A Technique for Verifying the Smoothness of Subdivision Schemes

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Abstract. In recent years, subdivision schemes for surfaces of arbitrary topology were developed that do not generalize box-splines subdivision schemes. Examples for this kind of subdivision schemes are the $\sqrt{3}$-scheme and certain averaging schemes for triangular and hexahedral nets. In order to analyze the smoothness of the limit surface, it is necessary to know if its characteristic map is regular and injective. For box-spline based schemes this can be done based on the explicit piecewise polynomial representation.

In this paper, a general approach is introduced that allows analyzing the characteristic map even if no explicit representation is available. The proposed technique requires that the first divided difference schemes are scalar and use convex combinations. Then simple geometric properties of the sub-dominant eigenvectors of the subdivision matrix can be used to prove regularity and injectivity of the characteristic map for any valence. This is demonstrated for a midpoint scheme for triangular nets.

§1. Introduction

Subdivision algorithms have become a popular technique in computer graphics for the modeling of free-form curves and surfaces. A vast variety of subdivision algorithms designed for different needs in computer graphics has evolved over the last decades. These subdivision algorithms can coarsely be classified into two categories. First there are subdivision algorithms that have been developed by generalizing the subdivision rules for spline surfaces in tensor-product B-spline or box-spline representation, for example [3, 5, 8, 11, 12, 15, 17, 21, 24]. For subdivision algorithms in this category it is well known that the limit surfaces can be represented in B-spline or box-spline form almost everywhere. The second category

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contains all those subdivision algorithms that have been developed as interpolatory schemes [7, 9] or from geometric considerations [10, 19]. For subdivision algorithms in this category no basis functions are known explicitly for the representation of the limit surface.

The analysis of the smoothness of the limit surface of a subdivision algorithm depends on the eigenvalues and -vectors of the subdivision matrix [1, 2, 5]. If its second-largest eigenvalue has algebraic and geometric multiplicity two, the two corresponding linear independent eigenvectors determine the characteristic map [18]. For a regular and injective characteristic map the limit surfaces are for (almost) all control nets regular surfaces with continuous normal everywhere [18, 22]. There are basically three techniques to check if the characteristic map satisfies these conditions:

1. **Polynomial representation:** The B-spline or box-spline representation of the characteristic map can be used to prove regularity and injectivity [8, 12, 13, 20].
2. **Linear approximation:** A linear approximation to the Jacobian of the characteristic map with error control is computed to prove regularity and injectivity [21, 22, 24].
3. **Visual inspection:** The control net of the characteristic map for some valences is drawn to judge its behavior visually [10].

The first and second techniques are more complicated to compute but can prove smoothness while the third is easy to compute but may lead to mistakes for higher valences.

It is the purpose of this paper to establish a set of simple geometric criteria for a certain class of subdivision algorithms that are as easy to compute as the information needed for the visual inspection. These criteria can be used to prove regularity and injectivity of the characteristic map for any valence.

The proposed technique will be demonstrated by the midpoint scheme of order 2 on triangular nets described in Section 2. Its subdivision matrix is analyzed in Section 3. In Section 4 the geometric criteria to prove regularity and injectivity of its characteristic map are discussed in detail.

§2. The Midpoint Scheme on Triangular Nets

One subdivision of a regular quadrilateral control net $\mathcal{C}$ with the algorithm of Lane and Riesenfeld for uniform tensor-product B-spline surfaces can conveniently be formulated in two steps (cf. [16]):

Refinement: Refine the net $\mathcal{C}$ by adding edge midpoints and face midpoints (for quadrilateral nets only) to the net.

Midpoints: Repeat m-times: Compute the net of face midpoints.

Because of the computation of midpoints this scheme is called the midpoint scheme of order m. It is well known that it generates for regular quadrilateral nets piecewise polynomial C^{m-1}-surfaces of bi-degree m.

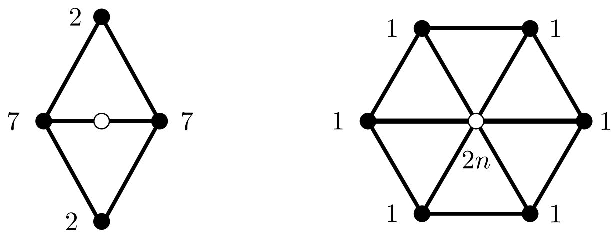

Fig. 1. The stencils of the midpoint scheme of order 2 on triangular nets, where n denotes the valence of the central vertex.

Using the above formulation of the midpoint scheme of order m it is obvious that it can be applied to arbitrary nets. In case of a net with triangular faces the midpoint schemes with even order do not agree with the respective algorithms for box-splines [24]. This can be seen for example by comparing the stencils of the midpoint scheme of order 2 on triangular nets shown in Figure 1 with the stencils of the subdivision scheme for quartic box-splines over the three-directional grid [11]. Nevertheless, the limit surfaces of the midpoint scheme of order 2 on regular triangular nets can be regarded as surfaces over the three-directional grid spanned by $\mathbf{e}_1 = [1\ 0]^t$, $\mathbf{e}_2 = [0\ 1]^t$ and $\mathbf{e}_3 = [1\ 1]^t$, see Figure 2.

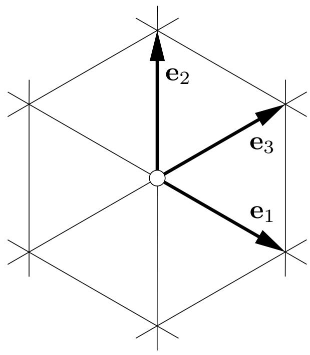

Fig. 2. The three-directional grid.

Let $\mathcal{C}^0$ be a regular triangular net and $(\mathcal{C}^l)_{l\geq 0}$ the sequence of control nets generated by a subdivision scheme from $\mathcal{C}^0$. Then the directional difference ∇_k of the points $\mathbf{c}^l_{\mathbf{i}} \in \mathcal{C}^l, \mathbf{i} \in \mathbb{Z}^2$, is defined as

$$\nabla_k \mathbf{c}^l_{\mathbf{i}} = \mathbf{c}^l_{\mathbf{i}+\mathbf{e}_k} - \mathbf{c}^l_{\mathbf{i}}, \qquad k = 1, 2, 3.$$

In general, there exists a matrix subdivision scheme relating the directional differences of different refinement levels, see [4, 6].

In the sequel we will focus on the midpoint scheme of order 2 on triangular nets. For this scheme there exists a scalar subdivision scheme that maps ∇_k-differences of $\mathcal{C}^l$ to ∇_k-differences of $\mathcal{C}^{l+1}, l \geq 0$. This subdivision scheme is the so-called ∇_k-difference scheme. For $k = 1$ this scheme is given by

$$\nabla_1 \mathbf{c}^{l+1}_{2\mathbf{i}+\mathbf{v}} = (6\nabla_1 \mathbf{c}^l_{\mathbf{i}-\mathbf{v}} + \nabla_1 \mathbf{c}^l_{\mathbf{i}+\mathbf{v}-\mathbf{e}_1} + \nabla_1 \mathbf{c}^l_{\mathbf{i}+\mathbf{e}_2} + \nabla_1 \mathbf{c}^l_{\mathbf{i}-\mathbf{e}_3})/18$$

for $\mathbf{v} \in \{[0\ 0]^t, \mathbf{e}_1\}$ and

$$\nabla_1 \mathbf{c}^{l+1}_{2\mathbf{i}+\mathbf{v}} = (5\nabla_1 \mathbf{c}^l_{\mathbf{i}+\mathbf{v}} + 2\nabla_1 \mathbf{c}^l_{\mathbf{i}} + 2\nabla_1 \mathbf{c}^l_{\mathbf{i}-\mathbf{e}_1})/18$$

for $\mathbf{v} \in \{\mathbf{e}_2, -\mathbf{e}_3\}$. For $k = 2, 3$ the ∇_k-difference scheme is represented by similar rules. Using the standard technique of [4, 6] it can been seen that this scheme generates C^1-surfaces for regular triangular nets. The divided ∇_k-differences of $\mathcal{C}^l$ converge towards the directional derivatives of the limit surface with respect to $\mathbf{e}_k$ for all k. Note that the divided ∇_k-difference schemes use only convex combinations.

§3. Analyzing the Subdivision Matrix

The midpoint scheme of order 2 on triangular nets uses only convex combinations. Thus there is a square subdivision matrix S mapping the 3-ring neighborhood of an n-valent vertex in $\mathcal{C}^l$ to the 3-ring neighborhood of the corresponding vertex in $\mathcal{C}^{l+1}, l \geq 0$. In order to analyze the smoothness of the limit surface of a subdivision scheme all three techniques described in §1. Introduction use the spectral analysis of the subdivision matrix.

Because S can be arranged in a block-cyclic form, the discrete Fourier transform $\hat{S} = \operatorname{diag}(\hat{S}_0, \ldots, \hat{S}_{n-1})$ can be used to compute the eigenvalues and eigenvectors of S. The Fourier blocks $\hat{S}_i, i = 0, \ldots, n-1$, are given by

$$\hat{S}_i = \frac{1}{18} \left[\begin{array}{c|c|cc|ccc}
12\delta_{i,0} & 6\delta_{i,0} & & & & & \\ \cline{1-2}
7\delta_{i,0} & 7 + 4c_i & & & & & \\ \cline{1-4}
\delta_{i,0} & 12 + 2c_i & 1 & 1 + \omega_{-i} & & & \\
2\delta_{i,0} & 7 + 7\omega_i & 0 & 2 & & & \\ \hline
0 & 7 & 7 & 2 + 2\omega_{-i} & 0 & 0 & 0 \\
0 & 7 + 2\omega_i & 2 & 7 & 0 & 0 & 0 \\
0 & 2 + 7\omega_i & 2\omega_i & 7 & 0 & 0 & 0
\end{array} \right]$$

with $c_i = \cos(2\pi i/n), s_i = \sin(2\pi i/n), \omega_i = c_i + s_i\sqrt{-1}$ and $\delta_{0,0} = 1$ and $\delta_{i,0} = 0$ for $i \neq 0$. Furthermore, the midpoint scheme of order 2 on triangular nets is a symmetric subdivision algorithm, i.e. it is invariant under rotations and reflections of the labeling of the control points of the 3-ring neighborhood (see [13]). This condition restricts the set of feasible eigenvalue distributions.

The subdivision matrix S has the single eigenvalues 1 and 5/18, the n-fold eigenvalues 1/9 and 1/18, the $(4n-1)$-fold eigenvalue 0 and the eigenvalues $\lambda_i = (7+4c_i)/18, i = 1, \ldots, n-1$. Since the largest eigenvalue of S is the single eigenvalue 1 with corresponding eigenvector $\mathbf{v}_1 = [1 \ldots 1]^t$ the subdivision scheme is guaranteed to converge towards a uniquely defined limit point.

If the second largest (so-called sub-dominant) eigenvalue of S has algebraic and geometric multiplicity 2, the characteristic map $\mathbf{c}$ is defined by the two real, linear independent eigenvectors corresponding to the sub-dominant eigenvalue. For this particular scheme $\lambda_1 = \lambda_{n-1}$ is the sub-dominant eigenvalue for $n \geq 4$. It originates from the Fourier blocks $\hat{S}_1$ and $\hat{S}_{n-1}$, respectively. This is a necessary condition for the characteristic map $\mathbf{c}$ of a symmetric subdivision scheme to be injective, see [13]. An example control net for the characteristic map is shown for $n = 5$ in Figure 3.

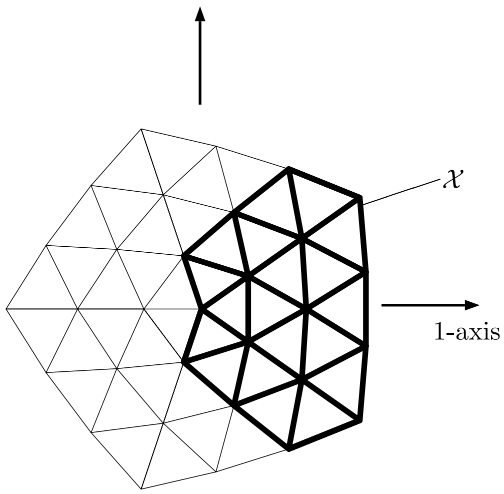

Fig. 3. The control net of the characteristic map of the midpoint scheme of order 2 on triangular nets for $n = 5$.

§4. Analyzing the Characteristic Map

For a symmetric subdivision scheme the characteristic map $\mathbf{c}$ is made up of n rotationally symmetric so-called segments. A segment $\mathbf{c}^0$ is defined by a subset of control points $\mathcal{X}$ of the control points of $\mathbf{c}$. In case of the midpoint scheme of order 2 on triangular nets these points are arranged as in the net drawn with heavy lines in Figure 3. A necessary condition for the characteristic map to be regular and injective is that $\mathbf{c}^0$ is regular and injective [13, 20]. Note that $\mathcal{X}$ depends on n and can be normalized such that it is symmetric with respect to the 1-axis.

The divided ∇_k-differences of $\mathcal{X}$ converge towards directional derivatives of $\mathbf{c}^0$ with respect to $\mathbf{e}_k$ for $k = 1, 2, 3$. The ∇_k-differences of $\mathcal{X}$ lie within cones B_k. These can be characterized by the angles between the ∇_k-differences and the 1-axis

$$B_k := [\inf_{\substack{\mathbf{c_i} \in \mathcal{X} \\ \text{all } n}} (\angle(\nabla_k \mathbf{c_i}, \text{1-axis}), \sup_{\substack{\mathbf{c_i} \in \mathcal{X} \\ \text{all } n}} (\angle(\nabla_k \mathbf{c_i}, \text{1-axis})].$$

For the midpoint scheme of order 2 on triangular nets and for $n \geq 4$ the net $\mathcal{X}$ defining the segment $\mathbf{c}^0$ is given by

$$\mathcal{X} = \begin{array}{c} \mathbf{c}_{0,1} \\ \mathbf{c}_{0,0} \\ \mathbf{c}_{0,-1} \end{array} \quad \begin{array}{c} \mathbf{c}_{1,2} \\ \mathbf{c}_{1,1} \\ \mathbf{c}_{1,0} \\ \mathbf{c}_{1,-1} \end{array} \quad \begin{array}{c} \mathbf{c}_{2,3} \\ \mathbf{c}_{2,2} \\ \mathbf{c}_{2,1} \\ \mathbf{c}_{2,0} \\ \mathbf{c}_{2,-1} \end{array} \quad \begin{array}{c} \mathbf{c}_{3,3} \\ \mathbf{c}_{3,2} \\ \mathbf{c}_{3,1} \\ \mathbf{c}_{3,0} \end{array}$$

$$= \begin{array}{c} \begin{bmatrix} \alpha\beta\gamma\delta(-1+2c_1) \\ \widehat{\alpha}\beta\gamma(9+14c_1+4c_2) \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} \alpha\beta\gamma\delta(-1+2c_1) \\ -\widehat{\alpha}\beta\gamma(9+14c_1+4c_2) \end{bmatrix} \end{array} \begin{array}{c} \begin{bmatrix} 14\alpha\beta\gamma c_1 \\ 14\widehat{\alpha}\beta\gamma(1+c_1) \end{bmatrix} \\ \begin{bmatrix} \alpha\beta\gamma\delta \\ \widehat{\alpha}\beta\gamma\delta \end{bmatrix} \\ \begin{bmatrix} \alpha\beta\gamma\delta \\ -\widehat{\alpha}\beta\gamma\delta \end{bmatrix} \\ \begin{bmatrix} 14\alpha\beta\gamma c_1 \\ -14\widehat{\alpha}\beta\gamma(1+c_1) \end{bmatrix} \end{array} \begin{array}{c} \begin{bmatrix} \alpha(315+560c_1+166c_2+8c_3) \\ \widehat{\alpha}(685+844c_1+182c_2+8c_3) \end{bmatrix} \\ \begin{bmatrix} \alpha\gamma(39+36c_1+2c_2) \\ \widehat{\alpha}\gamma(39+36c_1+2c_2) \end{bmatrix} \\ \begin{bmatrix} 14\alpha\beta\gamma \\ 0 \end{bmatrix} \\ \begin{bmatrix} \alpha\gamma(39+36c_1+2c_2) \\ -\widehat{\alpha}\gamma(39+36c_1+2c_2) \end{bmatrix} \\ \begin{bmatrix} \alpha(315+560c_1+166c_2+8c_3) \\ -\widehat{\alpha}(685+844c_1+182c_2+8c_3) \end{bmatrix} \end{array} \begin{array}{c} \begin{bmatrix} 14\alpha(37+39c_1+5c_2) \\ 14\widehat{\alpha}(37+39c_1+5c_2) \end{bmatrix} \\ \begin{bmatrix} \alpha(543+466c_1+40c_2) \\ \widehat{\alpha}(173+182c_1+24c_2) \end{bmatrix} \\ \begin{bmatrix} \alpha(543+466c_1+40c_2) \\ -\widehat{\alpha}(173+182c_1+24c_2) \end{bmatrix} \\ \begin{bmatrix} 14\alpha(37+39c_1+5c_2) \\ -14\widehat{\alpha}(37+39c_1+5c_2) \end{bmatrix} \end{array}$$

with $\alpha = c_{1/2}, \widehat{\alpha} = s_{1/2}, \beta = (3+2c_1), \gamma = (7+4c_1), \delta = (5+4c_1)$.

Note that up to this point all techniques to analyze a subdivision scheme agree.

The first coordinates of the points of $\mathcal{X}$ increase in direction $\mathbf{e}_1$ while the second coordinates decrease in direction $\mathbf{e}_1$ and increase in direction $\mathbf{e}_2$. Thus the first coordinates of the ∇_1-differences and the second coordinates of the ∇_2-differences are positive while the second coordinates of the ∇_1-differences are negative for $n \geq 4$. This yields for the cones B_k for $n \geq 4$

$$B_1 \subseteq (-\pi/2, 0), \tag{1}$$

$$B_2 \subseteq (0, \pi), \tag{2}$$

$$B_3 \subseteq (0, \pi/2). \tag{3}$$

Denote by $|B_k|$ the "length" of the interval B_k. The cones B_k provide a criterion for the characteristic map $\mathbf{c}$ to be regular and injective:

Theorem 1. *For a symmetric subdivision scheme the characteristic map is regular and injective, if*

1. *the divided ∇_1- and ∇_3-difference schemes are scalar and use only convex combinations,*
2. *none of the ∇_1- and ∇_3-differences of $\mathcal{X}$ vanish and*
3. *the cones B_1 and B_3 satisfy the conditions*

$$|B_1 \cup B_3| < \pi \tag{4}$$

and

$$B_1 \cap B_3 = \emptyset. \tag{5}$$

Proof: We prove first regularity and then injectivity of $\mathbf{c}^0$. Let $k = 1$ or $k = 3$.

Condition (4) implies $|B_k| < \pi$. Thus all directional derivatives of $\mathbf{c}^0$ with respect to $\mathbf{e}_k$ are computed as convex combinations of divided ∇_k-differences and do not vanish because of *2.*

Denote by $\widehat{B}_k$ the cone of the $(-\nabla_k)$-differences. Because the divided ∇_k-difference schemes use only convex combinations the respective directional derivatives lie within B_k or within $\widehat{B}_k$. Condition (4) implies that $\widehat{B}_k$ does not lie in the half-space of $B_1 \cup B_3$. Therefore condition (5) implies that the set

$$(B_1 \cap B_3) \cup (B_1 \cap \widehat{B}_3) \cup (\widehat{B}_1 \cap B_3) = \emptyset$$

is empty. Thus there are always two linear independent, non-vanishing directional derivatives of $\mathbf{c}^0$, so that $\mathbf{c}^0$ is regular.

Assume that there are two different points $\mathbf{p} \neq \mathbf{q}$ in the domain Ω of $\mathbf{c}^0$ with $\mathbf{c}^0(\mathbf{p}) = \mathbf{c}^0(\mathbf{q})$. Then $\mathbf{c}^0(t\mathbf{p} + (1-t)\mathbf{q})$ for $t \in [0, 1]$ is a closed curve on $\mathbf{c}^0$. Therefore its tangent $\mathbf{t}(t)$ covers an angle of at least π. If $\mathbf{p} - \mathbf{q} = \alpha\mathbf{e}_1 + \beta\mathbf{e}_3$ with $\alpha, \beta \geq 0$, then the angle between $\mathbf{t}(t)$ and the 1-axis lies within $B_1 \cup B_3$. (The other cases for $\mathbf{p} - \mathbf{q}$ are treated analogously.) Because of condition (4) this angle is smaller than π, so that $\mathbf{t}(t)$ can cover only an angle smaller than π. Thus $\mathbf{p}$ and $\mathbf{q}$ with the assumed properties cannot exist and $\mathbf{c}^0$ is injective. $\square$

Remark 1. *Note that only the last condition in Theorem 1 depends on the valence of the irregularity.*

For the midpoint scheme of order 2 on triangular nets both conditions (4) and (5) are fulfilled for $n \geq 4$, see (1) and (3). Because the divided ∇_k-difference schemes of this particular scheme use only convex combinations and the points of $\mathcal{X}$ are pairwise distinct, Theorem 1 implies:

Theorem 2. *The midpoint scheme of order* 2 *on triangular nets generates for (almost) all initial control nets regular surfaces with continuous normal everywhere.*

Because the analysis technique in Theorem 1 makes assumptions on the subdivision scheme and on the control points of the characteristic map, it should apply to other schemes that lack a rigorous proof of regularity and injectivity of the characteristic map. For example, this technique applies to the subdivision schemes of [3, 5, 11] and makes the respective smoothness proofs much simpler. Furthermore, already in [14] it is used to prove the smoothness of the limit surface for a 4-3 scheme.

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