Analysis and Tuning of Subdivision Algorithms

Georg Umlauf*
Computer Science Department
University of Kaiserslautern
Germany

Abstract

This paper surveys the current state in analyzing and tuning of subdivision algorithms. These two aspects of subdivision algorithms are very much intertwined with the differential geometry of the subdivision surface. This paper deals with the interconnection of these different aspects of subdivision algorithms and surfaces.

The principal idea for the analysis of a subdivision algorithm dates back to the late 70s although the overall technique is only well understood since the early 90s. Most subdivision algorithms are analyzed today but the proofs involve time consuming computations. Only recently simple proofs for a certain class of subdivision algorithms were developed that are based on geometric reasoning. This allows for easier smoothness proofs for new developed or tuned subdivision algorithms.

The analysis of the classical algorithms, such Catmull-Clark, Loop, etc., shows that the subdivision surfaces at the extraordinary points are not as smooth as the rest of the surface. It was also shown that the subdivision surfaces of these classical algorithms cannot model certain basic shapes. One way to tune a stationary subdivision algorithms to overcome this problem is to drop the stationarity while at the same time using the smoothness proof of the stationary algorithms.

Keywords: subdivision algorithms, smoothness, shape of subdivision surfaces

1 Introduction

In computer graphics subdivision algorithms are a well established technique to represent and compute curves and surfaces of arbitrary topology. The first subdivision algorithm for the generation of curves was described in the 19th century [Haase 1870; Boehm 1993]: the de Casteljau algorithm. Other early references date back to the 40s of the 20th century [Rahm 1947]. The breakthrough of subdivision algorithms in computer graphics came with an article by G.M. Chaikin [Chaikin 1974] for the fast computation of quadratic spline curves. Chaikin's algorithm was immediately generalized to arbitrary degrees [Lane and Riesenfeld 1980] and to surfaces of arbitrary topology [Catmull and Clark 1978]. This triggered the development of further subdivision algorithms for surfaces of arbitrary topology for different purposes which still continues today, e.g. [Loop 1987; Dyn et al. 1990; Kobbelt 2000; Velho and Zorin 2001; Peters and Shiue 2004].

*umlauf@informatik.uni-kl.de

At the same time when the Catmull-Clark algorithm was developed people got interested in the smoothness of so-called subdivision surfaces, i.e. surfaces that can be generated by subdivision algorithms [Doo and Sabin 1978]. Only after more than a decade U. Reif proved necessary and sufficient conditions for a subdivision surface to have a continuous normal everywhere [Reif 1993; Reif 1995]. These conditions where later generalized to arbitrary smoothness orders [Prautzsch 1998] and more general subdivision algorithms [Reif 1998a; Zorin 1998].

With these conditions proofs for normal-continuity for most known subdivision algorithms could be constructed. These proofs use either a polynomial representation of the subdivision surface [Peters and Reif 1997; Peters and Reif 1998; Habib and Warren 1999; Umlauf 2000] or extensive numerical computations [Velho and Zorin 2001; Zorin 2000; Zorin and Schröder 2001]. Some authors give evidence by visual inspection [Kobbelt 2000; Stam and Loop 2003]. Only recently some geometric criteria were developed that allow for a simple proof of normal-continuity for a certain class of subdivision surfaces [Umlauf 2003].

The proofs of normal-continuity for some of the known subdivision algorithms [Peters and Reif 1998; Umlauf 2000] verified the remarkable observation that subdivision surfaces have isolated so-called extraordinary points, where the smoothness is of lower order than for all other points on the surface. For example the Catmull-Clark algorithm generates piecewise bicubic C^2-surfaces everywhere except at the extraordinary points where the surfaces have diverging curvature. Although there exist subdivision algorithms that generate curvature continuous surfaces [Prautzsch 1997; Prautzsch and Umlauf 1998a; Prautzsch and Umlauf 1998b; Reif 1998b] or surfaces with bounded curvature at the extraordinary points [Sabin 1991; Holt 1996; Peters and Umlauf 2001; Loop 2002] these algorithms are more of academic interest. The reason for the strange behavior at the extraordinary points is the low polynomial degree of the subdivision surfaces [Reif 1996; Prautzsch and Reif 1999; Peters and Umlauf 2000].

Under certain assumptions it is not even possible to tune the algorithms of Catmull-Clark and Loop to generate surfaces with bounded Gaussian curvature of arbitrary sign at the extraordinary points [Peters and Reif 2004; Karciauskas et al. 2004]. Nevertheless, if some of the assumptions are dropped, it is possible to tune the algorithms of Catmull-Clark and Loop to generate surfaces with bounded curvature and arbitrary shape.

In order to survey the differential geometric properties of subdivision surfaces and to describe tuning methods Sections 2 and 3 give a brief introduction to the usual setting and notation of subdivision algorithms for regular and arbitrary control nets, respectively. Section 4 surveys the well-known smoothness conditions and describes a geometric analysis technique. In Section 5 the shape defects are discussed and possible tuning methods are presented in Section 6.

2 Subdivision algorithms for surfaces

Subdivision algorithms operate on polygons or nets of control points. That means a subdivision algorithm takes an initial control net $\mathscr{C}_0$ and computes a refined control net $\mathscr{C}_1$. We assume here that

Assumption 2.1 *the control points of $\mathscr{C}_1$ are computed as finite affine combinations of control points of $\mathscr{C}_0$.*

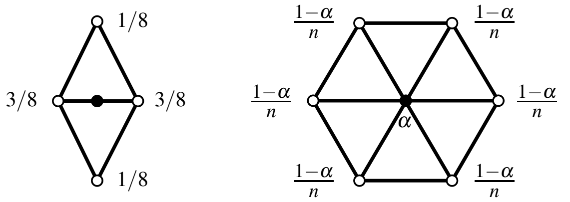

Figure 1: The stencils of the algorithm of Loop at a vertex of valence n, where $\alpha = 3/8 + (3/8 + \cos(2\pi/n)/4)^2$.

These finite affine combinations are called *stencils* and the number of non-vanishing weights in a stencil is its *size*. An example for the stencils of the algorithm of Loop is shown in Figure 1. Of course, this can be iterated to generate a sequence of ever finer control nets $(\mathscr{C}_m)_{m\geq 0}$. This sequence will eventually converge towards a limit surface $\mathbf{s}$, the so-called *subdivision surface*. An example is shown in Figure 2 for the algorithm of Loop [Loop 1987]. This subdivision algorithm is an example for a class of subdivision algorithms satisfying the following assumption:

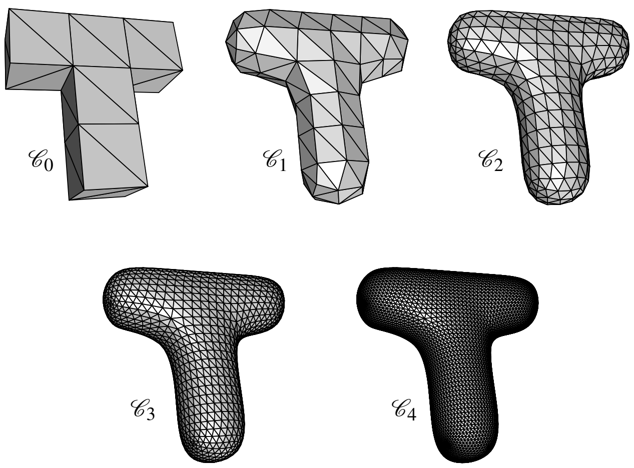

Figure 2: The initial triangular net $\mathscr{C}_0$ (top left) and the nets $\mathscr{C}_1, \ldots, \mathscr{C}_4$ of the first four iteration steps of the algorithm of Loop.

Assumption 2.2 *A subdivision algorithm is called* stationary *if it uses the same stencils in every iteration.*

Regular quadrilateral control nets contain only interior vertices of valence 4, i.e. interior vertices with 4 emanating edges. *Regular triangular control nets* contain only interior vertices with valence 6. For these nets the standard analysis tools, see e.g. [Cavaretta et al. 1991; Dyn 1992], can be used to explore the properties of the subdivision surface. The control points of a regular quadrilateral or triangular control net can be arranged in a matrix $\mathscr{C}_m = [\mathbf{c}_\mathbf{i}^m]_\mathbf{i}$ with $\mathbf{i} = [i\, j] \in \mathbb{Z}^2$. For a binary subdivision algorithm the stencils can be described by the so-called *refinement equation*

$$\mathbf{c}_\mathbf{i}^{m+1} = \sum_{\mathbf{j}\in\mathbb{Z}^2} \mathbf{c}_\mathbf{j}^m \alpha_{\mathbf{i}-2\mathbf{j}}, \quad \alpha_j \in \mathbb{R}.$$

Note that the refinement equation combines four different stencils for $\mathbf{i} \in 2\mathbb{Z}^2, 2\mathbb{Z}^2+\mathbf{e}_1, 2\mathbb{Z}^2+\mathbf{e}_2, 2\mathbb{Z}^2+\mathbf{e}_3$, where $\mathbf{e}_1 = [1\,0], \mathbf{e}_2 = [0\,1]$ and $\mathbf{e}_3 = \mathbf{e}_1 + \mathbf{e}_2$. Then, from differences of control points in direction $\mathbf{e}_k, k = 1,2,3$,

$$\nabla_k \mathscr{C}_m = [\mathbf{c}_\mathbf{i}^m - \mathbf{c}_{\mathbf{i}-\mathbf{e}_k}^m]_\mathbf{i}, \quad k = 1,2,3,$$

convergence towards a smooth subdivision surface can be concluded:

Theorem 2.3 ([Dyn 1992]) *The sequence $\mathscr{C}_m$ converges uniformly to a uniform continuous subdivision surface* $\mathbf{s}$ *if and only if the differences $\nabla_k \mathscr{C}_m$ converge uniformly to zero, where $k = 1,2$ for quadrilateral and $k = 1,2,3$ for triangular control nets.*

If the stencils of a subdivision algorithm are factorizable with respect to a direction $\mathbf{v} \in \mathbb{Z}^2$, i.e.

$$\sum_{\mathbf{i}\in\mathbb{Z}^2} \alpha_\mathbf{i} \mathbf{x}^\mathbf{i} / (1 + \mathbf{x}^\mathbf{v}), \quad \mathbf{x}^\mathbf{v} = x_1^{v_1} x_2^{v_2},$$

is a polynomial, there exists a subdivision algorithm to compute the sequence of differences $\nabla_\mathbf{v} \mathscr{C}_m$. This subdivision algorithm is the so-called $\nabla_\mathbf{v}$-*difference scheme*, which maps $\nabla_\mathbf{v} \mathscr{C}_m$ onto $\nabla_\mathbf{v} \mathscr{C}_{m+1}$.

Remark 2.4 *If the ∇_k-difference schemes satisfy Theorem 2.3 for the control nets $2^m \nabla_k \mathscr{C}_m$, the subdivision surface is C^1.*

In case of convergence the differences also provide directional derivatives:

Theorem 2.5 ([Dyn 1992]) *If the sequences $\mathscr{C}_m$ and*

$$2^m \nabla_\mathbf{v} \mathscr{C}_m = 2^m [\mathbf{c}_\mathbf{i}^m - \mathbf{c}_{\mathbf{i}-\mathbf{v}}^m], \quad \mathbf{v} \in \mathbb{Z}^2,$$

converge uniformly to continuous functions $\mathbf{s}$ *and* $\mathbf{d}$*, then* $\mathbf{d}$ *is the directional derivative of* $\mathbf{s}$ *with respect to* $\mathbf{v}$.

3 Subdivision algorithms for surfaces of arbitrary topology

For the modeling of surfaces of arbitrary topology control nets with an arbitrary topology must be used. The respective control nets contain *irregularities* of order n which are either n-sided facets or n-valent vertices with arbitrary $n \geq 3$. The respective subdivision algorithms are generalizations of subdivision algorithms for regular nets [Catmull and Clark 1978; Dyn et al. 1990; Loop 1987]. Thus, in net regions without irregularities the subdivision surfaces are known, see Section 2.

The number of vertices in the sequence of control nets $(\mathscr{C}_m)_{m\geq 0}$ grows. The subdivision algorithms are usually designed to generate no additional irregularities and the number of irregularities is constant in $(\mathscr{C}_m)_{m\geq 0}$. Therefore, irregularities get separated by growing regular regions. Because of Assumption 2.1 there exists an $m \geq 0$ such that irregularities in $\mathscr{C}_m$ do not influence each other in subsequent control nets $\mathscr{C}_k$ with $k > m$. As a consequence only the subdivision surfaces for initial control nets $\mathscr{C}_0$ with a single irregularity of order n need to be analyzed.

Let $\mathscr{C}_0$ be a control net with a single irregularity of order n such that the regular regions define a complete surface ring $\mathbf{s}_0$ around the irregularity. An example is shown in Figure 3. Subsequently, the

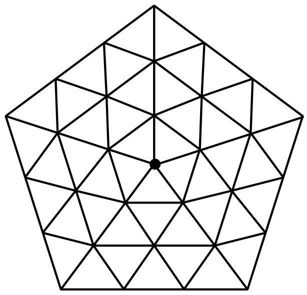

Figure 3: An initial triangular net $\mathscr{C}_0$ with an irregular vertex of valence $n = 5$ (marked by •) surrounded by three rings of regular vertices for the algorithm of Loop.

regular region of a control net $\mathscr{C}_m$ defines a surface $\mathbf{s}_m$ that contains the preceding surface $\mathbf{s}_{m-1}$

$$\mathbf{s}_{m-1} \subset \mathbf{s}_m, \quad m \geq 1.$$

The union of all surfaces $\mathbf{s}_m$ is the subdivision surface

$$\mathbf{s} = \bigcup_{m \geq 0} \mathbf{s}_m.$$

Thus, in the m-th step of the iteration the surface ring

$$\mathbf{r}_m = \mathbf{s}_m \setminus \mathbf{s}_{m-1}, \quad m \geq 1,$$

is added to the surface. Examples for the surfaces $\mathbf{s}_0, \mathbf{s}_1$ and the surface rings $\mathbf{r}_1, \mathbf{r}_2$ are shown schematically in Figure 4.

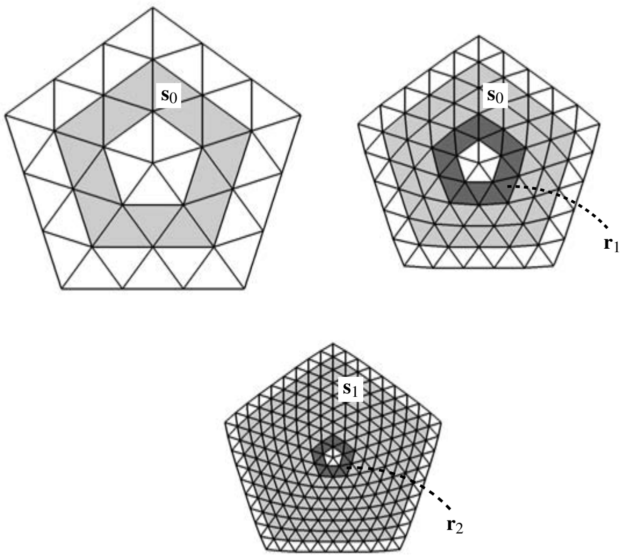

Figure 4: The surfaces $\mathbf{s}_0, \mathbf{s}_1$ (light gray) and the surface rings $\mathbf{r}_1, \mathbf{r}_2$ (dark gray) generated by the algorithm of Loop from an initial control net $\mathscr{C}_0$ with one irregular vertex of valence $n = 5$ (schematically).

Each surface ring $\mathbf{r}_m, m \geq 1$, is determined by a regular subnet $\mathscr{R}_m$ of the control net $\mathscr{C}_m$. All control nets $\mathscr{R}_m, m \geq 1$, contain the same number l of control points that are connected in the same way

$$\mathscr{R}_m = \begin{bmatrix} \mathbf{c}_0^m \\ \vdots \\ \mathbf{c}_{l-1}^m \end{bmatrix} \in \mathbb{R}^{l \times 3}.$$

Because the subdivision algorithm uses only finite affine combinations, there is a square $l \times l$-matrix S relating the control net $\mathscr{R}_{m+1}$ to the control net $\mathscr{R}_m$

$$\mathscr{R}_{m+1} = S \cdot \mathscr{R}_m.$$

The matrix S is the so-called *subdivision matrix*.

The numbering of the control points $\mathbf{c}_i^m$ is arbitrary. Nevertheless, choosing a special numbering can simplify further computations. A possible numbering for the control points of $\mathscr{R}_m$ for the algorithm of Loop is shown in Figure 5.

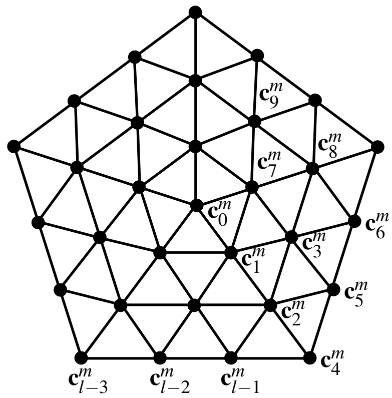

Figure 5: A numbering for the control points of $\mathscr{C}_m$ for the algorithm of Loop.

4 The analysis of subdivision surfaces

The analysis of the subdivision surfaces of a given subdivision algorithm depends on the eigenvalues and eigenvectors of the subdivision matrix [Doo and Sabin 1978]. The computation of eigenvalues and eigenvectors simplifies if only symmetric subdivision algorithms are considered:

Assumption 4.1 *A subdivision algorithm is called* symmetric *if there is a numbering of the control points of $\mathscr{R}_m$ such that S has a block-circulant structure*

$$S = \begin{bmatrix} S_0 & S_1 & \cdots & S_{n-1} \\ S_{n-1} & S_0 & \cdots & S_{n-2} \\ \vdots & & \ddots & \vdots \\ S_1 & \cdots & S_{n-1} & S_0 \end{bmatrix}$$

for an irregularity of order n.

Thus, S is similar to a block-diagonal matrix $\widehat{S}$ via discrete Fourier transform

$$\widehat{S} = \begin{bmatrix} \widehat{S}_0 & & \\ & \ddots & \\ & & \widehat{S}_{n-1} \end{bmatrix},$$

where

$$\widehat{S}_i = \sum_{j=0}^{n-1} \omega_n^{-ij} S_j, \quad \omega_n = \exp(2\pi\sqrt{-1}/n), i = 0, \ldots, n-1.$$

Note that the matrices S and $\widehat{S}$ have the same eigenvalues.

For simplicity assume that

Assumption 4.2 *S is diagonalizable with eigenvalues λ_i such that*

$$|\lambda_0| \geq |\lambda_1| \geq \cdots \geq |\lambda_{\ell-1}| \geq 0.$$

The corresponding eigenvectors are denoted by $\mathbf{v}_i$, i.e. $S\mathbf{v}_i = \lambda_i \mathbf{v}_i, i = 0, \ldots, \ell-1$. The eigenvectors $\mathbf{v}_i$ represent scalar control nets whose regular regions define scalar surface rings σ_i. The σ_i are also called *eigenfunctions*. For subdivision algorithms with more general subdivision matrix S the reader is referred to [Reif 1998a; Zorin 1998].

The subnets $\mathscr{R}_m$ can be expressed in terms of eigenvectors

$$\mathscr{R}_m = \sum_{i=0}^{\ell-1} \lambda_i^m \mathbf{v}_i \mathbf{b}_i, \quad \mathbf{b}_i \in \mathbb{R}^3.$$

Analogously, the surface rings $\mathbf{r}_m$ can be expressed in terms of eigenfunctions using the same coefficients $\mathbf{b}_i$.

Note that $1 = \lambda_0 > |\lambda_1|$ is a simple eigenvalue of S with eigenvector $[1, \ldots, 1]^t$. Therefore, the sequence $\mathscr{R}_m$ converges to $\mathbf{b}_0$ for $m \to \infty$. This is the limit point $\mathbf{s}_\infty$ on the subdivision surface $\mathbf{s}$, the so-called *extraordinary point*.

A translation of the subdivision surface yields $\mathbf{b}_0 = \mathbf{0}$. Under the additional assumption that

Assumption 4.3 $1 = \lambda_0 > \lambda := \lambda_1 = \lambda_2 > |\lambda_3|$ *and the so-called* sub-dominant eigenvalue λ *is real with algebraic and geometric multiplicity two*

the control nets $\mathscr{R}_m/\lambda^m$ converge to

$$\mathscr{R}_m/\lambda^m = \sum_{i=1}^{\ell-1} (\lambda_i/\lambda)^m \mathbf{v}_i \mathbf{b}_i \overset{m\to\infty}{\longrightarrow} \mathbf{v}_1 \mathbf{b}_1 + \mathbf{v}_2 \mathbf{b}_2.$$

This implies that the tangent plane at $\mathbf{s}_\infty$ is spanned by $\mathbf{n} = (\mathbf{b}_1 \times \mathbf{b}_2)/\|\mathbf{b}_1 \times \mathbf{b}_2\|$. Moreover, the subdivision surface $\mathbf{s}$ is regular with continuous normal at $\mathbf{s}_\infty$ under the following conditions:

Theorem 4.4 ([Reif 1995]) *Under Assumptions 4.2 and 4.3 the subdivision surfaces* $\mathbf{s}$ *are regular C^1 surfaces for almost all initial control nets $\mathscr{C}_0$ if the* characteristic map $\mathbf{c} = (\sigma_1, \sigma_2)$ *is regular and injective.*

Note that the characteristic map is the planar surface ring defined by the regular regions of the control net $[\mathbf{v}_1, \mathbf{v}_2]$ and does not depend on the initial control net $\mathscr{C}_0$. This theorem has been extended to arbitrary smoothness order $C^r, r \geq 0$, by [Prautzsch 1998]. Examples of the characteristic map of the algorithm of Loop and of the algorithm of Catmull-Clark for an irregular vertex of valence 7 are shown in Figure 6.

Remark 4.5 *If $\mathbf{b}_1$ and $\mathbf{b}_2$ are linear dependent or vanish Theorem 4.4 does not hold.*

Remark 4.6 *If $\widehat{\mathbf{v}}$ is an eigenvector of $\widehat{S}_i, i = 0, \ldots, n-1$, corresponding to eigenvalue μ, the eigenvector $\mathbf{v}$ of S corresponding to eigenvalue μ is*

$$\mathbf{v} = \begin{bmatrix} \widehat{\mathbf{v}} \\ \omega_n^i \widehat{\mathbf{v}} \\ \vdots \\ \omega_n^{i(n-1)} \widehat{\mathbf{v}} \end{bmatrix}.$$

This implies that the characteristic map $\mathbf{c}$ is not injective, if λ is not an eigenvalue of $\widehat{S}_1$ or $\widehat{S}_{n-1}$.

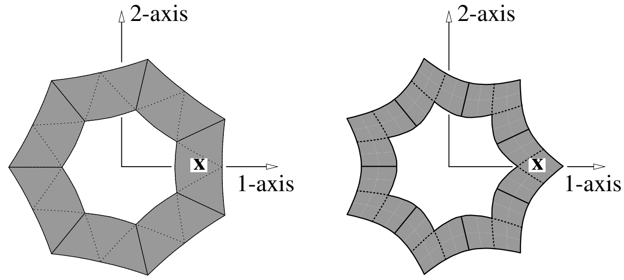

Figure 6: The characteristic map of the algorithm of Loop (left) and the algorithm of Catmull-Clark (right) for an irregular vertex of valence 7. The dashed lines show the boundaries of the polynomial pieces. The solid lines show the boundaries of the segments, especially the normalized segment **x**.

Because eigenvectors are unique up to scaling and because of rotational symmetry (cf. Remark 4.6) only one so-called *normalized segment* $\mathbf{x}$ of the characteristic map $\mathbf{c}$ needs to be analyzed in order to prove regularity and injectivity of $\mathbf{c}$. Such a normalized segment is defined by a subnet $\mathscr{X}$ of $\mathscr{R}_1$, that is symmetric with respect to the 1-axis, such that $\mathbf{c}$ can be constructed of rotated versions of $\mathbf{x}$ (cf. [Peters and Reif 1998]). Normalized segments $\mathbf{x}$ of the characteristic maps of the algorithm of Catmull-Clark and of Loop are shown in Figure 6.

Note that up to this point all techniques to prove the smoothness of the limit surface at an extraordinary point coincide, e.g. [Peters and Reif 1998; Umlauf 2000; Zorin 2000; Kobbelt 2000]. To simplify the subsequent analysis assume further that

Assumption 4.7 *the stencils of the ∇_k-difference schemes, $k = 1, 2, 3$, that map $\nabla_k \mathscr{C}_m$ to $\nabla_k \mathscr{C}_{m+1}$ have only non-negative entries that sum to one.*

If the ∇_k-differences of $\mathscr{X}$ span a pointed cone Γ_k, Theorem 2.5 and Assumption 4.7 imply that the directional derivatives of $\mathbf{x}$ with respect to $\mathbf{e}_k$ also lie in this cone. Thus, if two cones do not intersect and do not contain vectors with opposing orientations, the cross-product of any pair of vectors from these cones does not vanish. This implies that the cross-product of any pair of directional derivatives, i.e. any normal, does not vanish, too. So the respective map is regular. For injectivity consider an open curve in the domain of $\mathbf{c}$ that is mapped to a closed curve on $\mathbf{c}$. Its tangent is a convex combination of certain directional derivatives and must cover an angle of at least 180°. If the respective cones do not contain vectors with opposing orientations such a tangent cannot exist. This implies that any open curve in the domain of $\mathbf{c}$ is mapped to an open curve on $\mathbf{c}$. These arguments are made precise in the following theorem that simplifies the analysis of the characteristic map:

Theorem 4.8 ([Umlauf 2003]) *Under the additional Assumption 4.7 the characteristic map of a symmetric subdivision algorithm is regular and injective, if*

1. *none of the ∇_k-differences of $\mathscr{X}$ vanish for $k \in \{1,s\}$,*

2. *$\Gamma_1 \cup \Gamma_s$ lies in an open half-plane and*

3. *$\Gamma_1 \cap \Gamma_s = \emptyset$,*

where $s = 2$ for quadrilateral and $s = 3$ for triangular control nets.

Based on this theorem the analysis of a subdivision surface proceeds in three steps as follows:

Procedure 4.9

1. *Verify Theorem 2.3 and Remark 2.4 for regular control nets.*

2. *Compute the subdivision matrix and its eigenvalues and eigenvectors to verify Assumptions 4.2 and 4.3 and Remark 4.6.*

3. *Verify Assumption 4.7 and Theorem 4.8.*

Several constructions for free form surfaces have been shown to generate smooth surfaces using the above analysis procedure [Umlauf 2003; Peters and Shiue 2004; Prautzsch and Umlauf 2005].

Remark 4.10 *Steps 1. and 2. of Procedure 4.9 have to be performed for any of the analysis procedures mentioned in the Introduction, too. The first part of 3. is usually a byproduct of step 1.*

5 The shape of subdivision surfaces

One aspect of the shape of a subdivision surface are artifacts. These are features of a subdivision algorithm which are unexpected or undesired and cannot be controlled by the position of the control points [Sabin and Barthe 2002]. The so-called *polar artifacts* of a subdivision surface at an extraordinary point are caused by the sub-dominant eigenvalue λ which gets larger for the classical subdivision algorithms as n increases. This means that the facets of a control net shrink slower in the vicinity of an irregularity as in regular regions. Although, this does not influence the subdivision surface the distribution of facets sizes becomes inhomogeneous, making the control net look distorted. Note that the normal at the extraordinary point is only loosely coupled to the control net [Ginkel et al. 2005].

To avoid polar artifacts the sub-dominant eigenvalue λ should be for any valence n the same as the sub-dominant eigenvalue for regular control nets. An example for a polar artifact is shown in Figure 7 for the algorithm of Loop at an irregular vertex of valence $n = 15$.

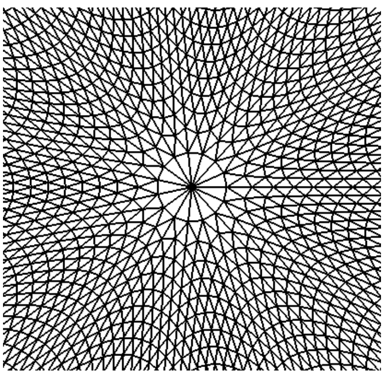

Figure 7: Polar artifacts for the algorithm of Loop at an irregular vertex of valence $n = 15$.

Polar artifacts and the convergence of a subdivision algorithm to a regular subdivision surface with continuous normal are controlled by the dominant and sub-dominant eigenvalues $\lambda_0 > \lambda_1 = \lambda_2$ of the subdivision matrix S and their corresponding eigenvectors. These are low order terms. It seems natural that the curvature of a subdivision surface at an extraordinary point is controlled by subsequent (higher order) eigenvalues λ_k, $k \geq 3$. Therefore, it is assumed that

Assumption 5.1 *the* subsub-dominant *eigenvalue $\mu := \lambda_3$ has algebraic and geometric multiplicity $M-2, M \geq 3$, such that $1 > \lambda > \mu = \lambda_3 = \cdots = \lambda_M > 0$.*

In fact, a coarse classification of the behavior of the Gaussian curvature gives the following theorem:

Theorem 5.2 ([Peters and Umlauf 2000; Peters and Reif 2004]) *The Gaussian curvature of a regular C^1-subdivision surface* $\mathbf{s}$ *at an extraordinary point $\mathbf{s}_\infty$ is*

1. *zero, if $\mu/\lambda^2 < 1$,*

2. *divergent, if $\mu/\lambda^2 > 1$,*

3. *bounded, if $\mu/\lambda^2 = 1$.*

Thus, a subdivision surface is C^2, if either $\mathbf{s}_\infty$ is a flat point or if $\mathbf{s}$ is locally at $\mathbf{s}_\infty$ a quadratic function over the tangent plane:

Theorem 5.3 ([Prautzsch 1998]) *A regular C^1-subdivision surface* $\mathbf{s}$ *is C^2 and not flat at $\mathbf{s}_\infty$ for almost all initial control nets, if and only if $\mu = \lambda^2$ and $\sigma_k \in span\{\sigma_1^2, \sigma_1\sigma_2, \sigma_2^2\}$ for $k = 3, \ldots, M$.*

This theorem implies that the minimal degree of a polynomial C^2-subdivision surface is at least bi-degree 6 or total degree 8 [Reif 1996; Prautzsch and Reif 1999; Peters and Umlauf 2000].

Remark 5.4 ([Peters and Reif 2004]) *Theorem 5.3 and Remark 4.6 imply that a subdivision surface can only be C^2, if μ is an eigenvalue of $\widehat{S}_i$ with $i \in \{0, 2, n-2\}$.*

The known C^2-subdivision surfaces [Prautzsch 1997; Reif 1998b] are not used in practice, like e.g. computer graphics or CAD. The reason for this might be that the algorithms of Loop and of Catmull-Clark are simpler to implement and the surface quality is sufficient for computer graphics applications. Nevertheless, the surface quality of the classical subdivision algorithms is not sufficient for CAD applications, because they do not allow arbitrary shapes.

In order to study the shape of the subdivision surface a local coordinate system in the tangent plane at the extraordinary point can be used

$$\mathbf{s}_\infty = \mathbf{b}_0 = \mathbf{0}, \quad \mathbf{e}_1 = \mathbf{b}_1/\|\mathbf{b}_1\|, \quad \mathbf{e}_2 = \mathbf{n} \times \mathbf{e}_1, \quad \mathbf{e}_3 = \mathbf{n}.$$

Thus, $[\mathbf{e}_1, \mathbf{e}_2]^t = L \cdot [\mathbf{b}_1, \mathbf{b}_2]^t$ with a 2×2 matrix L. The influence of the subsub-dominant eigenvectors can then be described by the so-called *central surface* $\mathbf{s}_c$ (cf. [Peters and Reif 2004])

$$\mathbf{s}_c = (\mathbf{s}_c^1, \mathbf{s}_c^2, \mathbf{s}_c^3), \quad (\mathbf{s}_c^1, \mathbf{s}_c^2) = \mathbf{c} \cdot L, \quad \mathbf{s}_c^3 = \sum_{i=3}^{M} \sigma_i \, \mathbf{n}\mathbf{b}_i.$$

Note that the central surface $\mathbf{s}_c$ depends on the initial control net $\mathscr{C}_0$ and contains information about higher order terms. Thus, it can be used to identify higher order artifacts depending on eigenvalues λ_i for $i \geq 3$ influencing the shape of a subdivision surface.

An extraordinary point $\mathbf{s}_\infty$ on a regular C^1-subdivision surface $\mathbf{s}$ is called *convex*, if $\mathbf{s}$ intersects its tangent plane in a sufficiently small neighborhood of $\mathbf{s}_\infty$ only in $\mathbf{s}_\infty$. The sign of $\mathbf{s}_c^3$ gives information about the convexity of an extraordinary point:

Theorem 5.5 ([Peters and Reif 2004]) *An extraordinary point* $\mathbf{s}_\infty$ *on a regular* C^1*-subdivision surface* $\mathbf{s}$ *is convex, if* $\mathbf{s}_c^3 < 0$ *or* $\mathbf{s}_c^3 > 0$. *If* $\mathbf{s}_c^3$ *changes sign, then* $\mathbf{s}_\infty$ *is not convex.*

This theorem in combination with Remark 4.6 implies that certain constellations of eigenvectors are necessary for arbitrary shapes:

Theorem 5.6 ([Peters and Reif 2004])

1. *If* $\mathbf{b}_i \neq \mathbf{0}$ *for at least one* $i \in \{3,\ldots,M\}$ *the corresponding extraordinary point* $\mathbf{s}_\infty$ *is not convex, if* μ *is not an eigenvalue of* $\widehat{S}_0$.
2. *If* $\mathbf{b}_i \neq \mathbf{0}$ *for at least one* $i \in \{3,\ldots,M\}$ *and* $\mathbf{b}_i = \mathbf{0}$ *for all* $i > M$, *the corresponding extraordinary point* $\mathbf{s}_\infty$ *is always convex, if* μ *is not an eigenvalue of* $\widehat{S}_i$ *for* $i \in \{2,\ldots,n-2\}$.

Therefore, a subdivision algorithm designed to generate high-quality surfaces should have a triple subsub-dominant eigenvalue corresponding to $\widehat{S}_0, \widehat{S}_2$ and $\widehat{S}_{n-2}$. The classical subdivision algorithms of Loop and Catmull-Clark do not have this property for $n \neq 6$ and $n \neq 4$, respectively [Karciauskas et al. 2004]. Thus, the corresponding subdivision surfaces cannot guarantee convexity or a saddle if that is intended.

An extraordinary point $\mathbf{s}_\infty$ on a regular C^1-subdivision surface $\mathbf{s}$ with Gaussian curvature K is called *elliptic*, if $K > 0$ in a sufficiently small neighborhood of $\mathbf{s}_\infty$. If $K < 0$, the extraordinary points $\mathbf{s}_\infty$ is called *hyperbolic* at $\mathbf{s}_\infty$. If K changes sign, the extraordinary points $\mathbf{s}_\infty$ is called *hybrid* at $\mathbf{s}_\infty$. The curvature behavior of a subdivision surface at an extraordinary point can be determined by the Gaussian curvature K_c of the central surface $\mathbf{s}_c$.

Theorem 5.7 ([Peters and Reif 2004]) *An extraordinary point* $\mathbf{s}_\infty$ *on a regular* C^1*-subdivision surface* $\mathbf{s}$ *is*

1. *elliptic, if* $K_c > 0$,
2. *hyperbolic, if* $K_c < 0$,
3. *hybrid, if* K_c *changes sign.*

In the third case of this theorem higher order artifacts occur at the extraordinary points. Thus, for high-quality subdivision surfaces hybrid shapes must be avoided. An example of a hybrid extraordinary points on a subdivision surface generated by the algorithm of Catmull-Clark is shown in Figure 8.

Figure 8: A hybrid extraordinary points on a subdivision surface generated by the algorithm of Catmull-Clark.

6 Tuning of subdivision algorithms

A subdivision algorithm suitable for CAD applications must generate C^1-subdivision surfaces with bounded non-vanishing curvature and arbitrary shape. Then, Theorems 5.2, 5.6 and 5.7 imply three constraints:

Constraint 6.1 *The subsub-dominant eigenvalue is* $\mu = \lambda^2$.

Constraint 6.2 *The subsub-dominant eigenvalue* μ *is a simple eigenvalue of* $\widehat{S}_0, \widehat{S}_2$ *and* $\widehat{S}_{n-2}$ *and no other* $\widehat{S}_i, i \in \{3,\ldots,n-3\} \cup \{1, n-1\}$.

Constraint 6.3 K_c *must not change sign.*

Thus, the general concept to tune a subdivision algorithm to fulfill Constraints 6.1 and 6.2 is based on modifying the leading terms in the eigenvector-decomposition of $\mathscr{R}_m$

$$\mathscr{R}_m = \mathbf{b}_0 + \lambda^m(\mathbf{v}_1\mathbf{b}_1 + \mathbf{v}_2\mathbf{b}_2) + \lambda_3^m\mathbf{v}_3\mathbf{b}_3 + \lambda_4^m\mathbf{v}_4\mathbf{b}_4 + \lambda_5^m\mathbf{v}_5\mathbf{b}_5 + o(\lambda_5^m).$$

The subdivision algorithm of Catmull-Clark and Loop cannot be tuned using only the free parameters of these algorithms to satisfy Constraints 6.1 and 6.2 simultaneously [Karciauskas et al. 2004]. The algorithm of Catmull-Clark can only be tuned to generate regular C^1-surfaces of arbitrary shape but without bounded curvature. The algorithm of Loop can only be tuned for some valences to generate regular C^1-surfaces with bounded curvature but not arbitrary shape.

The subdivision algorithms described in [Sabin 1991; Holt 1996] generate subdivision surfaces with bounded curvature and arbitrary sign of the Gaussian curvature by enforcing Constraints 6.1 and 6.2 on the eigenvalues $\lambda_3, \lambda_4, \lambda_5$ of the algorithms of Catmull-Clark and Loop, respectively. This is done by a judicious change of some weights of the stencils for regular regions adjacent to an irregularity. However, the eigenvectors change such that it is not clear if the subdivision surfaces satisfy all conditions of Section 4.

The subdivision algorithms described in [Prautzsch and Umlauf 1998a; Prautzsch and Umlauf 1998b] generate regular C^1-surface with flat extraordinary points. This is achieved by the following sketched procedure:

Procedure 6.4

i) *Diagonalize the subdivision matrix* S *by diagonalizing the relevant parts of the relevant* $\widehat{S}_k$: $V_k^{-1} \cdot \widehat{S}_k \cdot V_k = \operatorname{diag}(\lambda_i)$.

ii) *Change the relevant eigenvalues to* $\tilde{\lambda}_i$.

iii) *Compute the new subdivision matrix* $\widehat{\tilde{S}}_k = V_k \cdot \operatorname{diag}(\tilde{\lambda}_i) \cdot V_k^{-1}$.

This is done for the subdivision matrices of the algorithms of Catmull-Clark and Loop and the butterfly algorithm ([Dyn et al. 1990]) such that $\mu < \lambda^2$. To influence the shape of the subdivision surface an optimization procedure can be integrated to step ii) of Procedure 6.4 [Umlauf 1999]. It is clear that Procedure 6.4 can also be used to satisfy Constraints 6.1 and 6.2, see [Peters and Umlauf 2001; Loop 2002].

Because Procedure 6.4 does not change the dominant and subdominant eigenvectors, the subdivision algorithm corresponding to the new subdivision matrix $\tilde{S}$ satisfies all conditions of Section 4 if the subdivision algorithm corresponding to S does. However, some of the resulting stencils grow in size and have negative weights. For the algorithm of Loop negative weights can be avoided [Loop 2002].

The subdivision algorithm described in [Barthe and Kobbelt 2004] generates subdivision surfaces with tuned curvature behavior. In a optimization process the stencils are computed to satisfy either of the Constraints 6.1 and 6.2 or to avoid polar artifacts. The initialization is done manually and it is not clear if the subdivision surfaces are regular C^1-surfaces.

The subdivision algorithm described in [Zorin et al. 1996] generates regular C^1-surfaces and satisfies Constraints 6.1 and 6.2 and does not show polar artifacts. Although for this subdivision algorithm there is no closed form for the eigenfunctions, it is possible to prove regularity and injectivity of the characteristic map using extensive numerical computations [Zorin 1998].

None of the above tuned subdivision algorithms satisfy Constraint 6.3. There are different approaches to satisfy Constraint 6.3. All need to take the data $\mathbf{b}_i, i = 3,4,5$, into account because the central surface $\mathbf{s}_c$ depends on this data.

First, Procedure 6.4 can be used to tune the subdivision algorithm to satisfy Constraints 6.1 and 6.3 by making some of the eigenvalues of Constraint 6.2 smaller. Which of these eigenvalues is made smaller depends on the data. This can also be done gradually leading to a fading out of the eigenvalues that cause the hybrid shape. This gives rise to a non-stationary data-dependent subdivision scheme that has a non-hybrid central surface after a certain number of subdivision steps. A different approach to satisfy Constraint 6.3 using the classical subdivision algorithms is to change the $\mathbf{b}_i$ for $i = 3,4,5$. Thus, the control net is preprocessed to remove hybrid situations.

Note that these approaches have in common that the eigenvectors $\mathbf{v}_i, i = 0,\dots,5$ are not changed. So the characteristic map and the range for $\mathbf{b}_i, i = 3,4,5$, where $\mathbf{s}_c$ becomes hybrid do not change.

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