Convex Combinations of Stable Polynomials

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ABSTRACT: *Sufficient conditions are given under which convex combinations of stable (complex and real) polynomials are stable.*

I. Introduction

In this paper we consider *stable* polynomials, i.e. polynomials having all their zeros in the open left half-plane.

Let f_0, f_1 be two real stable polynomials

$$f_0(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n, \qquad a_0 \neq 0, \tag{1}$$

$$f_1(x) = c_0x^n + c_1x^{n-1} + \cdots + c_{n-1}x + c_n, \qquad c_0 \neq 0, \tag{2}$$

and put for $\lambda \in [0, 1]$

$$f_\lambda(x) = (1-\lambda)f_0(x) + \lambda f_1(x). \tag{3}$$

Then it does not follow that f_λ is stable for all $\lambda \in [0, 1]$.

Example 1. Put

$$f_0(x) = x^3 + x^2 + 2x + 1,$$

$$f_1(x) = x^3 + 0.001x^2 + 0.001x + 10^{-8}.$$

By the Routh–Hurwitz Criterion [e.g. (1), p. 231] f_0 and f_1 are stable but $f_{2/3}$ is not stable.

In this paper we present some sufficient conditions for the stability of f_λ. We give the statement at first for complex polynomials and as corollaries for real polynomials. We note that analogous statements are true for convex combinations of stable polynomials $f_1, f_2, \ldots, f_m$, $m \geqslant 2$, i.e.

$$\sum_{i=1}^{m} \lambda_i f_i(x),$$

where

$$\sum_{i=1}^{m} \lambda_i = 1 \quad \text{and} \quad \lambda_i \geqslant 0,\ i = 1(1)m.$$

For the sake of simplicity only the statements and the proofs for the case of two stable polynomials are given.

II. Main Results

Theorem I. Let the complex polynomials f_0, f_1,

$$f_0(x) = \sum_{\nu=0}^{n} (a_\nu + ib_\nu)x^{n-\nu}, \qquad |a_0|+|b_0| > 0, \tag{1'}$$

$$f_1(x) = \sum_{\nu=0}^{n} (c_\nu + id_\nu)x^{n-\nu}, \qquad |c_0|+|d_0| > 0,$$

be stable. For $k = 0, 1$, let $f_k(i\omega)$ be represented by the two real polynomials $h_k(\omega)$ and $g_k(\omega)$,

$$f_k(i\omega) = h_k(\omega)+ig_k(\omega).$$

Then for each $\lambda \in [0, 1]$ the polynomial $f_\lambda = (1-\lambda)f_0 + \lambda f_1$ is stable if $h_0 = h_1$ or $g_0 = g_1$.

Proof. We give the proof of the statement here only for the case $h_0 = h_1 = h$. The proof for $g_0 = g_1$ follows by multiplying f_k by $-i$. For $\lambda \in [0, 1]$ we represent $f_\lambda(i\omega)$ by the two real polynomials $h_\lambda(\omega)$ and $g_\lambda(\omega)$

$$f_\lambda(i\omega) = h_\lambda(\omega)+ig_\lambda(\omega). \tag{4}$$

Then by writing out h, g_0 and g_1 explicitly, we see that $h_\lambda = h$ and

$$g_\lambda(\omega) = (1-\lambda)g_0(\omega)+\lambda g_1(\omega). \tag{5}$$

We now assume the contrary of the statement: there is a λ' such that $f_{\lambda'}$ is not stable, i.e. $f_{\lambda'}$ has a root with non-negative real part. There is at most one $\lambda'' \in [0, 1]$ such that the degree of $f_{\lambda''}$ is less than n. We may assume without loss of generality that $\lambda' \leqslant \lambda''$. The roots of f_λ, regarded as functions of its coefficients, vary continuously for $0 \leqslant \lambda \leqslant \lambda'$, whereas if $\lambda' = \lambda''$ any new roots emerge from a neighbourhood of the point at infinity (see (3)). Hence there is a point $\mu \in (0, \lambda')$ such that $f_\mu(x)$ has a purely imaginary root, say $i\tau$. By (4), (5), we have

$$0 = f_\mu(i\tau) = h(\tau)+i[(1-\mu)g_0(\tau)+\mu g_1(\tau)],$$

hence $h(\tau) = 0$ and $(1-\mu)g_0(\tau)+\mu g_1(\tau) = 0$. This implies that

$$g_0(\tau) = 0 \quad \text{and} \quad g_1(\tau) = 0 \tag{6}$$

or

$$\operatorname{sign}(g_0(\tau)) \operatorname{sign}(g_1(\tau)) < 0. \tag{7}$$

By the Hermite–Biehler Theorem (2) the roots of the real polynomials h and g_0 as well as those of h and g_1 are distinct, real, and interlace each other, and the inequalities

$$h(\omega)g_k'(\omega)-h'(\omega)g_k(\omega) > 0 \quad \text{for all real } \omega, k = 0, 1, \tag{8}$$

hold. Setting $\omega = \tau$, we obtain $h'(\tau)g_0(\tau) < 0$ and $h'(\tau)g_1(\tau) < 0$, a contradiction to (6) and (7). The proof is complete.

Corollary 1. Let the real polynomials f_0, f_1 given by (1), (2) be stable. Then for each $\lambda \in [0,1]$ f_λ given by (3) is stable iff†

$$a_{2\nu+1} = c_{2\nu+1}, \qquad \nu = 0(1)\left[\frac{n-1}{2}\right],$$

or

$$a_{2\nu} = c_{2\nu}, \qquad \nu = 0(1)\left[\frac{n}{2}\right].$$

Example 2. By the Routh–Hurwitz Criterion (e.g. (**1**), p. 231), a polynomial $x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$ is stable iff the following conditions hold

$$a_\nu > 0,\ \nu = 1,2,3,4; \quad a_1a_2 - a_3 > 0; \quad a_3(a_1a_2 - a_3) - a_4a_1^2 > 0.$$

The two polynomials f_0, f_1,

$$f_0(x) = x^4 + 3x^3 + 4x^2 + x + 1,$$

$$f_1(x) = x^4 + 4x^3 + 4x^2 + 14x + 1,$$

are stable. Thus, by Corollary 1, each polynomial f_λ, $\lambda \in [0,1]$, is stable.

From Corollary 1, we obtain:

Corollary 2. Let the real polynomials f_0, f_1 given by (1), (2) be represented in the form

$$f_k(x) = h_k(x^2) + xg_k(x^2), \qquad k = 0, 1.$$

Then the polynomial

$$(1-\lambda)h_0(x^2) + \lambda h_1(x^2) + x[(1-\delta)g_0(x^2) + \delta g_1(x^2)]$$

is stable for all λ, $\delta \in [0,1]$ iff the four polynomials $f_0(x)$, $f_1(x)$,

$$h_0(x) + xg_1(x) = a_n + c_{n-1}x + a_{n-2}x^2 + c_{n-3}x^3 + a_{n-4}x^4 + \dots,$$

$$h_1(x) + xg_0(x) = c_n + a_{n-1}x + c_{n-2}x^2 + a_{n-3}x^3 + c_{n-4}x^4 + \dots$$

are stable.

Proof. It suffices to prove sufficiency. By Corollary 1, the following two polynomials are stable for all $\delta \in [0,1]$

$$h_0(x^2) + x[(1-\delta)g_0(x^2) + \delta g_1(x^2)],$$

$$h_1(x^2) + x[(1-\delta)g_0(x^2) + \delta g_1(x^2)].$$

Applying again Corollary 1, the assertion follows.

Remark. Putting $\lambda = \delta$ we obtain a sufficient condition for the stability of f_λ, $\lambda \in [0,1]$.

Theorem II. Let the real polynomials f_0 given by (1) and f_1,

$$f_1(x) = c_1x^{n-1} + c_2x^{n-2} + \cdots + c_{n-1}x + c_n, \qquad c_1 \neq 0,$$

† For real s, $[s]$ denotes the greatest integer less than or equal to s.

be stable. Then for each $\lambda \in [0,1]$ f_λ given by (3) is stable if $a_{2\nu+1} = c_{2\nu+1}$, $\nu = 0(1)\left[\frac{n-1}{2}\right]$.

Proof. We give the proof here only for n odd. If n is even one may proceed analogously. Let $n = 2m+1$. We represent f_k in the form

$$f_k(x) = h_k(x^2) + xg_k(x^2), \qquad k = 0, 1.$$

Then $h_0(u)$, $h_1(u)$, $g_0(u)$ are polynomials of degree m and $g_1(u)$ is a polynomial of degree $m-1$. From the assumption it follows that $h_0(u) = h_1(u) = h(u)$. Denote the roots of h by $u_1, u_2, \ldots, u_m$, those of g_0 by $u'_1, u'_2, \ldots, u'_m$, and those of g_1 by $v'_1, v'_2, \ldots, v'_{m-1}$. By a special case of the Hermite–Biehler Theorem (cf. (**1**), p. 271) the stability of f_0, f_1 is equivalent to that the roots of h and g_0 as well as those of h and g_1 are distinct, real, negative, and interlace in the following manner:

$$u'_1 < u_1 < u'_2 < u_2 < \cdots < u'_m < u_m < 0,$$

$$u_1 < v'_1 < u_2 < \cdots < v'_{m-1} < u_m < 0.$$

W.l.o.g. we may assume that all coefficients of g_0, g_1, are positive (e.g. (**1**), p. 262). Therefore, if u is large enough sign $(g_0(-u)) = (-1)^m$ and sign $(g_1(-u)) = (-1)^{m-1}$. The polynomial g_0 changes its sign for the first time in the interval $(-\infty, u_1)$, hence sign $(g_0(u_1)) = (-1)^{m-1} =$ sign $(g_1(u_1))$. Similarly,

$$\text{sign}\,(g_k(u_\mu)) = (-1)^{m-\mu}, \quad \mu = 1(1)m, \quad k = 0, 1. \tag{9}$$

Now consider an arbitrary polynomial $f_\lambda(x) = h_\lambda(x^2) + xg_\lambda(x^2)$. Then $h_\lambda(u) = h(u)$ and $g_\lambda(u) = (1-\lambda)g_0(u) + \lambda g_1(u)$. We have sign $(g_\lambda(-u)) = (-1)^m$ for u large enough and sign $(g_\lambda(u_1)) = (-1)^{m-1}$, hence g_λ has a root in $(-\infty, u_1)$. By the Mean Value Theorem, it follows from (9) that g_λ has exactly one root in each interval $(u_\mu, u_{\mu+1})$, $\mu = 1(1)m-1$. Hence all roots of g_λ are distinct, negative and interlace with those of $h = h_\lambda$. By the cited special case of the Hermite–Biehler Theorem, f_λ is stable.

The condition of Theorem II cannot be replaced by the condition $a_{2\nu} = c_{2\nu}$ as the following example shows.

Example 3. Let f_0 be given as in Example 1 and $f_1(x) = x^2 + 2x + 7$. Both polynomials are stable but $f_{1/2}$ is not stable.

Conclusion

We have given sufficient conditions under which convex combinations of stable (complex and real) polynomials are stable.

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References

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