounds for determinant of Hadamard product of H-matrices. *J. Comput. Math.*, 19(4):365–370, 2001.
[11] A Liénard and M.H. Chipart. Sur le signe de la partie réelle des racines d'une équation algébraique. *J. Math. Pures Appl.*, 10(6):291–346, 1914.
[12] M. Lin. An Oppenheim type inequality for a block Hadamard product. *Linear Algebra Appl.*, 452:1–6, 2014.
[13] J. Liu and L. Zhu. Some improvement of Oppenheim's inequality for M-matrices. *SIAM J. Matrix Anal. Appl.*, 18(2):305–311, 1997.
[14] A. Oppenheim. Inequalities connected with definite Hermitian forms. *J. Lond. Math. Soc.*, s1-5(2):114–119, 1930.
[15] F. Zhang. *Matrix Theory: Basic Results and Techniques.* 2nd ed., Springer, New York, NY, 2011.
[16] X.-D. Zhang and C.-X. Ding. The equality cases for the inequalities of Oppenheim and Schur for positive semi-definite matrices. *Czechoslowak Math. J.*, 59(134):197–206, 2009.