Solving Linear Systems with Polynomial Parameter Dependency with Application to the Verified Solution of Problems in Structural Mechanics

Jürgen Garloff, Evgenija D. Popova, and Andrew P. Smith

Abstract We give a short survey on methods for the enclosure of the solution set of a system of linear equations where the coefficients of the matrix and the right hand side depend on parameters varying within given intervals. Then we present a hybrid method for finding such an enclosure in the case that the dependency is polynomial or rational. A general-purpose parametric fixed-point iteration is combined with efficient tools for range enclosure based on the Bernstein expansion of multivariate polynomials. We discuss applications of the general-purpose parametric method to linear systems obtained by standard finite element analysis of mechanical structures and illustrate the efficiency of the new parametric solver.

1 Introduction

In this chapter we consider linear systems

$$A(x) \cdot s \;=\; d(x), \tag{1a}$$

where the coefficients of the $m \times m$ matrix $A(x)$ and the vector $d(x)$ are functions of n parameters $x_1, \ldots, x_n$ varying within given intervals $[x_1], \ldots, [x_n]$

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$$a_{ij}(x) = a_{ij}(x_1,\dots,x_n), \quad d_i(x) = d_i(x_1,\dots,x_n), \quad i,j = 1,\dots,m, \tag{1b}$$
$$x \in [x] = ([x_1],\dots,[x_n])^\top. \tag{1c}$$

The set of solutions to (1a–1c), called the *parametric solution set*, is

$$\Sigma = \Sigma\,(A(x),d(x),[x]) := \{s \in \mathbb{R}^m \mid A(x)\cdot s = d(x) \text{ for some } x \in [x]\}\,. \tag{2}$$

Engineering problems that involve such parametric linear systems may stem from structural mechanics, e.g., [3, 4, 21, 26, 29, 38, 42], the design of electrical circuits [5, 6], resistive networks [13], and robust Monte Carlo simulation [17], to name but a few examples. The source of parametric uncertainty is often the lack of precise data which may result from a lack of knowledge due to, e.g., measurement imprecision or manufacturing imperfections, or an inherent variability in the parameters, e.g., physical constants are only known to within certain bounds.

The parametric solution set can be described explicitly only in very simple cases. Therefore, one attempts to find the smallest axis-aligned box in $\mathbb{R}^m$ containing Σ. Since even this set can only be found easily in some special cases, it is more practical to attempt to compute a tight outer approximation to this box.

The chapter is organized as follows. In Section 2 we introduce the basic definitions and rules of interval arithmetic, which is a fundamental tool of our approach. In this section we also compare the interval solution set with the parametric solution set and give a short overview of methods for its enclosure. In Subsection 3.1 we present a method for the enclosure of the parametric solution set, called the *parametric residual iteration method*. This method needs tight bounds on the range of multivariate functions. In the applications we will present later in this chapter the coefficient functions (1b) are polynomials or rational functions. To find the range of a multivariate polynomial, we recall in Subsection 3.2 a method which is based on the expansion of a polynomial into Bernstein polynomials, termed the *Bernstein form*. Implementation issues concerning the combination of the parametric residual iteration method with the Bernstein form are discussed in Subsection 3.3. We apply the combined approach in Section 4 to some problems of structural mechanics and draw some conclusions in Section 5. [1]

2 The Parametric Solution Set

2.1 Interval Arithmetic

Let $\mathbb{IR}$ denote the set of the compact, nonempty real intervals. The arithmetic operation $\circ \in \{+,-,\cdot,/\}$ on $\mathbb{IR}$ is defined in the following way.

If $a = [\underline{a},\overline{a}], b = [\underline{b},\overline{b}] \in \mathbb{IR}$, then

[1] Preliminary results were presented at the 2nd International Conference on Uncertainty in Structural Dynamics, Sheffield, UK, June 15–17, 2009.

$$\begin{aligned}
a+b &= [\underline{a}+\underline{b}, \overline{a}+\overline{b}],\\
a-b &= [\underline{a}-\overline{b}, \overline{a}-\underline{b}],\\
a\cdot b &= [\min\{\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}\}, \max\{\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}\}],\\
a/b &= [\min\{\underline{a}/\underline{b}, \underline{a}/\overline{b}, \overline{a}/\underline{b}, \overline{a}/\overline{b}\},\\
&\qquad \max\{\underline{a}/\underline{b}, \underline{a}/\overline{b}, \overline{a}/\underline{b}, \overline{a}/\overline{b}\}], \text{ if } 0 \notin b.
\end{aligned}$$

As a consequence of these definitions we obtain the inclusion isotonicity of the interval arithmetic operations: If $a_1, b_1 \in \mathbb{IR}$ with $a_1 \subseteq a$ and $b_1 \subseteq b$ then it holds that

$$a_1 \circ b_1 \subseteq a \circ b.$$

Note that some relations known to be true in the set $\mathbb{R}$, e.g., the distributive law, are not valid in $\mathbb{IR}$. Here we have the weaker subdistributive law

$$a\cdot(b+c) \subseteq a\cdot b + a\cdot c \text{ for } a,b,c \in \mathbb{IR}.$$

The width of an interval $a = [\underline{a}, \overline{a}]$ is defined as

$$\omega(a) = \overline{a} - \underline{a}.$$

By $\mathbb{IR}^n$ and $\mathbb{IR}^{n\times n}$ we denote the set of n-vectors and n-by-n matrices with entries in $\mathbb{IR}$, respectively. For a nonempty bounded set $\mathscr{S} \subseteq \mathbb{R}^n$, define its interval hull by $\square\mathscr{S} := [\inf\mathscr{S}, \sup\mathscr{S}] = \cap\{[s] \in \mathbb{IR}^n \mid \mathscr{S} \subseteq [s]\}$.

Where the endpoints of an interval are stored as floating-point numbers, it is necessary to use *outward rounding* in all operations, viz. the infimum is rounded down and the supremum is rounded up. In this way, interval operations deliver guaranteed results even in the presence of rounding errors with floating-point arithmetic.

Further details on arithmetic with intervals may be found in [1, 22].

2.2 The Interval Solution Set vs. the Parametric Solution Set

A system of linear interval equations is a collection of systems

$$A\cdot s \;=\; d,\; A \in [A],\; d \in [d],\; \text{where } [A] \in \mathbb{IR}^{m\times m},\; [d] \in \mathbb{IR}^m; \tag{3}$$

its solution set

$$\{s \in \mathbb{R}^m \;\mid\; \exists A \in [A],\, \exists d \in [d] : A\cdot s = d\} \tag{4}$$

is called here the *interval solution set*. There are many methods for the enclosure of the interval solution set, cf. [1, 22]. With the parametric linear system (1a) a system (3) is associated which is obtained when each entry in (1b) is replaced by an enclosure for the range of the functions a_{ij} and d_i over $[x]$. In general, the resulting interval system can be more easily solved than the parametric system. However, the dependencies between the parameters are lost and so the interval solution set is in general much larger than the parametric solution set.

2.3 Prior Work on the Parametric Solution Set

One of the earliest papers on the solution of linear systems with nonlinear parameter dependencies is [8], cf. [9]. Later works focus on the solution of systems of linear equations whose coefficient matrices enjoy a special structure. Here the interval solution set (4) is restricted in such a way that only matrices which have this special structure are considered. The restricted solution set can also often be represented as a parametric solution set (2), cf. [11] for examples and references. In the sequel we survey some methods for the enclosure of the parametric solution set which have a wider range of applicability.

A method which is applicable to parameter dependencies which can be represented as

$$A(x) \;=\; \sum_{k=1}^{n} x_k A^{(k)},\; d(x) \;=\; \sum_{k=1}^{n} x_k d^{(k)},\; A^{(k)} \in \mathbb{R}^{m\times m},\; d^{(k)} \in \mathbb{R}^{m},\; k = 1,\ldots,n,$$

was recently given in [14]. This parameter dependency covers the (skew-) symmetric, Toeplitz, and Hankel matrices and was also considered in [4].

In [13] parametric linear systems are considered where the uncertain parameters x_i enter the system (1a) in a rank-one manner. As an example, any planar resistive network has the property that with resistances associated with the parameters x_i the resulting system of linear equations, corresponding to application of Kirchhoff's laws, has a rank-one structure. Such systems are solved in [5, 6] by application of the Sherman-Morrison formula. For systems with a rank-one structure, results are obtained in [13] which allow one to decide which parameters influence components of the solution

$$s(x) \;=\; A(x)^{-1} d(x)$$

in a monotone, convex, or concave manner. Such information greatly facilitates the computation of an enclosure of the solution set (2).

Another direct method is presented in [15]. Here the coefficient functions of (1a) are assumed only to be continuous. They are approximated by linear functions in such a way that one obtains a superset of (2). An interval enclosure for this superset is determined as an interval vector whose midpoint is obtained as the solution of a certain system of linear equations. The vector which contains the (half-) widths of the component intervals is computed as the solution of another system and therefore must be positive, which is a restriction of the method.

In [36] a direct method is proposed for the case of linear parameter dependency based on inclusion theorems of Neumaier [22]. However a prerequisite for this method is that a matrix of coefficients generated from the inverse of the midpoint of the interval matrix A must be an H-matrix [22], a condition which seems to be rarely satisfied for typical problems.

The method which presently seems to have the widest range of applicability is the parametric linear solver developed by the second author (E. D. P.); see Subsection 3.1 for details.

3 Methodology

3.1 The Residual Iteration Method

In this section we consider a self-verified method for bounding the parametric solution set. This is a general-purpose method since it does not assume any particular structure among the parameter dependencies. The method originates in the inclusion theory for nonparametric problems, which is discussed in many works (cf. [34] and the literature cited therein). The basic idea of combining the Krawczyk-operator [16] and the existence test by Moore [20] is further elaborated by S. Rump [33] who proposes several improvements leading to inclusion theorems for the interval solution (4). In [34, Theorem 4.8] S. Rump gives a straightforward generalization to (1a) with affine-linear dependencies in the matrix and the right hand side. With obvious modifications, the corresponding theorems can also be applied directly to linear systems involving nonlinear dependencies between the parameters in $A(x)$ and $d(x)$. This is demonstrated in [26, 29]. The following theorem is a general formulation of the enclosure method for linear systems involving arbitrary parametric dependencies.

Theorem 1. *Consider a parametric linear system defined by (1a–1c). Let* $R \in \mathbb{R}^{m\times m}$, $[y] \in \mathbb{IR}^m$, $\tilde{s} \in \mathbb{R}^m$ *be given and define* $[z] \in \mathbb{IR}^m$, $[C] \in \mathbb{IR}^{m\times m}$ *by*

$$[z] := \square\{R\,(d(x) - A(x)\tilde{s}) \mid x \in [x]\},$$
$$[C] := \square\{I - R\cdot A(x) \mid x \in [x]\},$$

where I denotes the identity matrix. Define $[v] \in \mathbb{IR}^m$ *by means of the following Gauss-Seidel iteration*

$$1 \le i \le m \;:\; [v_i] := \{[z] + [C]\cdot([v_1],...,[v_{i-1}],[y_i],\dots,[y_m])^\top\}_i.$$

If $[v] \subseteq [y]$ *and* $[v]_i \neq [y]_i$ *for* $i = 1,\dots,n$, *then R and every matrix* $A(x)$ *with* $x \in [x]$ *are regular, and for every* $x \in [x]$ *the unique solution* $\widehat{s} = A^{-1}(x)d(x)$ *of (1a–1c) satisfies* $\widehat{s} \in \tilde{s} + [v]$.

In the examples we present in Section 4, we have chosen $R \approx A(\check{x})^{-1}$ and $\tilde{s} \approx R^{-1}d(\check{x})$, where $\check{x}$ is the midpoint of $[x]$.

The above theorem generalises [34, Theorem 4.8] by stipulating a sharp enclosure of $C(x) := I - R\cdot A(x)$ for $x \in [x]$, instead of using the interval extension $C([x])$.

A sharp enclosure of the iteration matrix $C(x)$ for $x \in [x]$ is also required by other authors (who do not refer to [34]), e.g., [4], without addressing the issue of rounding errors. Examples demonstrating the extended scope of application of the generalized inclusion theorem can be found in [23, 24, 31]. It should be noted that the above theorem provides strong regularity (cf. [23]), which is a weaker but sufficient condition for regularity of the parametric matrix.

When aiming to compute a self-verified enclosure of the solution to a parametric linear system by the above inclusion method, a fixed-point iteration scheme is proven to be very useful. A detailed presentation of the computational algorithm can be found in [26, 33].

In case of arbitrary nonlinear dependencies between the uncertain parameters, computing $[z]$ and $[C]$ in Theorem 1 requires a sharp range enclosure of nonlinear functions. This is a key problem in interval analysis and there exists a huge number of methods and techniques devoted to this problem, with no one method being universal. In this work we restrict ourselves to linear systems where the elements of $A(x)$ and $d(x)$ are rational functions of the uncertain parameters. In this case the coefficients of $z(x) = R(d(x) - A(x)\tilde{s})$ and $C(x)$ are also rational functions of x. The quality of the range enclosure of $z(x)$ will determine the sharpness of the parametric solution set enclosure. In [26] the above inclusion theorem is combined with a simple interval arithmetic technique providing inner and outer bounds for the range of monotone rational functions. The arithmetic of generalised (proper and improper) intervals is considered as an intermediate computational tool for eliminating the dependency problem in range computation and for obtaining inner estimations by outwardly rounded interval arithmetic. Since this methodology is not efficient in the general case of non-monotone rational functions, in this work we combine the parametric fixed-point iteration with range enclosing tools based on the Bernstein expansion of multivariate polynomials.

3.2 Bernstein Enclosure of Polynomial Ranges

In this section we recall some properties of the Bernstein expansion which are fundamental to our approach, cf. [2, 10, 41] and the references therein.

Firstly, some notation is introduced. We define multiindices $i = (i_1, \ldots, i_n)^T$ as vectors, where the n components are nonnegative integers. The vector 0 denotes the multiindex with all components equal to 0. Comparisons are used entrywise. Also the arithmetic operators on multiindices are defined componentwise such that $i \odot l := (i_1 \odot l_1, \ldots, i_n \odot l_n)^T$, for $\odot = +, -, \times$, and $/$ (with $l > 0$). For instance, i/l, $0 \leq i \leq l$, defines the Greville abscissae. For $x \in \mathbb{R}^n$ its monomials are

$$x^i := \prod_{\mu=1}^{n} x_\mu^{i_\mu} . \tag{5}$$

For the n-fold sum we use the notation

$$\sum_{i=0}^{l} := \sum_{i_1=0}^{l_1} \ldots \sum_{i_n=0}^{l_n} . \tag{6}$$

The generalised binomial coefficient is defined by

$$\binom{l}{i} := \prod_{\mu=1}^{n} \binom{l_\mu}{i_\mu}. \tag{7}$$

For reasons of familiarity, the Bernstein coefficients are denoted by b_i; this should not be confused with components of the right hand side vector b of (1a). Hereafter, a reference to the latter will be made explicit.

3.2.1 The Bernstein Form

An n-variate polynomial p,

$$p(x) = \sum_{i=0}^{l} a_i x^i, \quad x = (x_1, \ldots, x_n), \tag{8}$$

can be represented over

$$[x] := [\underline{x}_1, \overline{x}_1] \times \ldots \times [\underline{x}_n, \overline{x}_n], \tag{9}$$
$$\underline{x} = (\underline{x}_1, \ldots, \underline{x}_n),\ \overline{x} = (\overline{x}_1, \ldots, \overline{x}_n),$$

as

$$p(x) = \sum_{i=0}^{l} b_i B_i(x), \tag{10}$$

where B_i is the *i-th Bernstein polynomial of degree* $l = (l_1, \ldots, l_n)$,

$$B_i(x) = \binom{l}{i} \frac{(x - \underline{x})^i (\overline{x} - x)^{l-i}}{(\overline{x} - \underline{x})^l}, \tag{11}$$

and the so-called *Bernstein coefficients* b_i of the same degree are given by

$$b_i = \sum_{j=0}^{i} \frac{\binom{i}{j}}{\binom{l}{j}} (\overline{x} - \underline{x})^j \sum_{\kappa=j}^{l} \binom{\kappa}{j} \underline{x}^{\kappa - j} a_\kappa, \quad 0 \le i \le l. \tag{12}$$

The essential property of the Bernstein expansion is the *range enclosing property*, namely that the range of p over $[x]$ is contained within the interval spanned by the minimum and maximum Bernstein coefficients:

$$\min_i \{b_i\} \le p(x) \le \max_i \{b_i\},\ x \in [x]. \tag{13}$$

It is also worth noting that the values attained by the polynomial at the vertices of $[x]$ are identical to the corresponding vertex Bernstein coefficients, for example $b_0 = p(\underline{x})$ and $b_l = p(\overline{x})$. The *sharpness property* states that the lower (resp. upper) bound provided by the minimum (resp. maximum) Bernstein coefficient is sharp, i.e. there is no underestimation (resp. overestimation), if and only if this coefficient corresponds to a vertex of $[x]$.

The traditional approach (see, for example, [10, 41]) requires that all of the Bernstein coefficients are computed, and their minimum and maximum is determined. By use of an algorithm (cf. [10, 41]) which is similar to de Casteljau's algorithm (see, for example, [32]), this computation can be made efficient, with time complexity $\mathrm{O}(n\hat{l}^{n+1})$ and space complexity (equal to the number of Bernstein coefficients) $\mathrm{O}((\hat{l}+1)^n)$, where $\hat{l} = \max_{i=1}^{n} l_i$. This exponential complexity is a drawback of the traditional approach, rendering it infeasible for polynomials with moderately many (typically, 10 or more) variables.

In [37] a new method for the representation and computation of the Bernstein coefficients is presented, which is especially well suited to sparse polynomials. With this method the computational complexity typically becomes nearly linear with respect to the number of the terms in the polynomial, instead of exponential with respect to the number of variables. This improvement is obtained from the results surveyed in the following subsections. For details and examples the reader is referred to [37].

3.2.2 Bernstein Coefficients of Monomials

Let $q(x) = x^r$, $x = (x_1, \ldots, x_n)$, for some $0 \le r \le l$. Then the Bernstein coefficients of q (of degree l) over $[x]$ (9) are given by

$$b_i = \prod_{m=1}^{n} b_{i_m}^{(m)}, \tag{14}$$

where $b_{i_m}^{(m)}$ is the i_mth Bernstein coefficient (of degree l_m) of the univariate monomial x^{r_m} over $[\underline{x}_m, \overline{x}_m]$. If the box $[x]$ is restricted to a single orthant of $\mathbb{R}^n$ then the Bernstein coefficients of q over $[x]$ are monotone with respect to each variable x_j, $j = 1, \ldots, n$.

With this property, for a single-orthant box, the minimum and maximum Bernstein coefficients must occur at a vertex of the array of Bernstein coefficients. This also implies that the bounds provided by these coefficients are sharp; see the aforementioned sharpness property. Finding the minimum and maximum Bernstein coefficients is therefore straightforward; it is not necessary to explicitly compute the whole set of Bernstein coefficients. Computing the component univariate Bernstein coefficients for a multivariate monomial has time complexity $\mathrm{O}(n(\hat{l}+1)^2)$. Given the exponent r and the orthant in question, one can determine whether the monomial (and its Bernstein coefficients) is increasing or decreasing with respect to each coordinate direction, and one then merely needs to evaluate the monomial at these two vertices.

Without the single orthant assumption, monotonicity does not necessarily hold, and the problem of determining the minimum and maximum Bernstein coefficients is more complicated. For boxes which intersect two or more orthants of $\mathbb{R}^n$, the box can be bisected, and the Bernstein coefficients of each single-orthant sub-box can be computed separately.

3.2.3 The Implicit Bernstein Form

Firstly, we can observe that since the Bernstein form is linear, if a polynomial p consists of t terms, as follows,

$$p(x) = \sum_{j=1}^{t} a_{i_j} x^{i_j}, \quad 0 \leq i_j \leq l, \quad x = (x_1, \ldots, x_n), \tag{15}$$

then each Bernstein coefficient is equal to the sum of the corresponding Bernstein coefficients of each term, as follows:

$$b_i = \sum_{j=1}^{t} b_i^{(j)}, \quad 0 \leq i \leq l, \tag{16}$$

where $b_i^{(j)}$ are the Bernstein coefficients of the jth term of p. (Hereafter, a superscript in brackets specifies a particular term of the polynomial. The use of this notation to indicate a particular coordinate direction, as in the previous subsection, is no longer required.)

Therefore one may implicitly store the Bernstein coefficients of each term, and compute the Bernstein coefficients as a sum of t products, only as needed. The implicit Bernstein form thus consists of computing and storing the n sets of univariate Bernstein coefficients (one set for each component univariate monomial) for each of t terms. Computing this form has time complexity $\mathrm{O}(nt(\hat{l}+1)^2)$ and space complexity $\mathrm{O}(nt(\hat{l}+1))$, as opposed to $\mathrm{O}((\hat{l}+1)^n)$ for the explicit form. Computing a ingle Bernstein coefficient from the implicit form requires $(n+1)t-1$ arithmetic operations.

3.2.4 Determination of the Bernstein Enclosure for Polynomials

We consider the determination of the minimum Bernstein coefficient; the determination of the maximum Bernstein coefficient is analogous. For simplicity we assume that $[x]$ is restricted to a single orthant.

We wish to determine the value of the multiindex of the minimum Bernstein coefficient in each direction. In order to reduce the search space (among the $(\hat{l}+1)^n$ Bernstein coefficients) we can exploit the monotonicity of the Bernstein coefficients of monomials and employ uniqueness, monotonicity, and dominance tests, cf. [37] for details. As the examples in [37] show, it is often possible in practice to dramatically reduce the number of Bernstein coefficients that have to be computed.

3.3 Software Tools

In our implementation we have combined software for the parametric residual iteration method with software developed for the enclosure of the range of a multivariate polynomial using the implicit Bernstein form. In the case of a rational, non-polynomial parameter dependency, the ranges of the numerator and the denominator have to be bounded independently at the expense of some overestimation. In both packages interval arithmetic is used throughout, such that the resulting enclosure for the parametric solution set can be *guaranteed* also in the presence of rounding errors. The software tools for the residual iteration are implemented in a *Mathematica* [40] environment by the second author (E. D. P); this software is publically available [25, 26]. The software for the Bernstein form is written by the last author (A. P. S.) and utilises the C++ interval library `filib++` [18, 19]. Since this is a specialized software exhibiting good performance there is no reason for its re-implementation in *Mathematica*. In order to shorten the development time and to preserve the beneficial properties of both implementation environments, we have connected both software packages into a new parametric solver via the *MathLink* [40] communication protocol, for details see [11]. However, this connection leads to longer computing times compared to an implementation in a single environment. For details of the implementation and the accessibility of the combined software see [11].

4 Application to Structural Mechanics

A standard method for solving problems in structural mechanics, such as linear static problems, is the finite element method (FEM). In the case of linearised geometric displacement equations and linear elastic material behaviour, the method leads to a system of linear equations which in the presence of uncertain parameters becomes a parametric system. Treating the parametric system as an interval system and using a typical interval method for the enclosure of (4) in general results in intervals for the quantities sought which are too wide for practical purposes.

In [21, 42] the authors combine an element-by-element (EBE) formulation, where the elements are kept disassembled, with a penalty method for imposing the necessary constrains for compatibility and equilibrium, in order to reduce the overestimation in the solution intervals. This approach should be applied simultaneously with FEM and affects the construction of the global stiffness matrix and the right-hand side vector, making them larger. A non-parametric fixed-point iteration is then used to solve the parametric interval linear system. While special construction methods are applied in [21], the parametric system obtained by standard FEM applied to a structural steel frame with partially constrained connections is solved by a sequence of interval-based (but not parametric) methods [3].

In the sequel we illustrate the usage of the new parametric solver based on bounding polynomial ranges by the implicit Bernstein form as described in Subsection 3.2. The improved efficiency is demonstrated by comparing both the computing time and the quality of the enclosure of the parametric solution set for the new solver and a previous solver which is based on the combination of the parametric residual iteration with the method for bounding the range of a rational function presented in [26], cf. Subsection 3.1. To compare the quality of two enclosures $[a]$ and $[b]$ with $[a] \subseteq [b]$ we employ a measure $\mathscr{O}_\omega$ for the overestimation of $[a]$ by $[b]$ which is defined by

$$\mathscr{O}_\omega([a],[b]) := 100(1-\omega([a])/\omega([b])), \tag{17}$$

where ω denotes the width of an interval.

The following examples were run on a PC with an AMD Athlon-64 3GHz processor.

4.1 One-Bay Steel Frame

We consider a simple one-bay structural steel frame, as shown in Figure 1, which was initially studied by interval methods in [3]. Following standard practice, the authors have assembled a parametric linear system of order eight and involving eight uncertain parameters. The typical nominal parameter values and the corresponding worst case uncertainties, as proposed in [3] but converted to SI-units, are shown in Table 1. The explicit analytic form of the given system involving polynomial parameter dependencies can be found in [3, 29].

Table 1 Parameters involved in the steel frame example.

parameter		nominal value	uncertainty
Young modulus	E_b	$1.999*10^8$ kN/m^2	$\pm 2.399*10^7$ kN/m^2
	E_c	$1.999*10^8$ kN/m^2	$\pm 2.399*10^7$ kN/m^2
Second moment	I_b	$2.123*10^{-4}$ m^4	$\pm 2.123*10^{-5}$ m^4
	I_c	$1.132*10^{-4}$ m^4	$\pm 1.132*10^{-5}$ m^4
Area	A_b	$6.645*10^{-3}$ m^2	$\pm 6.645*10^{-4}$ m^2
	A_c	$9.290*10^{-3}$ m^2	$\pm 9.290*10^{-4}$ m^2
External force	H	23.600 kN	± 9.801 kN
Joint stiffness	α	$3.135*10^5$ kNm/rad	$\pm 1.429*10^5$ kNm/rad
Length	L_c	3.658 m, L_b 7.316 m	

As in [3, 29], we solved the system first with parameter uncertainties which are 1% of the values presented in the last column of Table 1.

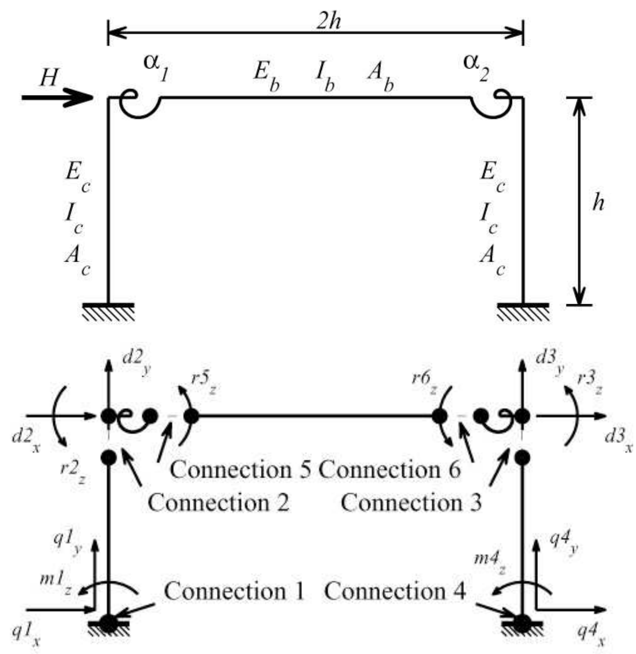

Fig. 1 One-bay structural steel frame [3].

The previous parametric solver finds an enclosure for the solution set in about 0.34 s, whereas the new solver needs only 0.05 s. The quality of the enclosures provided by both solvers is comparable. As shown in [26, 29], the solution enclosure obtained by the parametric solver is better by more than one order of magnitude than the solution enclosure obtained in [3].

Based on the runtime efficiency of the new parametric solver, we next attempt to solve the same parametric linear system for the worst case parameter uncertainties in Table 1 ranging between about 10% and 46%. Firstly, we notice that the parametric solution depends linearly on the parameter H, so that we can obtain a better solution enclosure if we solve two parametric systems with the corresponding end-points for H. Secondly, enclosures of the hull of the solution set are obtained by subdivision of the worst case parameter intervals $(E_b, E_c, I_b, I_c, A_b, A_c, \alpha)^\top$ into $(2,2,2,2,1,1,6)^\top$ subintervals of equal width, respectively. We use more subdivision with respect to α since α is subject to the greatest uncertainty. The solution enclosure, obtained within 11 s, is given in Table 2. Moreover, the quality of the solution enclosure $[u]$ of the respective eight quantities is compared to the combinatorial solution $[\tilde{h}]$, i.e. the convex hull of the solutions to the point linear systems obtained when the parameters take all possible combinations of the interval end-points. The combinatorial solution serves as an *inner* estimation of the solution enclosure.

Table 2 One-bay steel frame example with worst-case parameter uncertainties (Table 1). Solution enclosure $[u]$ found by dividing the parameter intervals $(E_b, E_c, I_b, I_c, A_b, A_c, \alpha)^\top$ into $(2,2,2,2,1,1,6)^\top$ subintervals of equal width, respectively. All interval end-points are multiplied by 10^5. The enclosure $[u]$ is compared to the combinatorial solution $[\tilde{h}]$.

	$10^5 *$ solution enclosure $[u]$	$\mathscr{O}_\omega([\tilde{h}],[u])$
$d2_x$:	[138.54954789, 627.59324779]	12.5
$d2_y$:	[0.29323100807, 2.1529383383]	8.0
$r2_z$:	[-129.02427835, -22.381136355]	23.7
$r5_z$:	[-113.21398401, -17.95789860]	25.6
$r6_z$:	[-105.9680866, -17.64526946]	25.0
$d3_x$:	[135.25570695, 616.85512710]	12.7
$d3_y$:	[-3.7624790816, -0.41629803684]	13.2
$r3_z$:	[-122.3361772, -21.69878778]	23.5

These results show that by means of a small number of subdivisions the new parametric solver provides a good solution enclosure very quickly for the difficult problem of worst-case parameter uncertainties. Note that sharper bounds, close to the exact hull, can be obtained by proving the monotonicity properties of the parametric solution [28].

4.2 Two-Bay Two-Story Frame Model with 13 Parameters

We consider a two-bay two-story steel frame with IPE 400 beams and HE 280 B columns, as shown in Figure 2, after [29]. The frame is subjected to lateral static forces and vertical uniform loads. Beam-to-column connections are considered to be semi-rigid and they are modelled by single rotational spring elements. Applying conventional methods for the analysis of frame structures, a system of 18 linear equations is obtained, where the elements of the stiffness matrix and of the right hand side vector are rational functions of the model parameters. We consider the parametric system resulting from a finite element model involving the following 13 uncertain parameters: A_c, I_c, E_c, A_b, I_b, E_b, c, $w_1, \ldots, w_4$, F_1, F_2. Their nominal values, taken according to the European Standard Eurocode3 [7], are given in Table 3. The explicit analytic form of the given parametric system can be found in [30].

The parametric system is solved for the element material properties $(A_c, \ldots, E_b)$, which are taken to vary within a tolerance of 1% (that is $[x - x/200, x + x/200]$, where x is the corresponding parameter nominal value from Table 3) while the spring stiffness and all applied loadings are taken to vary within 10% tolerance intervals.

The previous parametric solver finds an enclosure for the solution set in about 7.4 s, whereas the new solver needs only about 1.3 s; here it is about six times faster. The solution enclosure provided by the new solver is also significantly tighter; the overestimation (17) of the components of the enclosure provided by the previous

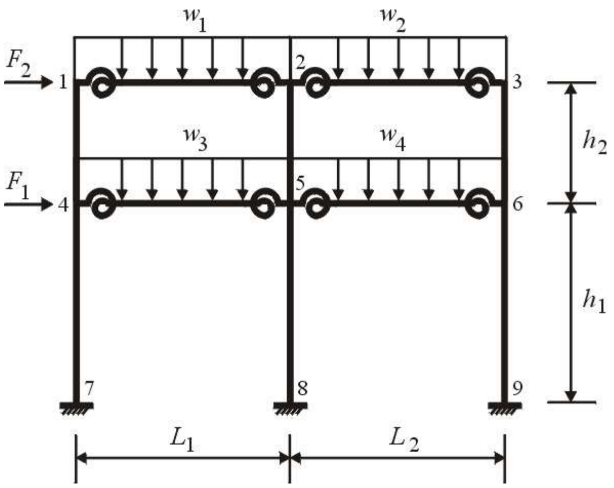

Fig. 2 Two-bay two-story steel frame [29].

Table 3 Parameters involved in the two-bay two-story frame example with their nominal values.

parameter	Columns (HE 280 B)	Beams (IPE 400)
Cross-sectional area	$A_c = 0.01314$ m^2	$A_b = 0.008446$m^2
Moment of inertia	$I_c = 19270 * 10^{-8}$ m^4	$I_b = 23130 * 10^{-8}$ m^4
Modulus of elasticity	$E_c = 2.1 * 10^8$kN/m^2	$E_b = 2.1 * 10^8$ kN/m^2
Length	$L_c = 3$ m	$L_b = 6$ m
Rotational spring stiffness	$c = 10^8$ kN	
Uniform vertical load	$w_1 = \ldots = w_4 = 30$ kN/m	
Concentrated lateral forces	$F_1 = F_2 = 100$ kN	

solver relative to the respective components found by the new solver ranges between 53.46 and 92.92.

An algebraic simplification applied to functional expressions in computer algebra environments may reduce the occurrence of interval variables, which could result in a sharper range enclosure. Such an algebraic simplification is expensive and when applied to complicated rational expressions usually does not result in a sharper range enclosure. For the sake of comparison, we have run the previous parametric solver in two ways: applying intermediate simplification during the range computation, and without any algebraic simplification. The above results were obtained when the range computation does not use any algebraic simplification. When the range computation of the previous solver uses intermediate algebraic simplification, the cost of this improvement is that the computing time is approximately doubled; the results are obtained in 14.4 s. This is much slower, but provided a tighter enclosure of the solution set than the rational solver, based on polynomial ranges, which did not account for all the parameter dependencies. Here the overestimation of the new solver relative to the modified previous solver ranges between 18.62 and 37.07. It should be noted that given the complicated rational expressions such an improvement is not at all typical (in the next example the improvement is only marginal at a much larger computation time possibly due to the more complicated expressions). Details may be found in [11].

4.3 Two-Bay Two-Story Frame Model with 37 Parameters

As a larger problem of a parametric system involving rational parameter dependencies, we consider the finite element model of the two-bay two-story steel frame from the previous example, where each structural element has properties varying independently within 1% tolerance intervals. This does not change the order of the system but it now depends on 37 interval parameters. The explicit analytic form of the given parametric system can be found in [30]. Here the right hand side vector is given to illustrate the dependencies.

$$\Bigg(f_2,\ -\frac{1}{2}w_1Lb_1,\ -\frac{w_1Lb_1^2}{12(1+\frac{2Eb_1Ib_1}{cLb_1})},\ 0,\ -\frac{w_1Lb_1}{2}-\frac{w_2Lb_2}{2},\ \frac{w_1Lb_1^2}{12(1+\frac{2Eb_1Ib_1}{cLb_1})}-$$
$$\frac{w_2Lb_2^2}{12(1+\frac{2Eb_2Ib_2}{cLb_2})},\ 0,\ -\frac{w_2Lb_2}{2},\ \frac{w_2Lb_2^2}{12(1+\frac{2Eb_2Ib_2}{cLb_2})},\ f_1,\ -\frac{1}{2w_3Lb_3},$$
$$-\frac{w_3Lb_3^2}{12(1+\frac{2Eb_3Ib_3}{cLb_3})},\ 0,\ -\frac{w_3Lb_3}{2}-\frac{w_4Lb_4}{2},\ \frac{w_3Lb_3^2}{12(1+\frac{2Eb_3Ib_3}{cLb_3})}-$$
$$\frac{w_4Lb_4^2}{12(1+\frac{2Eb_4Ib_4}{cLb_4})},\ 0,\ -\frac{w_4Lb_4}{2},\ \frac{w_4Lb_4^2}{12(1+\frac{2Eb_4Ib_4}{cLb_4})}\Bigg)^\top.$$

The previous solver finds an enclosure for the solution set in about 755 s and thereby exhibits performance approximately three times slower than the new solver (about 245 s). Also, the quality of the solution enclosure provided by the new solver is much better than the solution enclosure provided by the previous solver; here, the relative overestimation ranges between 28.4 and 95.46.

5 Conclusions

In this chapter, we demonstrated the advanced application of a general-purpose parametric method, combined with the Bernstein enclosure of polynomial ranges, to linear systems obtained by standard FEM analysis of mechanical structures, and illustrated the efficiency of the new parametric solver. Further applications, viz. to truss structures with uncertain node locations, can be found in [38].

It is shown that powerful techniques for range enclosure are necessary to provide tight bounds on the solution set, in particular when the parameters of the system are subject to large uncertainties and the dependencies are complicated.

The new self-verified parametric solvers can be incorporated into a general framework for the computer-assisted proof of global and local monotonicity properties of the parametric solution. Based on these properties, a guaranteed and highly accurate enclosure of the interval hull of the solution set can be computed [12, 28, 39]. The parametric solver for square systems also facilitates the guaranteed enclosures of the solution sets to over- and underdetermined parametric linear systems [27].

Being presently the only general-purpose parametric linear solver, the presented methodology and software tools are applicable in the context of any problem (stemming, e.g., from fuzzy set theory [35] or the other fields listed in the Introduction) that requires the solution of linear systems whose input data depend on uncertain (interval) parameters.

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