Convergence and Inclusion Isotonicity of the Rational Bernstein Form

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Abstract. A method is investigated by which tight bounds on the range of a multivariate rational function over a box can be computed. The approach relies on the expansion of the numerator and denominator polynomials in Bernstein polynomials. Convergence of the bounds to the range with respect to degree elevation of the Bernstein expansion, to the width of the box and to subdivision are proven and the inclusion isotonicity of the related enclosure function is shown.
Keywords: Bernstein polynomials·rational function·range bounding

1 Introduction

The expansion of a given (multivariate) polynomial p into Bernstein polynomials provides bounds on the range of p over a box. This is now a well-established tool as documented in [6]. In [8] the approach is extended to rational functions, however, without any proof of the convergence of the bounds to the range. In this paper we aim at filling this gap. Furthermore, we show that the related rational Bernstein form is inclusion isotone, a property which is of fundamental importance in interval computations, see, e.g., [9, Section 1.4]. The organization of our paper is as follows. In Sections 2 and 3 we recall the polynomial and the rational Bernstein forms. In Section 4 we present our main results.

2 The Polynomial Bernstein Form

In this section we briefly recall the most important properties of the Bernstein expantion, which will be used in the following sections. Let $\mathbb{I}(\mathbb{R})$ be the set of the compact, non-empty real intervals. We denote the distance q between two intervals $A = [\underline{a}, \overline{a}]$, $B = [\underline{b}, \overline{b}]$ by

$$q([\underline{a}, \overline{a}], [\underline{b}, \overline{b}]) := \max\{|\underline{a} - \underline{b}|, |\overline{a} - \overline{b}|\}.$$

Without loss of generality we may consider the unit box $I := [0, 1]^n$ since any compact non-empty box in $\mathbb{R}^n$ can be mapped thereupon by an affine transformation.

Comparisons and arithmetic operations on multiindices $i = (i_1, \ldots, i_n)^T$ are defined componentwise. For $x \in \mathbb{R}^n$ its monomials are $x^i := x_1^{i_1} \ldots x_n^{i_n}$. Using the compact notation $\sum_{i=0}^{k} := \sum_{i_1=0}^{k_1} \cdots \sum_{i_n=0}^{k_n}$, $\binom{k}{i} := \prod_{\mu=1}^{n} \binom{k_\mu}{i_\mu}$, an n-variate polynomial p, $p(x) = \sum_{i=0}^{l} a_i x^i$, can be represented as

$$p(x) = \sum_{i=0}^{k} b_i^{(k)}(p)\; B_i^{(k)}(x),\;\; x \in I, \tag{1}$$

where

$$B_i^k(x) = \binom{k}{i} x^i (1-x)^{k-i} \tag{2}$$

is the ith Bernstein polynomial of degree $k \geq l$, and the so-called *Bernstein coefficients* $b_i^{(k)}(p)$ are given by

$$b_i^{(k)}(p) = \sum_{j=0}^{i} \frac{\binom{i}{j}}{\binom{k}{j}} a_j,\;\; 0 \leq i \leq k,\; \text{where}\;\; a_j := 0\;\; \text{for}\; l \leq j, j \neq l. \tag{3}$$

In particular, we have the *endpoint interpolation property*

$$b_i^{(k)}(p) = p(\frac{i}{k}),\;\; \text{for all}\; i,\;\; 0 \leq i \leq k, \tag{4a}$$

$$\text{with}\quad i_\mu \in \{0, k_\mu\}. \tag{4b}$$

A fundamental property for our approach is the *convex hull property*, which states that the graph of p over I is contained within the convex hull of the control points derived from the Bernstein coefficients, i.e.,

$$\left\{ \begin{pmatrix} x \\ p(x) \end{pmatrix} : x \in I \right\} \subseteq conv \left\{ \begin{pmatrix} \frac{i}{k} \\ b_i^{(k)}(p) \end{pmatrix} : 0 \leq i \leq k \right\}, \tag{5}$$

where *conv* denotes the convex hull. This implies the *interval enclosing property* [1]

$$\min_{0 \leq i \leq k} b_i^{(k)}(p) \leq p(x) \leq \max_{0 \leq i \leq k} b_i^{(k)}(p),\;\; \text{for all}\;\; x \in I. \tag{6}$$

Equality holds on the left or right hand side of (6), if the minimum or maximum, respectively, is attained at an index i satisfying (4b). This condition is called the *vertex condition.* For an efficient computation of the Bernstein coefficients, see [4].

A disadvantage of the direct use of (6) is that the number of the Bernstein coefficients to be computed explicity grows exponentially with the number of variables n. Therefore, it is advantageous to use a method [11] by which the number of coefficients which are needed for the enclosure only grows approximately linearly with the number of the terms of the polynomial.

In many cases it is desired to calculate the Bernstein expansion of p over a general n-dimensional box X in the $\mathbb{I}(\mathbb{R})^n$,

$$X = [\underline{x}_1, \overline{x}_1] \times \cdots \times [\underline{x}_n, \overline{x}_n]$$

with

$$\underline{x}_\mu < \overline{x}_\mu, \;\; \mu = 1, \ldots, n.$$

The *width* of X is denoted by $w(X)$,

$$w(X) := \overline{x} - \underline{x}.$$

It is possible to firstly apply the affine transformation which maps X on the unit box I and to apply (3) using the coefficients of the transformed polynomial. However, in Section 4 it will be useful to consider the direct computation. Here, the ith Bernstein polynomial of degree $k \geq l$ is given by

$$B_i^{(k)}(x) = \binom{k}{i} (x - \underline{x})^i (\overline{x} - x)^{k-i} w(X)^{-k}, \;\; 0 \leq i \leq k. \tag{7}$$

The Bernstein coefficients $b_i^{(k)}$ of p of degree k over X are given by

$$b_i^{(k)}(p) = \sum_{j=0}^{i} \frac{\binom{i}{j}}{\binom{k}{j}} c_j, \;\; 0 \leq i \leq k, \tag{8}$$

$$\text{where} \quad c_j = w(X)^j \sum_{\tau=j}^{k} \binom{\tau}{j} a_\tau \underline{x}^{\tau-j} \tag{9}$$

with the convention $a_j := 0$ for $l \leq j, \; l \neq j$.

The interval

$$B^{(k)}(p, X) := [\min_{0 \leq i \leq k} b_i^{(k)}, \max_{0 \leq i \leq k} b_i^{(k)}]$$

encloses the range of p over X and is called the *polynomial Bernstein form* of p.

If the degree of the Bernstein expansion is elevated, the Bernstein coefficients of order $k+1$ can easily be computed as convex combinations of the coefficients of order k, e.g., [2, formula (13)], [4, formula (3.11)]. It follows that

$$B^{(k+1)}(p, X) \subseteq B^{(k)}(p, X). \tag{10}$$

The following theorem, see [10, formula (16)] for the univariate case and [4, Theorem 3] for its multivariate extension, will be used to derive our main results.

Theorem 1. For $l \leq k$, the following bound holds for the overestimation of the range $p(X)$ of p over X by the Bernstein form

$$q(p(X), B^{(k)}(p, X)) \leq \sum_{i=0}^{l} \sum_{\mu=1}^{n} \frac{[\max(0, i_\mu - 1)]^2}{k_\mu} |c_i|, \tag{11}$$

where the coefficients c_i are given by (9).

Remark 1. If $2 \leq k_\mu$ the bound on the right hand side of (11) can be improved slightly, see [10, formula (17)].

3 The Rational Bernstein Form

Let p and q be polynomials in variables $x_1, \ldots, x_n$ with Bernstein coefficients $b_i^{(k)}(p)$ and $b_i^{(k)}(q)$, $0 \leq i \leq k$, over a box X, respectively. We consider the rational function $f := p/q$. We may assume that both p and q have the same degree l since otherwise we can elevate the degree of the Bernstein expansion of either polynomial by component where necessary to ensure that their Bernstein coefficients are of the same order $k \geq l$. We call

$$b_i^{(k)}(f) := \frac{b_i^{(k)}(p)}{b_i^{(k)}(q)},\ 0 \leq i \leq k,$$

the *rational Bernstein coefficients* of f.

Theorem 2. [8, Theorem 3.1] Assume that all Bernstein coefficients $b_i^{(k)}(q)$ have the same sign and are non-zero (this implies that $q(x) \neq 0$, for all $x \in X$). Then the following enclosure for the range of f over X holds

$$\underline{m}^{(k)} := \min_{0 \leq i \leq k} b_i^{(k)}(f) \leq f(x) \leq \max_{0 \leq i \leq k} b_i^{(k)}(f) =: \overline{m}^{(k)}, \text{ for all } x \in X. \tag{12}$$

The interval spanned by the left and right hand sides of (12) constitutes the *rational Bernstein form* $B(f, X)$,

$$B^{(k)}(f, X) := [\underline{m}^{(k)}, \overline{m}^{(k)}].$$

Remark 2. The convex hull property (5) does not in general carry over to rational functions and control points formed from the rational Bernstein coefficients even in the univariate case ($n = 1$). For a counterexample see [8].

Remark 3. Theorem 2 carries over to the Bernstein polynomials over the standard simplex in $\mathbb{R}^n$ [4].

4 Main Results

Let throughout $f = p/q$ be a rational function, where p and q are polynomials of degree l and let the range of f over X be $f(X) = [\underline{f}, \overline{f}]$. Without loss of generality we assume that

$$0 < b_i^{(l)}(q), \text{ for all } i,\ 0 \leq i \leq l, \tag{13}$$

and prove the statements only for the upper bounds since the proofs for the lower bounds are entirely analogous. The polynomial r,

$$r := p - \overline{m}^{(k)} q, \tag{14}$$

will serve as a vehicle to convey the results from the polynomial to the rational case. Note that the Bernstein coefficients of a polynomial are linear, hence

$$b_i^{(k)}(r) = b_i^{(k)}(p) - \overline{m}^{(k)} b_i^{(k)}(q). \tag{15}$$

First we show that the vertex condition remains in force.

Proposition 3. It holds that $\underline{m}^{(k)} = \underline{f}$ $(\overline{m}^{(k)} = \overline{f})$ if and only if $\underline{m}^{(k)}$ $(\overline{m}^{(k)}) = b_i^{(k)}(f)$ with i satisfying (4b).

Proof. By (4a), $b_i^{(k)}(f)$ with i satisfying (4b) is a value of f at a vertex of X. If follows that $\overline{m}^{(k)}$ is sharp if it is attained at such a Bernstein coefficient. Conversely, assume that $\overline{m}^{(k)} = \overline{f}$,

$$\overline{m}^{(k)} = b_{i_0}^{(k)}(f), \text{ for some } i_0,\ 0 \leq i_0 \leq k, \tag{16}$$

and $\overline{f} = f(\hat{x})$ for some $\hat{x} \in X$. Then we can conclude that

$$\frac{r(\hat{x})}{q(\hat{x})} = f(\hat{x}) - \overline{m}^{(k)} = 0,$$

hence $r(\hat{x}) = 0$. Since r is nonpositive on X it attains its maximum at $\hat{x}$. On the other hand, we have by (15)

$$b_i^{(k)}(r) \leq 0, \text{ for all } i,\ 0 \leq i \leq k, \tag{17}$$

and by (16) $b_{i_0}^{(k)}(r) = 0$. So we can conclude that

$$\max_{x \in X} r(x) = b_{i_0}^{(k)}(r). \tag{18}$$

By the polynomial vertex condition if follows that the index i_0 satisfies (4b). $\square$

4.1 Linear Convergence with Respect to Degree Elevation

We start with the observation that the monotonicity property (10) carries over to the rational case.

Proposition 4. For $l \leq k$ it holds that $B^{(k+1)}(f, X) \subseteq B^{(k)}(f, X)$.

Proof. By application of (10) to polynomial r (14) and noting (15) we obtain for all j, $0 \leq j \leq k+1$,

$$b_j^{(k+1)}(p) - \overline{m}^{(k)} b_j^{(k+1)}(q) \leq \max_{0 \leq i \leq k+1} \{b_i^{(k+1)}(p) - \overline{m}^{(k)} b_i^{(k+1)}(q)\}$$

$$\leq \max_{0 \leq i \leq k} \{b_i^{(k)}(p) - \overline{m}^{(k)} b_i^{(k)}(q)\} \leq 0,$$

hence $\quad b_j^{(k+1)}(f) \leq \overline{m}^{(k)}$. $\square$

Theorem 5. For $l \leq k$ it holds that

$$q(f(X), B^{(k)}(f, X)) \leq \frac{\beta}{k}, \tag{19}$$

where β is a constant not depending on k.

Proof. Without loss of generality we consider only the case $0 \leq \overline{m}^{(k)}$. Since by (10) for $l \leq k$

$$\overline{m}^{(k)} \leq \overline{m}^{(l)} \leq \frac{\max b_i^{(l)}(p)}{\min b_i^{(l)}(q)} =: \beta' \tag{20}$$

we can conclude from Theorem 1 that

$$\begin{aligned} -r(x) &\leq \overline{m}^{(k)} q(x) - p(x) + (\max_{0\leq i\leq k} b_i^{(k)}(p) - \overline{m}^{(k)} \min_{0\leq i\leq k} b_i^{(k)}(q)) \\ &\leq \beta'(q(x) - \min_{0\leq i\leq k} b_i^{(k)}(q)) + (\max_{0\leq i\leq k} b_i^{(k)}(p) - p(x)) \qquad (21) \\ &\leq \frac{\beta''}{k}, \end{aligned}$$

where β'' is a constant not depending on k. Division by q results in

$$\overline{m}^{(k)} - f(x) \leq \frac{\beta''}{q(x)} \frac{1}{k} \leq \frac{\beta''}{\min b_i^{(l)}(q)} \frac{1}{k} \tag{22}$$

which completes the proof. $\square$

4.2 Quadratic Convergence with Respect to the Width of an Interval

Inspection of (21) shows that we can extract $\max_{\mu=1}^{n}(\overline{x}_\mu - \underline{x}_\mu)$ from the constant β in (19), cf. (9), (11). Therefore, we obtain the following extension of [12, Corollary 3.4.16].

Theorem 6. Let $A \in \mathbb{I}(\mathbb{R})^n$ be fixed. Then for all $X \in \mathbb{I}(\mathbb{R})^n$, $X \subseteq A$, and $l \leq k$ it holds that

$$q(f(X), B^{(k)}(f, X)) \leq \gamma \, ||w(X)||_\infty^2, \tag{23}$$

where γ is a constant not depending on X.

4.3 Quadratic Convergence with Respect to Subdivision

Since the convergence with respect to degree elevation is only linear we will choose $k = l$ in the sequel and reserve in this subsection the upper index of the Bernstein coefficients for the subdivision level. For simplicity we consider the unit box I. Repeated bisection of $I^{(0,1)} := I$ in all n coordinate directions results at subdivision level $1 \leq h$ in subboxes $I^{(h,\nu)}$ of edge length 2^{-h}, $\nu = 1, \ldots, 2^{nh}$. Denote the Bernstein coefficients of f over $I^{(h,\nu)}$ by $b_i^{(h,\nu)}(f)$. For their computation see [4], [13]. put

$$B^{(h)}(f) := [\min_{\substack{0\leq i\leq l,\\ 1\leq\nu\leq 2^{nh}}} b_i^{(h,\nu)}(f), \max_{\substack{0\leq i\leq l,\\ 1\leq\nu\leq 2^{nh}}} b_i^{(h,\nu)}(f)].$$

We obtain the following extension of [3, formula (23)].

Theorem 7. For each $1 \leq h$ it holds

$$q(f(X), B^{(h)}(f)) \leq \delta(2^{-h})^2, \tag{24}$$

where δ is a constant not depending on h.

Proof. Assume that

$$\max_{\substack{0\leq i\leq l,\\ 1\leq \nu\leq 2^{nh}}} b_i^{(h,\nu)} = \max_{0\leq i\leq l} b_i^{(h,\nu_0)}, \text{ for some } \nu_0,\ 0 \leq \nu_0 \leq 2^{nh}.$$

Then it follows by Theorem 6

$$\max_{\substack{0\leq i\leq l,\\ 1\leq \nu\leq 2^{nh}}} b_i^{(h,\nu)} - \max_{x\in I} f(x) \leq \max_{0\leq i\leq l} b_i^{(h,\nu_0)} - \max_{x\in I^{(h,\nu_0)}} f(x)$$

$$\leq \delta ||w(I^{(h,\nu_0)})||_\infty^2 = \delta\ (2^{-h})^2. \quad \square$$

Remark 4. Note that by (9), (11) the constants β, γ and δ in (19), (23) and (24) can be given explicity.

4.4 Inclusion Isotonicity

We continue with choosing $k = l$ and suppress therefore the upper index for the Bernstein coefficients. An interval function $F : \mathbb{I}(\mathbb{R})^n \longrightarrow \mathbb{I}(\mathbb{R})$ is called *inclusion isotone,* if, for all $X, Y \in \mathbb{I}(\mathbb{R})^n,\ X \subseteq Y$ implies $F(X) \subseteq F(Y)$.

In [7] it was shown by a lengthy proof that the polynomial Bernstein form is inclusion isotone. In [5] a brief proof of this property and an extension to the multivariate case are presented. We show that the inclusion isotonicity carries over to rational functions.

Theorem 8. The rational Bernstein form is inclusion isotone.

Proof. We consider without loss of generality the unit box I and denote the Bernstein coefficients of the rational function f over I by $b_i(f),\ 0 \leq i \leq l$. It suffices to show that the inclusion isotonicity holds if we shrink only one edge of I and this in turn separately at its left and right endpoint. Without loss of generality we consider only the first case and the first component interval of I and denote by $b_i^*(f),\ 0 \leq i \leq l$, the Bernstein coefficients of f over $[\epsilon, 1] \times [0, 1]^{n-1}, 0 < \epsilon < 1$. Put

$$\overline{m}^* := \max_{0\leq i\leq l} b_i^*(f).$$

We proceed by contradiction and assume that

$$\overline{m}^* = b_{i_0}^*(f), \text{ for some } i_0,\ \ 0 \leq i_0 \leq l, \tag{25}$$

and

$$\overline{m} := \max_{0 \le i \le l} b_i(f) < \overline{m}^*. \tag{26}$$

Since the Bernstein form of the polynomial $p - \overline{m}^* q$ is inclusion isotone we obtain from (26) that

$$b_{i_0}^*(p) - \overline{m}^* b_{i_0}^*(q) \le \max_{0 \le i \le l} \{b_i^*(p) - \overline{m}^* b_i^*(q)\}$$

$$\le \max_{0 \le i \le l} \{b_i(p) - \overline{m}^* b_i(q)\}$$

$$< \max_{0 \le i \le l} \{b_i(p) - \overline{m}\, b_i(q)\} \le 0$$

from which we get a contradiction to (25). □

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