Parametrizations for Triangular G^k Spline Surfaces of Low Degree

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In this article, we present regularly parametrized G^k free-form spline surfaces that extend box and half-box splines over regular triangular grids. The polynomial degree of these splines is $\max\{4k + 1, \lceil 3k/2 + 1\rceil r\}$, where $r \in \mathbb{N}$ can be chosen arbitrarily and determines the flexibility at extraordinary points. The G^k splines presented in this article depend crucially on low-degree (re-)parametrizations of piecewise polynomial hole fillings. The explicit construction of such parametrizations forms the core of this work and we present two classes of singular and regular parametrizations. Also, we show how to build box and half-box spline surfaces of arbitrarily high smoothness with holes bounded by only n patches, in principle.

Categories and Subject Descriptors: I.3.5 [**Computer Graphics**]: Computational Geometry and Object Modeling—*Curver, surface, solid, and object representations*; *splines*

General Terms: Algorithms

Additional Key Words and Phrases: Geometric modeling, CAD, curves, surfaces

1. INTRODUCTION

Smooth free-form surfaces are commonly built from polynomial patches. In particular, subdivision provides a powerful method to generate free-form surfaces. A subdivision surface is defined as the limiting surface of a mesh sequence. In general, subdivision surfaces need not be polynomial, as for example, the butterfly and $\sqrt{3}$-subdivision surfaces [Dyn et al. 1990; Kobbelt 2000], but many well-known subdivision algorithms are derived from regular box spline subdivision algorithms. Their limiting surfaces consist of infinitely many polynomial patches. For example, midpoint schemes [Prautzsch 1998; Zorin and Schröder 2001], which include the Doo-Sabin, Catmull-Clark, and Qu-Algorithm [Doo and Sabin 1978; Catmull and Clark 1978; Qu 1990], are based on the Lane-Riesenfeld algorithm [Lane and Riesenfeld 1980] for uniform tensor-product splines. Loop's algorithm [1987] is based on the subdivision algorithm for quartic box splines [Prautzsch 1984].

Box spline-based subdivision can also be understood as a process by which more and more polynomial patches are added to an initial box spline surface defined by a mesh being subdivided. The initial surface

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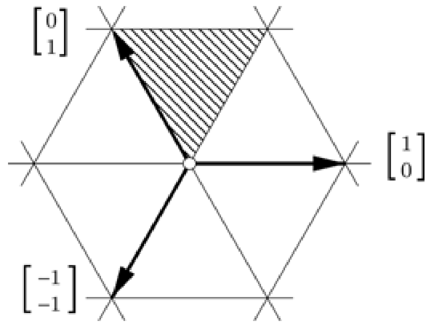

Fig. 1. A regular triangular grid.

consists of finitely many patches and has holes associated with irregularities in the mesh. Under subdivision, such a hole is filled with infinitely many patches surrounding a so-called extraordinary point.

While subdivision is elegant and simple, subdivision surfaces typically suffer from shape artifacts [Karciauskas et al. 2004; Peters and Reif 2004] and it has been shown [Reif 1996; Prautzsch and Reif 1999] that generating smoother subdivision surfaces with second- or higher-order smoothness at the extraordinary points cannot be as simple and elegant. Therefore, other methods are preferred to fill n-sided holes in piecewise polynomial spline surfaces.

The construction by Hahn [1989] is one of the oldest. There, the surfaces are piecewise polynomial of degree $\mathcal{O}(k^2)$. More recently, Reif presented singularly parametrized G^k surfaces with polynomial degree $2k+2$ (see Reif [1998]). Simultaneously, the same technique was used to construct regular G^k surfaces of the same degree $2k+2$ in Prautzsch [1997]. Further improvements were made in Peters [2002].

So far, the constructions in Prautzsch [1997] and Peters [2002] have been outlined only for $k = 2$. In this article, we show that these ideas can be extended to construct hole fillings for three-direction box and half-box splines of any smoothness-order k. The polynomial degree of our fillings is $\max\{4k+1, \lceil\frac{3}{2}k+1\rceil r\}$, where $r \in \mathbb{N}$ can be chosen arbitrarily. The number r controls the flexibility at extraordinary point, that is, the filling consists of a reparametrized, split, and modified polynomial of degree r. A crucial point is the construction of a parametrization for the filling polynomial. We present two different parametrizations. The first is singular in analogy to the parametrization for quadrilateral patches in Reif [1998]. We show that this singular parametrization is a special degenerate parametrization from a class of regular parametrizations we present second.

2. BOX SPLINE SURFACES

The symmetric box splines of order m over the triangular grid spanned by $[1\ 0]^t$, $[0\ 1]^t$, $[-1\ -1]^t$ are C^{2m} continuous, and piecewise polynomial of degree $3m+1$ on each triangle of the grid. The grid is shown in Figure 1. We are using only these box splines, since we need their symmetries.

In particular, let $B_0(\mathbf{u})$ be the piecewise linear box spline over this triangular grid defined by

$$B_0(\mathbf{i}) = \begin{cases} 1, & \text{if } \mathbf{i} = \mathbf{0} \\ 0, & \text{if } \mathbf{i} \in \mathbb{Z}^2 \backslash \{\mathbf{0}\} \end{cases}$$

and let

$$B_m(\mathbf{u}) = B_0(\mathbf{u}) * \overset{m}{\dots} * B_0(\mathbf{u})$$

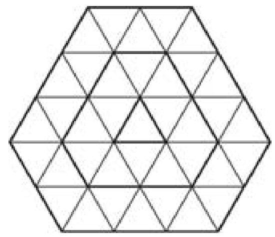

Fig. 2. A B-primitive of order two, schematically.

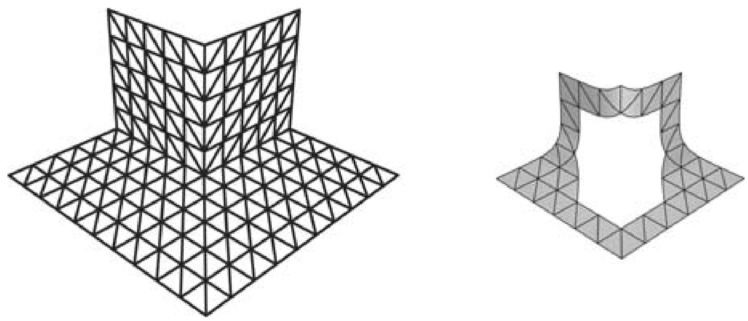

Fig. 3. A box spline surface of order two (right) and its control net (left).

be the m-fold convolution of this. A linear combination of these basis functions $B_m(\mathbf{u})$

$$\mathbf{s}(\mathbf{u}) = \sum_{\mathbf{i} \in \mathbb{Z}^2} \mathbf{c}_\mathbf{i} B_m(\mathbf{u} - \mathbf{i})$$

forms a box spline surface of order m. The control net of $\mathbf{s}(\mathbf{u})$ is a regular triangular net with the vertices $c_\mathbf{i}$. Any triangle of this net with the next m-rings of surrounding triangles is called a B-primitive of order m (see Figure 2). Every B-primitive determines one triangular polynomial patch of $\mathbf{s}(\mathbf{u})$ (see Prautzsch and Boehm [2002]).

By definition, a box spline surface has a regular control net. With symmetric box splines $B_m(\mathbf{u})$ it is possible to extend this definition. In this article, a *box spline surface of order m* with an arbitrary, that is, not necessarily regular, triangular net consists of the patches defined by all B-primitives of order m contained in the net. For simplicity, we only consider nets without boundary. Then, all vertices with valence $n \neq 6$ are *irregular*. If all irregular vertices are surrounded by at least $2m$-rings of regular vertices, each of them corresponds bijectively to an n-sided hole in the box spline surface. Figure 3 shows a box spline surface of order two and its control net. Note that it is impossible to define box spline surfaces with B-primitives which are not symmetric if the control net has irregular vertices.

The "size" of an n-sided hole in a box spline surface $\mathbf{s}(\mathbf{u})$ depends on the order m. The hole boundary is formed by $n \cdot m$ patches of $\mathbf{s}(\mathbf{u})$, where we do not count patches with only one corner on the hole boundary. We wish to fill these holes smoothly and first show how to reduce their size. The k-ring of a vertex c consists of all vertices that are not further away from $\mathbf{c}$ than k edges, that is, it consists of $\mathbf{c}$ and the next k-rings of vertices around it. In particular, the 0-ring of $\mathbf{c}$ consists only of $\mathbf{c}$ and a (-1)-ring is empty. Let $k \leq m$ and let the k-ring of any irregular vertex coalesce into one multiple vertex, as shown in Figure 4.

Further, we treat irregular vertices specially. To explain how, it suffices to consider a net with one irregular vertex of valence $n \neq 6$. This net consists of n regular net segments $C_1, \dots, C_n$ that are

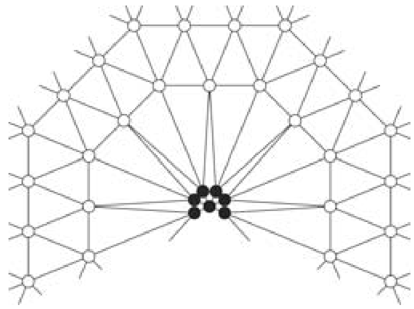

Fig. 4. An irregular vertex as a multiple vertex for $k = 1$.

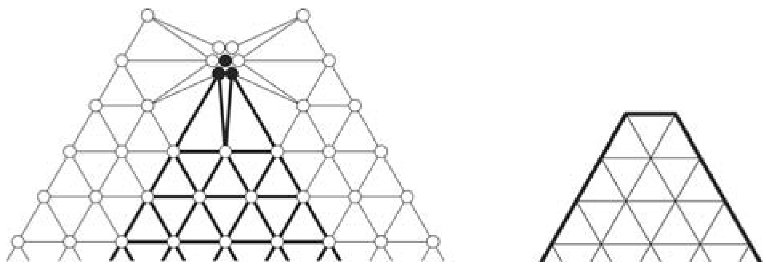

Fig. 5. A net segment D_i (left) and the associated box spline surface $\mathbf{d}_i$ for order $m = 2$ and $k = 1$ (right), schematically.

topologically equivalent to a regularly subdivided cone, as shown by heavy lines in the left of Figure 5. We count periodically, that is, $C_i = C_{i+n}$, and assume that C_i is adjacent to C_{i+1}.

To get to the surface, we momentarily remove the $(m - k - 1)$-ring of the irregular vertex. What remains of one segment C_i is equivalent to an obtuse cone. Then, we add to the remains of C_i the next m layers of control points (and thus also reinsert the points momentarily removed). Finally, we replace the irregular k-ring by a regular degenerate k-ring at the same position such that we obtain a regular net D_i, as shown schematically in the left of Figure 5. The net D_i defines a box spline surface $\mathbf{d}_i(\mathbf{u})$ of order m, as shown in the right of Figure 5.

Since D_i and D_{i+1} are parts of a regular net, $\mathbf{d}_i(\mathbf{u})$ and $\mathbf{d}_{i+1}(\mathbf{u})$ have a C^{2m} joint. Consequently, $\mathbf{d}_1(\mathbf{u}), \ldots, \mathbf{d}_n(\mathbf{u})$ form a C^{2m} surface with a hole whose boundary is formed by $n(m-k)$ patches. We call it a b-surface of type mk, or shortened, a b_{mk}-surface.

If $n < 6$, we obtain a k_{mk}-surface in exactly the same fashion. However, the net segments D_i have coalescing vertices in this case and are not just simple subsets of the entire net. Therefore, it is easier to see what happens if we double the net and view it as a two-sheeted net winding around the irregular vertex twice. Thus, the valence of the irregular vertex is doubled and as described before, we obtain a b_{mk}-surface. The surface has two sheets winding around its hole twice. Removing the extra sheet, we finally obtain a b_{mk}-surface with an n-sided hole.

Note that a b_{mm}-surface has no holes. Its derivatives up to order $2m$ are zero at extraordinary points, but, in general, it is not a C^{2m}-manifold. Since a b_{mm}-surface is subdividable, we can conclude from Prautzsch and Reif [1999] that the vicinities of extraordinary points have regular C^{2m} parametrizations only if they are planar. Also note that a generic b_{mk}-surface with $k < m$ has no singularities unless the control net further degenerates.

In the following, we assume that we have a given b_{mk}-surface, where $k = m - 1$ or $k = m - 2$. It is our goal to show that the holes of this surface can be filled smoothly for arbitrary high orders m.

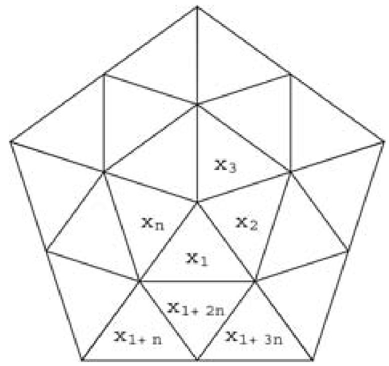

Fig. 6. The patches $\mathbf{x}_i$, schematically.

3. FILLING HOLES IN BOX SPLINE SURFACES

Let $r \geq 2$ be an arbitrary integer and consider an n-sided hole of a b_{mk}-surface, where $k = m-1$ or $k = m-2$. The hole can be filled smoothly with $4n$ triangular patches of degree $\max\{(3m+1)r, 8m+1\}$ obtained from a best filling polynomial of degree r that we reparametrize, split, and modify. The construction is based on the ideas introduced in Prautzsch [1997] and Prautzsch and Umlauf [2000].

To describe this construction, we need what we call a pre-C^p joint. Let $\mathbf{a}_{ijk}$ and $\mathbf{b}_{ijk}$ be the Bézier points of two triangular patches $\mathbf{a}$ and $\mathbf{b}$, respectively. Let $\mathbf{a}_{0jk} = \mathbf{b}_{0jk}$. The patches $\mathbf{a}$ and $\mathbf{b}$ have a pre-C^p joint along the corresponding boundary curve if $\mathbf{a}_{ijk}$ and $\mathbf{b}_{ijk}$ for $i \leq p$ and ($j \leq 2p$ or $k \leq 2p$) are as if $\mathbf{a}$ and $\mathbf{b}$ had a C^p joint.

Construction 3.1. *First*, we construct a planar, piecewise polynomial n-sided macropatch $\mathbf{x}(\mathbf{u})$ of degree $3m+1$ consisting of $4n$ triangular patches $\mathbf{x}_i(\mathbf{u})$ with C^{2m} joints, except between the inner patches $\mathbf{x}_1, \dots, \mathbf{x}_n$ (see Figure 6). For symmetry reasons, we construct $\mathbf{x}_i, \mathbf{x}_{i+n}, \mathbf{x}_{i+2n}$ and $\mathbf{x}_{i+3n}$ to be a rotation of $\mathbf{x}_1, \mathbf{x}_{1+n}, \mathbf{x}_{1+2n}$ and $\mathbf{x}_{1+3n}$ by $\frac{i}{n}2\pi$, respectively. The details are given in Section 4.

Second, let $\mathbf{q}(\mathbf{u})$ be polynomial of degree r and let

$$\mathbf{p}_i(\mathbf{u}) = \mathbf{q}(\mathbf{x}_i(\mathbf{u})), \qquad i = 1, \dots, 4n.$$

The C^{2m} joints of patches $\mathbf{x}_i(\mathbf{u})$ are carried over to the patches $\mathbf{p}_i(\mathbf{u})$. Note that $\mathbf{p}_i$ is of degree $(3m+1)r$.

If $\mathbf{q}(\mathbf{x})$ is determined appropriately, then the n-sided surface $\mathbf{p}$ formed by $\mathbf{p}_i$ lies “in” the n-sided hole of the b_{mk}-surface, but we need to modify the boundary of $\mathbf{p}$ to obtain a C^{2m} joint with the b_{mk}-surface. Locally, the b_{mk}-surface is a box spline surface which can be extended into the hole by further patches. This means that we can change any patch $\mathbf{p}_i$, $i \geq n+1$ such that it has a C^{2m} joint with the b_{mk}-surface, and even such that it has a pre-C^{2m} joint with any adjacent patch $\mathbf{p}_j$. The Bézier points not involved in pre-C^{2m} joints can be changed so as to obtain full C^{2m} joints (see, e.g., Prautzsch et al. [2002]).

Thus, for the modified patches $\mathbf{p}_i(\mathbf{u})$, some Bézier points depend on $\mathbf{q}(\mathbf{x})$, some depend on the b_{mk}-surface, and some are constrained by C^{2m} joints between the $\mathbf{p}_i(\mathbf{u})$. All other Bézier points can be chosen arbitrarily. These points, together with $\mathbf{q}(\mathbf{x})$, can be determined such that the $\mathbf{p}_i(\mathbf{u})$ minimize a fairness functional, such as thin plate energy or other functionals involving higher-order derivatives.

Next, we present two different parametrizations $\mathbf{x}(\mathbf{u})$.

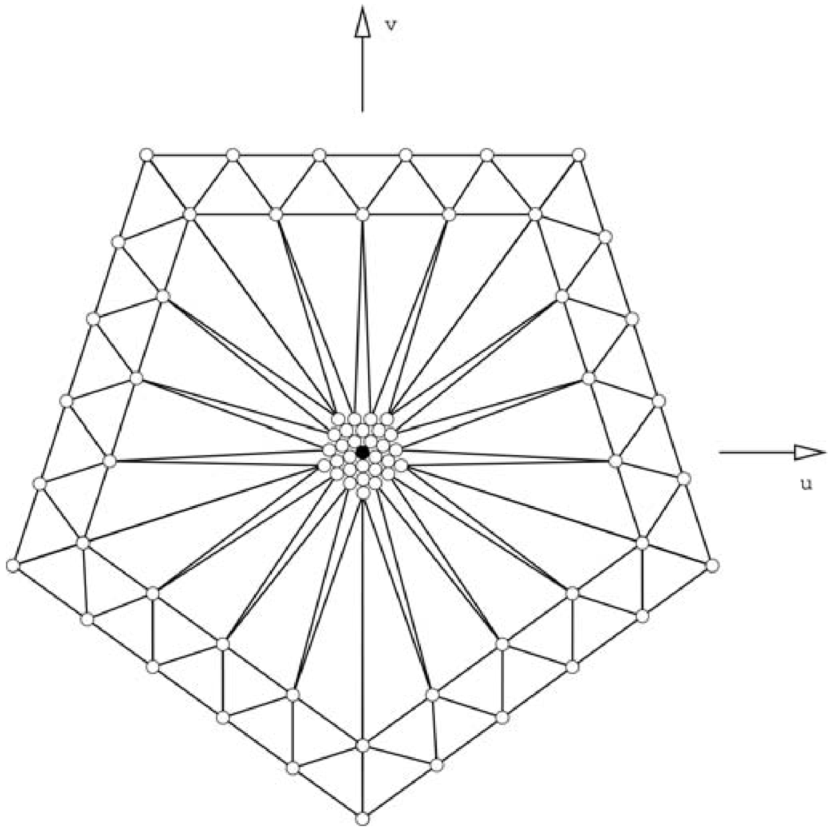

Fig. 7. The control net of a singular parametrization $\mathbf{x}^b$ for $n = 5$ and $m = 3$.

4. A SINGULAR PARAMETRIZATION FOR THE FILLING

First, we present a singular parametrization $\mathbf{x}^b = \mathbf{x}$ of degree $3m + 1$. It has similar properties as the parametrization used in Reif [1998]. The map $\mathbf{x}^b(\mathbf{u})$ is a b_{mm}-surface controlled by an $(m+2)$-ring. The $n(m + 2)$ boundary control points lie equally spaced on a regular planar n-gon and similarly the next ring of control points, while all other points lie at the center, as illustrated in Figure 7. The boundary triangles are isosceles with vertex angle $2\pi/n$.

THEOREM 4.1. *The map $\mathbf{x}^b(\mathbf{u})$ is injective and, except for its center point, regular.*

PROOF. It suffices to consider the patch $\mathbf{x}_1(\mathbf{u})$. We choose the multiple control point as the origin and the symmetry axis of $\mathbf{x}_1(\mathbf{u})$ as the v-axis of the coordinate system, as shown in Figure 7. We assume that the triangle $[0\,0]^t, [1\,1]^t, [0\,1]^t$ is the parameter domain of $\mathbf{x}_1(\mathbf{u})$ (see Figure 1). The partial derivatives $\frac{\partial}{\partial u}\mathbf{x}_1$ and $\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\mathbf{x}_1$ are controlled by certain edge directions of the control net. We call these the u- and uv-directions.

Any real interval I of angles defines a pointed cone

$$I_c = \left\{ r \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix} \middle| \, r \geq 0, \varphi \in I \right\}.$$

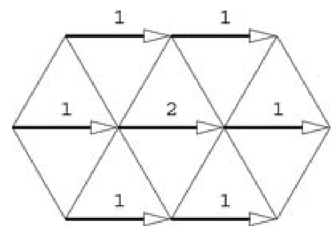

Fig. 8. Boehm's mask.

The u-direction of all u-edges in the left half-plane $u \leq 0$ lie in the cone

$$A = \begin{cases} \left(\frac{\pi}{n} - \frac{\pi}{2}, 0\right]_c & \text{, if } n \geq 6 \\ \left(\frac{\pi}{n} - \frac{\pi}{2}, \frac{3\pi}{n} - \frac{\pi}{2}\right)_c & \text{, if } n \leq 5 \end{cases} \tag{1}$$

and the uv-directions of all uv-edges starting in the half-plane $u \leq 0$ lie in the cone

$$B = \begin{cases} \left[\frac{2\pi}{n}, \frac{\pi}{2} + \frac{\pi}{n}\right)_c & \text{, if } n \geq 4 \\ \left[\frac{\pi}{2}, \frac{5}{6}\pi\right)_c & \text{, if } n = 3. \end{cases}$$

Subdividing the control net (with scaling factor two) means that edge directions are halved and averaged by Boehm's mask, shown in Figure 8. These operations preserve the aforementioned cone properties if all directions averaged lie in A or B. Because of symmetry, u-edges crossing the v-axis are perpendicular to this axis before and after subdivision. This implies that the directions of uv-edges starting at the v-axis lie in B (and the half-plane $u \leq 0$) before and after subdivision. Thus, in summary, subdivision preserves the cone properties, which implies that the derivatives $\frac{\partial}{\partial u}\mathbf{x}_1$ and $\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\mathbf{x}_1$ lie in A and B, respectively. Because of the nonzero control edges, the derivatives are nonzero, except at $\mathbf{0}$. Therefore, we can argue as in Umlauf [2004] and show that the "left" part of $\mathbf{x}_1(u, v)$, where $v \geq u$ and $(u, v) \neq \mathbf{0}$, is injective and regular. Due to symmetry, $\mathbf{x}_1(\mathbf{u})$ is injective and regular for all $\mathbf{u} \neq \mathbf{0}$.

Further, $\mathbf{x}_1(u, v) = \mathbf{x}_1(\mathbf{0})$ implies that $\mathbf{x}_1(u, v) = \mathbf{x}_1(v, u)$, since $\mathbf{x}_1(\mathbf{0}) = \mathbf{0}$ lies on the symmetry axis of $\mathbf{x}_1$. Because of injectivity, we obtain $u = v$. Since the half-open line segment $(\mathbf{0}, (1/2, 1/2)]$ is mapped injectively into the v-axis, the continuous map $\mathbf{x}_1$ is also injective on the closure. Hence, $\mathbf{0}$ is the only point mapped onto $\mathbf{0}$ under $\mathbf{x}_1$, which concludes the proof. □

5. A REGULAR PARAMETRIZATION FOR THE FILLING

Now, we present a regular parametrization $\mathbf{x}$ of degree $3m + 1$. This is a modification of the singular parametrization in Section 4.

Let $\mathbf{x}$ be as in Section 4 and let $\mathbf{x}_{ijk}, i + j + k = 3m + 1 = d$, be the Bézier points of $\mathbf{x}_1(\mathbf{u})$ such that $\mathbf{x}_{d00} = \mathbf{x}_1(\mathbf{0}) = \mathbf{0}$, $\mathbf{x}_{0d0} = \mathbf{x}_1(0, 1)$, and $\mathbf{x}_{00d} = \mathbf{x}_1(1, 1)$ are the corner points. We change the points $\mathbf{x}_{ijk}$ for $i < d$ by

$$\mathbf{x}_{ijk} := \frac{\varepsilon}{d}(j\, \mathbf{x}_{0d0} + k\, \mathbf{x}_{00d}),$$

where $\varepsilon > 0$.

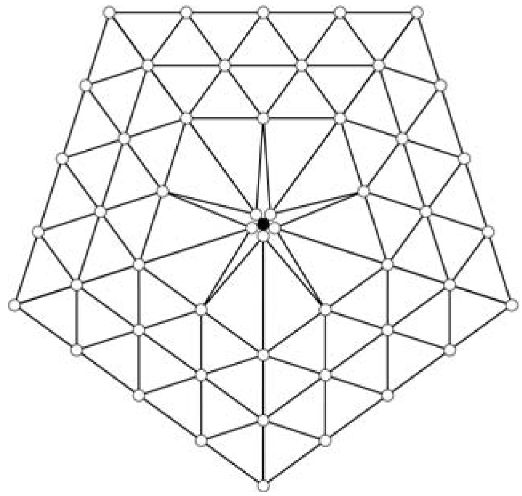

Fig. 9. The control net of the outer patches $\mathbf{x}_{n+1}, \ldots, \mathbf{x}_{4n}$ for $m = 2$.

LEMMA 5.1. *The map $\mathbf{x}_1$ is regular and injective for sufficiently small ε.*

PROOF. The curve $\mathbf{a}(v) = \mathbf{x}_1(\alpha v, v)$, $\alpha \in [0, 1/2]$ has the Bézier points

$$\mathbf{a}_i = \begin{cases} (1-\alpha)\mathbf{x}_{d-i,i,0} + \alpha \mathbf{x}_{d-i,0,i} & , i < d \\ \mathbf{x}_1(\alpha, 1) & , i = d. \end{cases}$$

Hence,

$$\Delta \mathbf{a}_i \in \left(\frac{\pi}{2} - \delta, \frac{\pi}{2} + \frac{\pi}{n} + \delta\right)_c$$

with δ, depending on ε, and $\lim_{\varepsilon \to 0} \delta = 0$. The cross-derivative curve $\frac{\partial}{\partial u} x_1(\alpha v, v)$ has the Bézier points

$$\mathbf{b}_i = \begin{cases} \varepsilon(\mathbf{x}_{00d} - \mathbf{x}_{0d0}) & , i < d-1 \\ \dfrac{\partial}{\partial u}\mathbf{x}_1(\alpha, 1) & , i = d-1. \end{cases}$$

For $\alpha \leq 1/2$, these points lie in cone A (see Eq. (1)). It is straightforward to conclude from these estimates that $\mathbf{x}_1$ is regular and injective for sufficiently small ε. □

Changing also $\mathbf{x}_2, \ldots, \mathbf{x}_n$ in a similar fashion, we obtain a regular and injective map $\mathbf{x}$.

Remark 5.2. In the preceding construction, the outer patches $\mathbf{x}_{n+1}, \ldots, \mathbf{x}_{4n}$ are determined by a control net, as shown in Figure 7. Instead, we could use a control net, as shown in Figure 9. Then, it seems possible to construct $\mathbf{x}_1, \ldots, \mathbf{x}_n$ as regular injective maps with C^{2m} contact to the outer patches $\mathbf{x}_{n+1}, \ldots, \mathbf{x}_{4n}$. For $m = 1$, this is done in Prautzsch and Umlauf [2000]. For $m = 2$ and $n = 5$, the Bézier points are shown in Figure 10.

6. HALF-BOX SPLINE SURFACES

A b_{mk}-surface has even smoothness-order. To obtain similar surfaces with odd smoothness-orders, we use symmetric half-box splines. In this section, we recall the definition of half-box splines and in the next section, we show how to fill a hole in a half-box spline surface, in analogy to the construction for b_{mk}-surfaces.

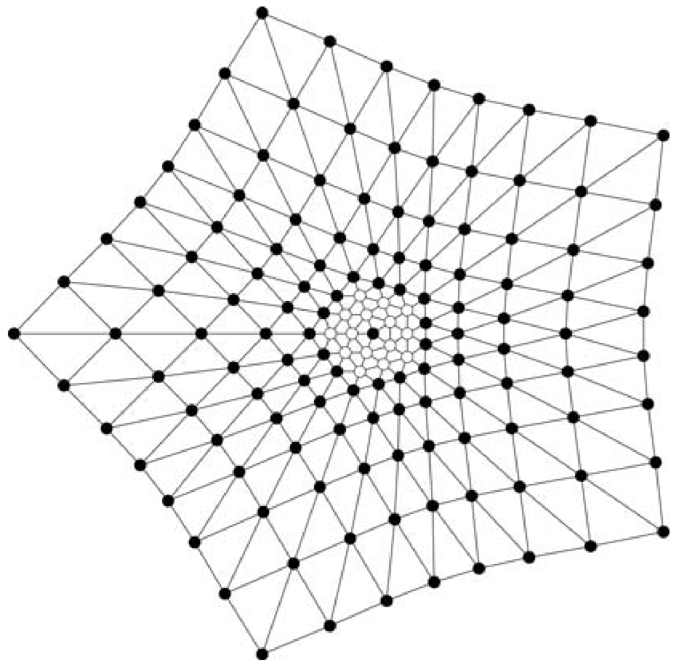

Fig. 10. The Bézier points of the inner patches $\mathbf{x}_1, \ldots, \mathbf{x}_n$ for $m = 2$.

The piecewise constant half-box splines over the triangular grid shown in Figure 1 are translates of the two functions

$$H_0^{\triangle}(\mathbf{u}) = \begin{cases} 1, & \text{if } \mathbf{u} \in \triangle \\ 0, & \text{else} \end{cases} \qquad \text{and} \qquad H_0^{\triangledown}(\mathbf{u}) = \begin{cases} 1, & \text{if } \mathbf{u} \in \triangledown \\ 0, & \text{else} \end{cases},$$

where $\triangle$ and $\triangledown$ are the two triangles

$$\triangle := \{\mathbf{u} \mid 0 \le u \le v < 1\} \qquad \text{and} \qquad \triangledown := \{\mathbf{u} \mid 0 \le v < u < 1\}$$

which form a partition of the unit square. A convolution with the box spline $B_m(\mathbf{u})$ gives the symmetric half-box splines

$$H_m^{\triangle}(\mathbf{u}) = H_0^{\triangle}(\mathbf{u}) * B_m(\mathbf{u}) \qquad \text{and} \qquad H_m^{\triangledown}(\mathbf{u}) = H_0^{\triangledown}(\mathbf{u}) * B_m(\mathbf{u})$$

of order m. They are C^{2m-1} continuous and polynomial of degree $3m$ on each triangle of the grid. We use them to build half-box spline surfaces

$$\mathbf{s}(\mathbf{u}) = \sum_{\mathbf{i} \in \mathbb{Z}^2} \left(\mathbf{c}_{\mathbf{i}}^{\triangle} H_m^{\triangle}(\mathbf{u} - \mathbf{i}) + \mathbf{c}_{\mathbf{i}}^{\triangledown} H_m^{\triangledown}(\mathbf{u} - \mathbf{i}) \right)$$

of order m. The control net of $\mathbf{s}(\mathbf{u})$ is a regular hexagonal net with the vertices $\mathbf{c}_{\mathbf{i}}^{\triangle}$ and $\mathbf{c}_{\mathbf{i}}^{\triangledown}$. Any vertex of this net with the next m-rings of surrounding hexagons is called an H-primitive of order m (see Figure 11). Every H-primitive determines one triangular polynomial patch of $\mathbf{s}(\mathbf{u})$ (see Prautzsch and Boehm [2002]).

Half-box spline surfaces can be generalized in analogy to box spline surfaces. For this we are using a duality. A net $\mathcal{N}$ and triangular net $\mathcal{T}$ are called *dual* if there is a one-to-one correspondence between vertices of $\mathcal{N}$ and faces of $\mathcal{T}$ such that vertices with a common edge correspond to faces with a common

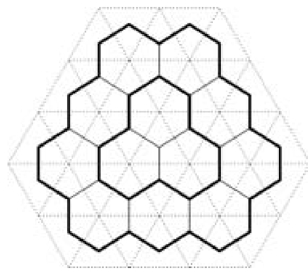

Fig. 11. An H-primitive of order two, schematically. It is dual to the dotted B-primitive of order two.

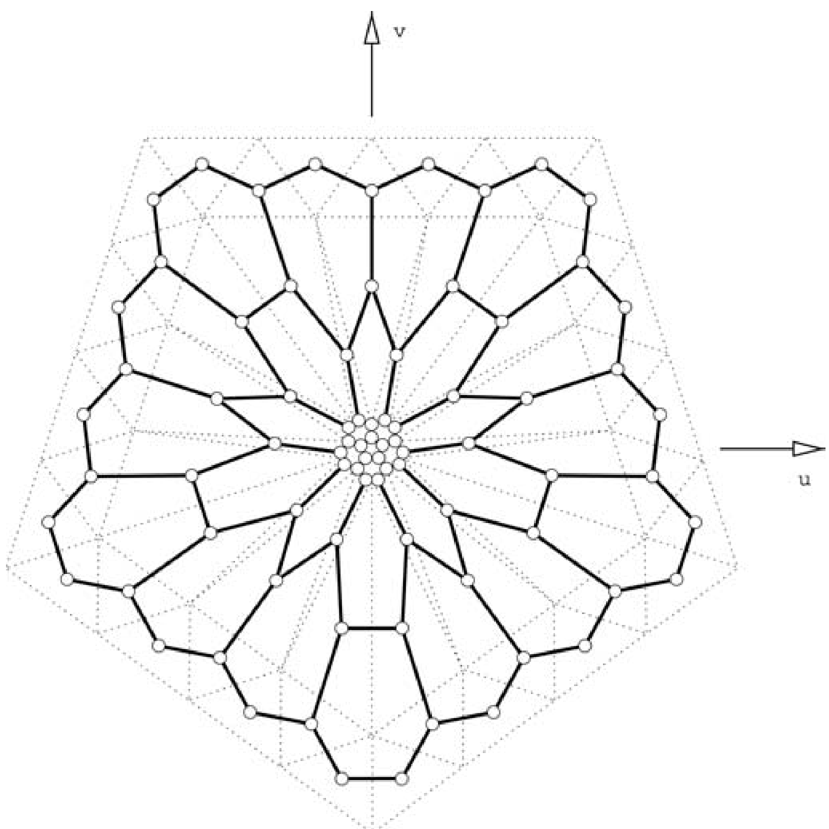

Fig. 12. The control net of a singular parametrization $\mathbf{x}$ for $n = 5$ and $m = 2$.

edge, and vice versa. If two triangles in $\mathcal{T}$ coincide, their dual vertices in $\mathcal{N}$ also coincide. Under this definition, H-primitives of order m are dual to B-primitives of order m. In particular, if the vertices of $\mathcal{N}$ are the centroids of their dual triangles in $\mathcal{T}$, we call $\mathcal{N}$ the *centroid net* of $\mathcal{T}$.

A general half-box spline surface of order mk, or shortened, an h_{mk}-surface, has a control net $\mathcal{N}$ that is dual to the control net of a b_{mk}-surface. It consists of all patches determined by the H-primitives in $\mathcal{N}$ that are dual to the B-primitives in $\mathcal{T}$. Thus, an h_{mk}-surface is piecewise polynomial of degree $3m$ and $2m - 1$ times continuously differentiable.

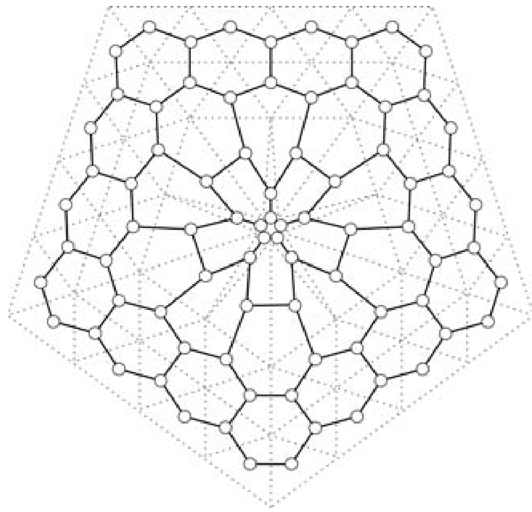

Fig. 13. The control net of the outer patches $\mathbf{x}_{n+1}, \ldots, \mathbf{x}_{4n}$ for $m = 2$.

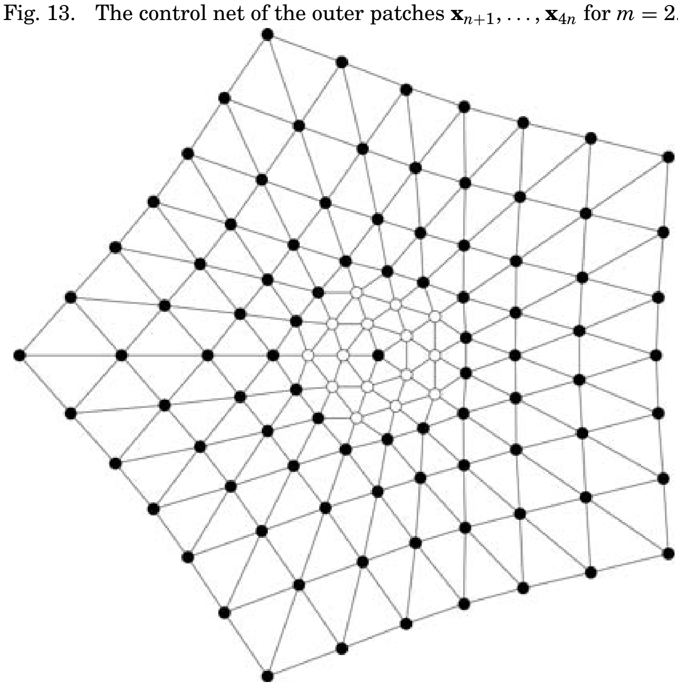

Fig. 14. The Bézier points of the inner patches $\mathbf{x}_1, \ldots, \mathbf{x}_n$ for $m = 2$.

As in Section 2, an h_{mm}-surface has no holes. Its derivatives up to order $2m-1$ are zero at extraordinary points, but, in general, it is not a C^{2m-1}-manifold.

7. FILLING HOLES IN HALF-BOX SPLINE SURFACES

Let r be an arbitrary integer and consider an n-sided hole of an $h_{m,k}$-surface, where $k = m - 1$ or $k = m - 2$. The hole can be filled smoothly with $4n$ triangular patches of degree $\max\{3mr, 8m - 3\}$ in complete analogy to Section 3. Next, we show how to "dualize" the singular and regular parametrizations $\mathbf{x}(\mathbf{u})$ given in Sections 4 and 5.

7.1 A Singular Parametrization for the Filling

Let map $\mathbf{x}^h(\mathbf{u})$ be the h_{mm}-surface controlled by the centroid net of the control net of $\mathbf{x}^b(\mathbf{u})$ given in Section 4 (see Figure 12). This is a singular parametrization as that to fill the hole of a h_{mk}-surface by Construction 3.1.

THEOREM 7.1. *The map $\mathbf{x}^h(\mathbf{u})$ is injective and, except for its center point, regular.*

PROOF. The control vectors of the derivative $\frac{\partial}{\partial u}\mathbf{x}^h$ form the centroid net of the control net of $\frac{\partial}{\partial u}\mathbf{x}^b$, and similarly for the derivative $\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\mathbf{x}^h$. This hexagonal centroid net can be split into two triangular nets (see Prautzsch and Boehm [2002]). Subdividing the hexagonal net means to duplicate its vertices and average each triangular net using Boehm's mask, shown in Figure 8. Therefore, we can continue exactly as in the proof of Theorem 4.1. This will prove this theorem also. □

7.2 A Regular Parametrization for the Filling

As shown in Section 5, we can change the preceding singular parametrization $\mathbf{x}^h$ into a regular and injective parametrization of the same degree. Again, it is possible to define the outer patches $\mathbf{x}_{n+1}, \ldots, \mathbf{x}_{4n}$ by a less degenerate control net, as seen in Figure 13. The Bézier net of $\mathbf{x}_1, \ldots, x_n$ for $m = 2$ and $n = 5$ is shown in Figure 14.

8. CONCLUSION

We have introduced box and half-box spline surfaces with multiple extraordinary points to minimize the holes of general box and half-box spline surfaces with arbitrary triangular control nets. Second, we have proved that the holes can be filled smoothly with a small number of polynomial patches of low degree. In this article we did not address the important issue of energy minimization for shape optimization.

We have presented two solutions for both box and half-box spline surfaces. The first solution consists of a singularly and the second of a regularly parametrized piecewise polynomial filling. It is simple to prove the correctness of the first solution, but it has the disadvantage of being singular. Therefore, the second solution should be preferred in practical applications.

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