Pivot Tightening for the Interval Cholesky Method

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The performance of the Cholesky decomposition in interval arithmetic is considered. In order to avoid the algorithm breaking down due to an interval pivot containing zero, a method is presented by which such a pivot can be tightened.

1 Introduction

Systems of linear interval equations arise when the entries of the coefficient matrix and the right hand side of a system of linear equations are varying in given intervals, cf. [7, Sect. 3.4]. The *solution set* of such a system $[A]x = [b]$, where $[A] = ([a_{ij}])_{i,j=1}^n = ([\underline{a}_{ij}, \overline{a}_{ij}])_{i,j=1}^n$ is an n-by-n interval matrix and $[b] = ([b_i])_{i=1}^n = ([\underline{b}_i, \overline{b}_i])_{i=1}^n$, is an interval vector, is the set

$$\Sigma\left([A],[b]\right) \;:=\; \{x \in \mathbf{R}^n \mid Ax = b,\, A \in [A],\, b \in [b]\}\,. \tag{1}$$

We will assume here throughout that all $A \in [A]$ are non-singular. A method to enclose the solution set (1) in an interval vector is interval Gaussian elimination, which is obtained from the usual (termed *ordinary* henceforth) Gaussian elimination by replacing the real numbers by the related intervals and the real operations by the respective interval operations, e. g., [7, Sect. 4.5]. However, interval Gaussian elimination may fail due to division by an interval pivot containing zero, even when ordinary Gaussian elimination works for all matrices $A \in [A]$. There are some classes of interval matrices for which interval Gaussian elimination cannot fail, e. g., the H-matrices. If ordinary Gaussian elimination is applied without pivoting, the pivots can be represented as the quotient of two successive leading principal minors. This property was used in [5] to avoid the breakdown of interval Gaussian elimination by tightening the (interval) pivots. This is accomplished by replacing the interval pivots by the ranges of the respective ordinary pivots over the interval matrix. These ranges can be given explicitly for some classes of matrices which have identically signed inverses, e. g., the inverse nonnegative matrices, the non-singular totally nonnegative matrices and the inverse M-matrices. As an additional advantage this tightening may lead to a shrinking of the enclosure of the solution set. In this paper we are concerned with *symmetric* interval matrices, i. e., $[A]^T = [A]$. If $[A]$ is not symmetric it is replaced by the largest symmetric interval matrix contained in $[A]$. Instead of the solution set in (1) we now consider the *symmetric solution set*

$$\Sigma_{sym}\left([A],[b]\right) \;:=\; \{x \in \mathbf{R}^n \;\mid\; Ax = b,\, A \in [A]_s,\, b \in [b]\} \quad \text{with} \quad [A]_s \;:=\; \left\{A \in [A] \;\mid\; A = A^T\right\}. \tag{2}$$

A method for the enclosure of the symmetric solution set (2) is the interval Cholesky method (abbreviated ICh henceforth) which is obtained from the ususal Cholesky decomposition by replacing the real numbers by the related intervals and the real operations by the corresponding interval operations [1]. It is feasible if and only if the lower endpoints of the diagonal entries of the Cholesky factor are all positive. It is known that the ICh may break down even if $[A]_s$ contains only positive definite matrices. In this paper we will present a method by which the breakdown of ICh can be avoided.

2 Interval Cholesky Method

We assume that all matrices contained in $[A]_s$ are positive definite, or equivalently [2,8] that the following *vertex matrices* A_{zz} of $[A]$, i. e., the real matrices contained in $[A]$ whose entries coincide with an endpoint of the respective component intervals of $[A]$, are positive definite:

$A_{zz} \;=\; A_c - \mathrm{diag}(z_1,\ldots,z_n)\cdot\triangle A\cdot\mathrm{diag}(z_1,\ldots,z_n)$, where A_c and $\triangle A$ are the midpoint and radius matrix of $[A]$, respectively, $[A] \;=\; [A_c - \triangle A, A_c + \triangle A]$, and $z = (z_1,\ldots,z_n) \in Y_n \;:=\; \{-1,1\}^n$.

Then the cardinality of the set of the vertex matrices to be tested for positive definiteness is at most 2^{n-1}. Define the triangular matrix $[L]$ by (since we are interested in the feasibility of the ICh, we only consider the computation of the interval Cholesky factor)

$$[l_{jj}] \;\;=\;\; \left([a_{jj}] - \sum_{k=1}^{j-1}[l_{jk}]^2\right)^{\frac{1}{2}}\;;\quad [l_{ij}] \;\;=\;\; \left([a_{ij}] - \sum_{k=1}^{j-1}[l_{ik}][l_{jk}]\right)/\,[l_{jj}],\quad i = j+1,\ldots,n,\; j = 1,\ldots,n.$$

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Here $[a]^{\frac{1}{2}} := \{a^{\frac{1}{2}} \mid a \in [a]\}$ for $0 \leq \underline{a}$. ICh is feasible if and only if $0 < \underline{l}_{ii}$, $i = 1, \ldots, n$. In analogy to interval Gaussian elimination, we call the diagonal entries $[l_{ii}]$ (interval) *pivots*. For the ordinary Cholesky decomposition, l_{jj} can be represented as the square root of the ratio of the leading principal minors of order j and $j-1$, cf. [4, formula (42) on p.38]. In the next section we will present a method for the construction of positive lower bounds for $[l_{jj}]$.

3 Pivot Tightening

W. l. o. g. we may consider $p(A) := l_{nn}^2(A) = \det A / \det A'$, where A' is the submatrix of A obtained by deletion of the last row and column of A. Since the eigenvalues of A and A' interlace, we obtain $\lambda_1(A) \leq p(A)$, where $\lambda_1(A)$ denotes the smallest eigenvalue of A. By [9], it is known that

$$\min_{A \in [A]_s} \lambda_1(A) = \min_{z \in Y_n} \lambda_1(A_{zz}).$$

We employ the following lower bounds for the smallest eigenvalue of a positive definite symmetric matrix A partitioned in the form

$$A = \begin{pmatrix} A' & b \\ b^T & c \end{pmatrix}.$$

Let β_{n-1} be any lower bound for $\lambda_1(A')$. Then we have the easily computable lower bound [3]

$$\beta_n = \frac{1}{2}\left(c + \beta_{n-1} - \sqrt{(c-\beta_{n-1})^2 + 4b^Tb}\right) \leq \lambda_1(A). \qquad \text{Dembo's bound} \tag{3}$$

This bound may not be positive. If this case occurs we use the following bound [6]

$$\tilde{\beta}_n = \frac{1}{2}\left(c + \beta_{n-1} - \sqrt{(c-\beta_{n-1})^2 + 4\beta_{n-1}b^T(A')^{-1}b}\right) \leq \lambda_1(A). \qquad \text{Ma and Zarowski's bound} \tag{4}$$

which is always positive. If $(A')^{-1}b$ is computed as the solution x of $A'x = b$, then the computation of $\tilde{\beta}_n$ can be arranged in a recursive way such that, starting with $\beta_1 = a_{11}$, the i-th step needs $O(i^2)$ arithmetic operations (and one square root), $i = 1, \ldots, n$. It should be noted that sharper bounds are given in [10] which require slightly more computational effort.

Example: We consider

$$[A] = \begin{pmatrix} [4,5] & [-3,-2] & 1 \\ [-3,-2] & 4 & [-3,-2] \\ 1 & [-3,-2] & [4,5] \end{pmatrix}.$$

$[A]_s$ contains only positive definite matrices but the ICh breaks down due to $\underline{l}_{33}^2 = -0.112\ldots$. The four matrices A_{zz}, $z \in Y_3$, together with the associated bounds according to (3) and (4) are as follows

	$\begin{pmatrix} 4 & -3 & 1 \\ -3 & 4 & -3 \\ 1 & -3 & 4 \end{pmatrix}$	$\begin{pmatrix} 4 & -3 & 1 \\ -3 & 4 & -2 \\ 1 & -2 & 4 \end{pmatrix}$	$\begin{pmatrix} 4 & -2 & 1 \\ -2 & 4 & -3 \\ 1 & -3 & 4 \end{pmatrix}$	$\begin{pmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{pmatrix}$
Dembo bound:	-1	$-0.192\ldots$	$-0.316\ldots$	$0.550\ldots$
Ma and Zarowski bound:	$0.177\ldots$	$0.658\ldots$	$0.619\ldots$	

Thus $\underline{l}_{33}^2$ can be improved by $0.177\ldots$. Since $[A]$ is inverse nonnegative the lower endpoint of the range of p over $[A]$ is given by $p(\underline{A}) = 6/7$, cf. [5].

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