Computation of the Bernstein Coefficients on Subdivided Triangles

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Abstract. We present a procedure for computing the coefficients of the expansion of a bivariate polynomial into Bernstein polynomials over subtriangles. These triangles are generated by partitioning the standard simplex of $\mathbb{R}^2$. The coefficients are computed directly from the coefficients on the subdivided triangle from the preceding subdivision level. This allows a recursive computation of the coefficients and facilitates the economical computation of bounds for the range of a bivariate polynomial over a given triangle.

Keywords: Range enclosure, Bernstein polynomials, subdivision

1. Introduction

Finding a tight enclosure for the range of a function over a compact set $D \subseteq \mathbb{R}^l$ is a basic problem of interval computations, cf. [5],[6] and the references therein. If the function is a multivariate polynomial and D is a box, there exist several methods, for references cf. [4]. Compared to boxes, simplices allow more flexibility since far more general geometries can be treated. In [4] we present a method for computing bounds for the range of a bivariate polynomial

$$p(x,y) = \sum_{\mu,\nu=0}^{n} a_{\mu\nu} x^\mu y^\nu \quad \text{with} \quad a_{\mu\nu} \in \mathbb{C}$$

over a triangle. This method is based on the expansion of p into Bernstein polynomials. If p has only real coefficients the minimum and the maximum of the coefficients of this expansion, the so–called Bernstein coefficients, provide lower and upper bounds for the range. All rounding errors appearing in the computation of the Bernstein coefficients can be taken into account similarly as in [1] for the univariate case. In the case

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that p has complex coefficients the convex hull of the Bernstein coefficients encloses the range, for details see [4]. Without loss of generality we can assume that the given triangle is the *unit triangle*

$$T = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; x, y \geq 0 \;\wedge\; x + y \leq 1 \right\}$$

since any nonempty triangle can be mapped by an affine transformation onto T. The bounds are improved by subdivision.

In partitioning T we are led by the following useful fact [2]: When we subdivide a square by successively halving in both coordinate directions and calculate the Bernstein coefficients on a generated subsquare then we obtain as a byproduct of the computation the Bernstein coefficients on the neighbouring subsquares.

This is one of the reasons why we partition T into squares *and* triangles, cf. Figures 2 and 3 in [4]. In this technical note we use the definitions and notations from [4] and present a procedure for computing the Bernstein coefficients on the subtriangles T_{2t-1}^{m+1} and T_{2t}^{m+1} of the triangle T_t^m, $t = 1, \ldots, 2^m$, $m \geq 1$, which are subtriangles generated by subdivision; in the beginning these are the triangle T and the subtriangles with vertex sets $\{(0,\frac{1}{2}), (\frac{1}{2},\frac{1}{2}), (0,1)\}$ and $\{(\frac{1}{2},0), (1,0), (\frac{1}{2},\frac{1}{2})\}$. The Bernstein coefficients are computed directly from the Bernstein coefficients on the triangle T_t^m. This allows a recursive computation of the Bernstein coefficients on the subtriangles and facilitates the economical computation of the bounds for the range of the given polynomial over T.

In passing we note that the procedure presented in [4] for computing the Bernstein coefficients on the four subsquares generated by halving $[0,1] \times [0,1]$ in both coordinate directions can be generalized to the l–dimensional case. The resulting algorithm is rather technical. The procedure is therefore omitted here and the interested reader is referred to [3].

2. Main Result

In this section we give the procedure for computing the Bernstein coefficients on the subtriangles T_{2t-1}^{m+1} and T_{2t}^{m+1} of a triangle T_t^m. In the following Δ stands for T_{2t-1}^{m+1} or T_{2t}^{m+1}.

PROCEDURE Let $m \geq 1$ and $t \in \{1, \ldots, 2^m\}$.
Start by setting

$$\beta_{\mu\nu}^{(0)}(\Delta) \qquad = b_{\mu\nu}(T_t^m) \quad \textit{for all} \;\; (\mu,\nu) \in I_T^{(2n)}\,;$$

then for $\kappa = 0, \ldots, n-1$

$$\gamma^{(\kappa)}_{\mu\nu}(\Delta) = \left(\beta^{(\kappa)}_{\mu,\nu+1}(\Delta) + \beta^{(\kappa)}_{\mu+1,\nu}(\Delta)\right) \Big/ 2 \ \textit{for} \ (\mu,\nu) \in I_T^{(2(n-\kappa)-1)}, \quad (1)$$

$$\left.\begin{aligned} \beta^{(\kappa+1)}_{\mu\nu}\left(T^{m+1}_{2t-1}\right) &= \left(\gamma^{(\kappa)}_{\mu\nu}\left(T^{m+1}_{2t-1}\right) + \gamma^{(\kappa)}_{\mu,\nu+1}\left(T^{m+1}_{2t-1}\right)\right) \Big/ 2\,, \\ \beta^{(\kappa+1)}_{\mu\nu}\left(T^{m+1}_{2t}\right) &= \left(\gamma^{(\kappa)}_{\mu\nu}\left(T^{m+1}_{2t}\right) + \gamma^{(\kappa)}_{\mu+1,\nu}\left(T^{m+1}_{2t}\right)\right) \Big/ 2\,, \end{aligned}\right\} \quad \textbf{for}$$

$(\mu,\nu) \in I_T^{(2(n-\kappa-1))}$ **,**

$$\delta^{(\kappa,1)}_{1\mu}(\Delta) = \gamma^{(\kappa)}_{\mu,\,2(n-\kappa)-1-\mu}(\Delta) \ , \ \mu = 0, \ldots, 2(n-\kappa)-1\,, \quad (2)$$

$$\delta^{(\kappa,1)}_{2\nu}\left(T^{m+1}_{2t-1}\right) = \left(\beta^{(\kappa)}_{0\nu}\left(T^{m+1}_{2t-1}\right) + \beta^{(\kappa)}_{0,\nu+1}\left(T^{m+1}_{2t-1}\right)\right) \Big/ 2 \ , \ \nu = 0, \ldots, 2(n-\kappa)-1,$$

$$\delta^{(\kappa,1)}_{2\mu}\left(T^{m+1}_{2t}\right) = \left(\beta^{(\kappa)}_{\mu 0}\left(T^{m+1}_{2t}\right) + \beta^{(\kappa)}_{\mu+1,0}\left(T^{m+1}_{2t}\right)\right) \Big/ 2 \ , \ \mu = 0, \ldots, 2(n-\kappa)-1;$$

finally for $\sigma = 1, 2$ **,** $\kappa = 0, \ldots, n-2$ **and** $\tau = 1, \ldots, 2(n-\kappa)-1$

$$\delta^{(\kappa,\tau+1)}_{\sigma\nu}(\Delta) = \left(\delta^{(\kappa,\tau)}_{\sigma\nu}(\Delta) + \delta^{(\kappa,\tau)}_{\sigma,\nu+1}(\Delta)\right) \Big/ 2 \ , \ \nu = 0, \ldots, 2(n-\kappa)-\tau-1.$$

Theorem The Bernstein coefficients on the subtriangles at subdivision level $m+1$ **can be obtained by the Procedure from the following relations with** $(i,j) \in I_T^{(2n)}$**:**

$$b_{ij}\left(T^{m+1}_{2t-1}\right) = \begin{cases} \delta^{(2n-i-j,j-2(n-i))}_{10}\left(T^{m+1}_{2t-1}\right) & if \quad 2(n-i) < j\,, \\ \beta^{(i)}_{0j}\left(T^{m+1}_{2t-1}\right) & if \quad 2(n-i) = j\,, \\ \delta^{(i,2(n-i)-j)}_{2j}\left(T^{m+1}_{2t-1}\right) & if \quad 2(n-i) > j\,, \end{cases}$$

$$b_{ij}\left(T^{m+1}_{2t}\right) = \begin{cases} \delta^{(2n-i-j,i-2(n-j))}_{1i}\left(T^{m+1}_{2t}\right) & if \quad 2(n-j) < i\,, \\ \beta^{(j)}_{i0}\left(T^{m+1}_{2t}\right) & if \quad 2(n-j) = i\,, \\ \delta^{(j,2(n-j)-i)}_{2i}\left(T^{m+1}_{2t}\right) & if \quad 2(n-j) > i\,. \end{cases}$$

First one shows by induction on κ that

$$\beta^{(\kappa)}_{\mu,2(n-\kappa)-\mu-\nu}\left(T^{m+1}_{2t-1}\right) = 4^{-\kappa} \sum_{u,v=0}^{\kappa} \binom{\kappa}{u}\binom{\kappa}{v} b_{\mu+u,2n-(\mu+u)-(\nu+v)}\left(T^m_t\right) \quad (3)$$

$$\beta^{(\kappa)}_{2(n-\kappa)-\mu-\nu,\nu}\left(T^{m+1}_{2t}\right) = 4^{-\kappa} \sum_{u,v=0}^{\kappa} \binom{\kappa}{u}\binom{\kappa}{v} b_{2n-(\mu+u)-(\nu+v),\nu+v}\left(T^m_t\right)$$

holds true for all $(\mu,\nu)\in I_T^{(2(n-\kappa))}$ for $\kappa=0,\dots,n$.

Then one proves by induction on $\tau=1,\dots,2(n-\kappa)$ that the following identity is true for all $\sigma=1,2$ and $\kappa=0,\dots,n-2$

$$\delta_{\sigma\nu}^{(\kappa,\tau)}(\Delta) \;=\; 2^{-\tau+1}\sum_{h=0}^{\tau-1}\binom{\tau-1}{h}\delta_{\sigma,\nu+h}^{(\kappa,1)}(\Delta)\;,\;\nu=0,\dots,2(n-\kappa)-\tau\,. \tag{4}$$

We show the statements of the theorem only for the subtriangle T_{2t-1}^{m+1} and only in the case $2(n-i)<j$. The proof for the two remaining cases and for the subtriangle T_{2t}^{m+1} is analogous. To simplify notation, we mark the dependency from T_{2t-1}^{m+1} and T_t^m of the quantities considered in the sequel by writing $m+1$ and m for short.

From (2) and (1) it follows that for $\mu=0,\dots,2(n-\kappa)-1$

$$\begin{aligned}
\delta_{1\mu}^{(\kappa,1)}(m+1) &= \gamma_{\mu,2(n-\kappa)-1-\mu}^{(\kappa)}(m+1)\\
&= \Big(\beta_{\mu,2(n-\kappa)-\mu}^{(\kappa)}(m+1)+\beta_{\mu+1,2(n-\kappa)-1-\mu}^{(\kappa)}(m+1)\Big)\Big/2\\
&= 2^{-2\kappa-1}\Big(\sum_{u,v=0}^{\kappa}\binom{\kappa}{u}\binom{\kappa}{v}b_{\mu+u,2n-(\mu+u)-v}(m)\\
&\qquad+\sum_{u,v=0}^{\kappa}\binom{\kappa}{u}\binom{\kappa}{v}b_{\mu+u+1,2n-(\mu+u+1)-v}(m)\Big)\\
&= 2^{-2\kappa-1}\sum_{u=0}^{\kappa+1}\sum_{v=0}^{\kappa}\binom{\kappa+1}{u}\binom{\kappa}{v}b_{\mu+u,2n-(\mu+u)-v}(m)\,,
\end{aligned}$$

where the last but one identity follows from (3).

Now we apply identity (4) to conclude from this result that

$$\delta_{10}^{(2n-i-j,j-2(n-i))}(m+1)$$

$$\begin{aligned}
&= 2^{-(j-2(n-i))+1}\sum_{h=0}^{j-2(n-i)-1}\binom{j-2(n-i)-1}{h}\delta_{1h}^{(2n-i-j,1)}(m+1)\\
&= 2^{j-2n}\sum_{h=0}^{j-2(n-i)-1}\binom{j-2(n-i)-1}{h}\times\\
&\quad\times\sum_{u=0}^{2n-i-j+1}\sum_{v=0}^{2n-i-j}\binom{2n-i-j+1}{u}\binom{2n-i-j}{v}b_{h+u,2n-(h+u)-v}(m)\\
&= 2^{-i-(2n-i-j)}\sum_{v=0}^{2n-i-j}\binom{2n-i-j}{v}\sum_{u=0}^{i}\sum_{h=0}^{\min\{u,j-2(n-i)-1\}}\binom{j-2(n-i)-1}{h}\times
\end{aligned}$$

$$\times \binom{2n-i-j+1}{u-h} b_{u,2n-u-v}(m)$$
$$= 2^{-i-(2n-i-j)} \sum_{u=0}^{i} \sum_{v=0}^{2n-i-j} \binom{i}{u} \binom{2n-i-j}{v} b_{u,2n-u-v}(m) \, .$$

To obtain the last but one identity we used the Vandermonde convolution formula, e.g., [7].

Now by Lemma 4.2 in [4] we can conclude

$$\delta_{10}^{(2n-i-j,j-2(n-i))}(m+1) = b_{ij}(m+1) \, .$$

To compute the Bernstein coefficients on the subtriangles T_{2t-1}^{m+1} and T_{2t}^{m+1} by the Procedure requires $\frac{16}{3}n^3 + 4n^2 + \frac{8}{3}n$ additions and the same amount of multiplications (binary shifts). Here we made use of the relations

$$\gamma_{\mu\nu}^{(0)}\left(T_{2t-1}^{m+1}\right) \;=\; \gamma_{\mu\nu}^{(0)}\left(T_{2t}^{m+1}\right) \qquad \textit{for} \;\; (\mu,\nu) \in I_T^{(2n-1)}$$

and

$$\delta_{1\mu}^{(0,\tau)}\left(T_{2t-1}^{m+1}\right) \;=\; \delta_{1\mu}^{(0,\tau)}\left(T_{2t}^{m+1}\right) \quad \textit{for} \;\; \tau = 1,\ldots,2n \;\; \textit{and} \;\; \mu = 0,\ldots,2n-\tau \, .$$

Then the cumulated operations count for the procedure for computing the Bernstein coefficients on all subsquares and subtriangles filling the unit triangle T at subdivision level i for all $i = 1,\ldots,m$ sums up to $\mathcal{O}(4^m n^3)$ additions and the same amount of multiplications (binary shifts).

3. Numerical Example

It was shown in [4] that the convex hull of the Bernstein coefficients on all subsquares and subtriangles generated at subdivision level m, denoted by C_m, provides an enclosure for the range of p over T. Obviously, a tighter enclosure for this range is given by taking the union of the convex hulls of the Bernstein coefficients on each subregion at level m, i.e.,

$$C'_m \;=\; \bigcup_{\substack{r=1,\ldots,2^m-1 \\ s=1,\ldots,2^m-r}} C_{S_{rs}^m} \;\cup\; \bigcup_{t=1,\ldots,2^m} C_{T_t^m} \, ,$$

where

$$C_{S_{rs}^m} \;:=\; \mathrm{conv}\left\{ b_{ij}(S_{rs}^m) \;\middle|\; (i,j) \in I_S^{(n)} \right\} \, , \; r = 1,\ldots,2^m-1 \, ,$$

$$s = 1, \ldots, 2^m - r$$

and

$$C_{T_t^m} := \operatorname{conv}\left\{ b_{ij}(T_t^m) \,\middle|\, (i,j) \in I_T^{(2n)} \right\}, \; t = 1, \ldots, 2^m .$$

We consider now the polynomial

$$p(x,y) \;=\; 3.1 + 1.2i + (-1.8 + 4i)y + (2.5 + i)x + (17 + 8i)xy .$$

Figures 1 and 2 show C_m' for $m = 2, 4$. Already three extreme points of C_2' are sharp, viz.

$$\begin{aligned}
b_{00}\left(S_{11}^2\right) &= a_{00} = p(0,0) = 3.1 + 1.2i\,, \\
b_{02}\left(T_1^2\right) &= a_{00} + a_{01} = p(0,1) = 1.3 + 5.2i\,, \\
b_{20}\left(T_4^2\right) &= a_{00} + a_{10} = p(1,0) = 5.6 + 2.2i\,.
\end{aligned}$$

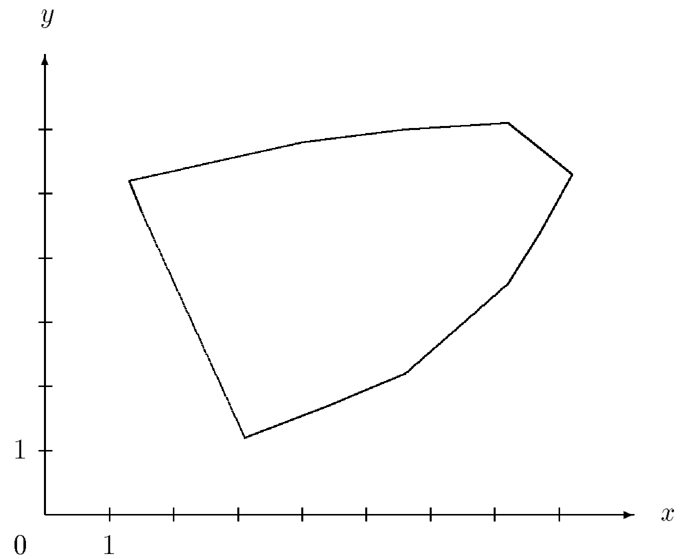

Figure 1. Enclosure C_2' for the range of p over T

The edges connecting $3.1{+}1.2i$ with the points $1.3{+}5.2i$ and $5.6{+}2.2i$ are part of the range of p and are therefore sharp. Yet the rest of the boundary of C_2' provides a crude approximation of the image of the straight line connecting $(0,1)$ with $(1,0)$.

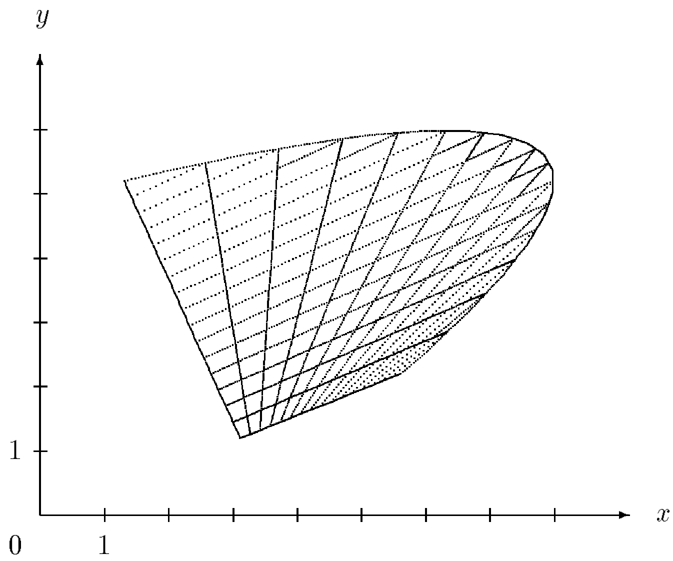

Figure 2. Enclosure C_4' for the range of p over T

The set C_4' gives a much better approximation of the range of p. For illustration, the convex hulls of the Bernstein coefficients which constitute C_4' are displayed. The convex hulls of the Bernstein coefficients on the subsquares are already identical to the respective ranges of p.

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