Robust Schur Stability of Polynomials with Polynomial Parameter Dependency

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Abstract. The paper considers the robust Schur stability verification of polynomials with coefficients depending polynomially on parameters varying in given intervals. A new algorithm is presented which relies on the expansion of a multivariate polynomial into Bernstein polynomials and is based on the decomposition of the family of polynomials into its symmetric and antisymmetric parts. It is shown how the inspection of both polynomial families on the upper half of the unit circle can be reduced to the analysis of two related polynomial families on the real interval $[-1,1]$. Then the Bernstein expansion can be applied in order to check whether both polynomial families have a zero in this interval in common.

1. Introduction

Bernstein expansion has proved to be a well established and important tool for solving robust stability problems such as checking a polynomial with polynomial parameter dependency for stability. Stability regions covered include the open left half of the complex plane (Hurwitz stability) [1], [2] and sectors centered around the negative real axis with vertices at the origin (damping) [2]. Also, exponentional parameter dependency can be treated [2]. A comparison of Bernstein expansion with methods based on interval analysis can be found in [3]. In this paper the stability region is the open unit disc, i.e., we are considering Schur stability:

The Robust Schur Stability Problem: Let the parameter set $\mathbf{Q}$ be an l-dimensional box, i.e., $\mathbf{Q} = [\underline{q}_1, \overline{q}_1] \times \cdots \times [\underline{q}_l, \overline{q}_l]$. Let a family of polynomials be given by

$$p(s, \mathbf{q}) = \sum_{k=0}^{m} a_k(\mathbf{q}) s^k, \qquad (1)$$

where the coefficients are depending polynomially on parameters q_i, $i = 1, \ldots, l$, $\mathbf{q} = (q_1, \ldots, q_l)$, i.e., for $k = 0, \ldots, m$

$$a_k(\mathbf{q}) = \sum_{i_1,\ldots,i_l=0}^{d} a^{(k)}_{i_1\ldots i_l} q_1^{i_1} \ldots q_l^{i_l}. \tag{2}$$

Since we can add terms with zero coefficients in (2) we may use a common upper summation bound d for all coefficients.

Question: *Is the family of polynomials (robustly) Schur stable for* $\mathbf{Q}$*, i.e., are the polynomials* $p(\mathbf{q})$ *Schur stable for all* $\mathbf{q} \in \mathbf{Q}$ *?*

Most of the methods known from literature for checking a family of polynomials for robust Schur stability have focused on the case of affine and multiaffine parameter dependency, e.g. the monographs [4], [5] and the review articles [6], [7] and the references therein, and can treat only problems with a few parameters and/or polynomials of lower degree. In [8] an approach is presented which is based on frequency domain considerations. While it is hard to check analytically the necessary and sufficient condition given therein in the general case, in simpler cases a plot can be generated which, upon inspection by eye, provides the relevant informations. In [9] Jury and Pavlidis give a criterion for robust Schur stability which requires the test of a special determinant for positivity over $\mathbf{Q}$. In the case of the polynomial parameter dependency (2) this determinant is a multivariate polynomial in the variables $q_1, \ldots, q_l$ and thus can be checked for positivity over $\mathbf{Q}$ efficiently by the Bernstein Algorithm [10]. However, working with this determinant causes a growing of complexity of the original problem so that this approach is restricted to problems with a moderate number of parameters and to lower degree polynomials. Focusing on larger robust control problems we propose in this paper a new algorithm which relies on the Bernstein expansion of the symmetric and antisymmetric parts of the polynomial family. An extension to the computation of stability regions can be found in [11].

As a consequence of the continuous dependency of the zeros of a polynomial on its coefficients it is sufficient for robust Schur stability to show that for the polynomial family (1)

$$0 \neq p(e^{j\omega}, \mathbf{q}) \text{ for all } \omega \in [0, 2\pi], \mathbf{q} \in \mathbf{Q}, \tag{3}$$

if there is a stable member $p(s, \mathbf{q}^0), \mathbf{q}^0 \in \mathbf{Q}$, of the family. This principle is often called the *Zero Exclusion Principle*, cf. p. 114 in [5]. For simplicity, we concentrate here on real coefficient functions $a_k(\mathbf{q})$ in (2) so that we can restrict the verification of (3) to $\omega \in [0, \pi]$.

The organization of our paper is as follows: In Sect. 2 we recall from [1] the Bernstein transformation of a polynomial and explain the sweep procedure which is fundamental for our algorithm. In Sect. 3 we use the decomposition of a polynomial into its symmetric and antisymmetric parts to recast the test for common zeros of both polynomials on the upper half of the unit circle as one for common zeros of two transformed polynomials in the interval $[-1, 1]$, [12] see also [13],[14]. In Sect. 4 we present the new algorithm. Numerical examples are given in Sect. 5.

2. Bernstein Expansion

We start this section with introducing some notation.
A *multi-index* I is an ordered l-tupel of nonnegative integers $(i_1, \ldots, i_l)$. We will use multi-indices e.g. to shorten power products: For $\mathbf{x} = (x_1, \ldots, x_l) \in \mathbf{R}^l$ $\mathbf{x}^I$ stands for $x_1^{i_1} x_2^{i_2} \cdot \ldots \cdot x_l^{i_l}$. For $N = (n_1, \cdots, n_l)$ we write $I \leq N$ if $0 \leq i_k \leq n_k$, $k = 1, \cdots, l$, and set $S = \{I : I \leq N\}$. Then we can write an l-variate polynomial p in the form

$$p(\mathbf{x}) = \sum_{I \in S} \mathbf{a}_I \mathbf{x}^I, \quad \mathbf{x} \in \mathbf{R}^l, \tag{4}$$

and refer to N as the *degree* of p. We define the *total degree* of polynomial (4) as

$$\hat{n} = \max\{n_i : i = 1, \ldots, l\}.$$

Also, we write I/N for $(i_1/n_1, \ldots, i_l/n_l)$ and $\binom{N}{I}$ for $\binom{n_1}{i_1} \cdot \ldots \cdot \binom{n_l}{i_l}$. With $I = (i_1, \ldots, i_r, \ldots, i_l)$ we associate the index $I_{r,k}$ given by $I_{r,k} = (i_1, \ldots, i_r + k, \ldots, i_l)$, where $0 \leq i_r + k \leq n_r$.

2.1. Bernstein Transformation of a Polynomial

In this subsection we expand a given l-variate polynomial (4) into Bernstein polynomials to obtain bounds for its range over an l-dimensional box, for references see [1], [15]. Without loss of generality we consider the unit box $\mathbf{U} = [0, 1]^l$ since any nonempty box of $\mathbf{R}^l$ can be mapped affinely onto this box.

For $\mathbf{x} = (x_1, \ldots, x_l) \in \mathbf{R}^l$ the Ith *Bernstein polynomial* of degree N is defined as

$$B_{N,I}(\mathbf{x}) = b_{n_1,i_1}(x_1) b_{n_2,i_2}(x_2) \cdot \ldots \cdot b_{n_l,i_l}(x_l),$$

where for $i_j = 0, \ldots, n_j$, $j = 1, \ldots, l$

$$b_{n_j,i_j}(x_j) = \binom{n_j}{i_j} x_j^{i_j}(1-x_j)^{n_j-i_j}.$$

The transformation of a polynomial from its power form (4) into its *Bernstein form* results in

$$p(\mathbf{x}) = \sum_{I\in S} b_I(\mathbf{U}) B_{N,I}(\mathbf{x}),$$

where the *Bernstein coefficients* $b_I(\mathbf{U})$ of p over $\mathbf{U}$ are given by

$$b_I(\mathbf{U}) = \sum_{J\le I} \frac{\binom{I}{J}}{\binom{N}{J}} a_J, \quad I \in S. \tag{5}$$

We collect the Bernstein coefficients in an array $B(\mathbf{U})$, i.e., $B(\mathbf{U}) = (b_I(\mathbf{U}))_{I\in S}$. A similar notation will be employed for other sets of related coefficients. For an efficient calculation of the Bernstein coefficients see [16].
In the following we will use a special subset of the index set S comprising those indices which correspond to the indices of the vertices of the array $B(\mathbf{U})$, i.e.,

$$S_0 = \{0, n_1\} \times \ldots \times \{0, n_l\}.$$

We list some useful properties of the Bernstein coefficients, e.g. [17]. As usual, we denote the convex hull of a set A by Conv$A$.

LEMMA 1. *Let p be a polynomial (4) of degree N. Then the following properties hold for its Bernstein coefficients $b_I(\mathbf{U})$ (5):*

i) Sharpness of special coefficients:

$$\forall I \in S_0 : b_I(\mathbf{U}) = p(I/N). \tag{6}$$

ii) Convex hull property:

$$Conv\{(\mathbf{x}, p(\mathbf{x})) : \mathbf{x} \in \mathbf{U}\} \subseteq Conv\{(I/N, b_I(\mathbf{U})) : I \in S\}. \tag{7}$$

2.2. Sweep Procedure

We define a sweep in the rth direction ($1 \le r \le l$) as recursively applied linear interpolation. Let $\mathbf{D} = [\underline{d}_1, \overline{d}_1] \times \ldots \times [\underline{d}_l, \overline{d}_l]$ be any subbox of $\mathbf{U}$ generated by sweep operations (at the beginning we have $\mathbf{D} = \mathbf{U}$). Starting with $B^{(0)}(\mathbf{D}) = B(\mathbf{D})$ we set for $k = 1, \ldots, n_r$

$$b_I^{(k)}(\mathbf{D}) = \begin{cases} b_I^{(k-1)}(\mathbf{D}) : i_r < k \\ (b_{I_{r,-1}}^{(k-1)}(\mathbf{D}) + b_I^{(k-1)}(\mathbf{D}))/2 : k \leq i_r. \end{cases}$$

To obtain the new coefficients, we apply the above formula for $i_j = 0, \ldots, n_j$, $j = 1, \ldots, r-1, r+1, \ldots, l$.
Then the Bernstein coefficients on $\mathbf{D}_0$, where the subbox $\mathbf{D}_0$ is given by

$$\mathbf{D}_0 = [\underline{d}_1, \overline{d}_1] \times \ldots \times [\underline{d}_r, \hat{d}_r] \times \ldots \times [\underline{d}_l, \overline{d}_l]$$

are obtained as $B(\mathbf{D}_0) = B^{(n_r)}(\mathbf{D})$, where $\hat{d}_r$ denotes the midpoint of $[\underline{d}_r, \overline{d}_r]$. The Bernstein coefficients $B(\mathbf{D}_1)$ on the neighbouring subbox $\mathbf{D}_1$

$$\mathbf{D}_1 = [\underline{d}_1, \overline{d}_1] \times \cdots \times [\hat{d}_r, \overline{d}_r] \times \cdots \times [\underline{d}_l, \overline{d}_r]$$

are intermediate values of this computation since for $k = 0, \ldots, n_r$ the following relation holds [15]

$$b_{i_1,\ldots,n_r-k,\ldots,i_l}(\mathbf{D}_1) = b_{i_1,\ldots,n_r,\ldots,i_l}^{(k)}(\mathbf{D}).$$

In analogy to Computer Aided Geometric Design we call the arrays of Bernstein coefficients $B(\mathbf{D}_0)$ and $B(\mathbf{D}_1)$ *patches*. A sweep needs $O(\hat{n}^{l+1})$ additions and multiplications, cf. [1].

2.3. Selection Procedures

The definition of the sweep procedure shows that we are free in choosing the sweep direction in the computation of the Bernstein coefficients. Our rules for selecting the sweep direction are based on an upper bound associated with the first ($h = 1$) or second ($h = 2$) partial derivative of a polynomial in Bernstein form. As sweep direction we choose that r for which

$$\tilde{I}_r^{(h)} = \max_{I \leq N_{r,-h}} \{ \frac{n_r!}{(n_r - h)!} ||\Delta_r^{(h)} b_I(\mathbf{D})||_2 \}$$

achieves its maximum. Here the difference operator $\triangle_r$ is defined recursively by

$$\begin{array}{ll} \triangle_r^{(0)} b_I(\mathbf{D}) = b_I(\mathbf{D}) & \text{, and for } k = 1, \ldots, h \\ \triangle_r^{(k)} b_I(\mathbf{D}) = \triangle_r^{(k-1)} b_{I_{r,1}}(\mathbf{D}) - \triangle_r^{(k-1)} b_I(\mathbf{D}) \ . & \end{array}$$

We conclude this section with a selection rule for the patches. After a sweep we have the choice between two patches D_0 and D_1, say. Let $p_i^{(w)}$,

$i = 1, \ldots, \mu_w$, generate Conv $B(D_w), w = 0, 1$, cf. p.131 in [5]. Then that patch is chosen for which $(|p_1^{(w)}| + \ldots + |p_{\mu_w}^{(w)}|)/\mu_w$ is minimal.

3. Transformation of the symmetric and antisymmetric part

We consider the family (1) for a fixed $\mathbf{q} \in \mathbf{Q}$ and suppress therefore the explicit reference to $\mathbf{q}$ for simplicity. In the sequel we use the decomposition of the polynomial p into its symmetric and antisymmetric parts h and g, respectively, i.e.,

$$p(s) = h(s) + g(s),$$

defined by

$$\begin{aligned} h(s) &= \frac{1}{2}(p(s) + s^m p(1/s)), \\ g(s) &= \frac{1}{2}(p(s) - s^m p(1/s)). \end{aligned}$$

Then we have with $n = [\frac{m+1}{2}]$, [ ] denoting the integer part,

$$\begin{aligned} h(s) &= \frac{1}{2}\Big(\sum_{k=0}^{n-1} \alpha_k (s^k + s^{m-k}) + \alpha_n s^n\Big), \\ g(s) &= \frac{1}{2}\sum_{k=0}^{n-1} \beta_k (s^k - s^{m-k}), \end{aligned}$$

where the coefficients α_k, β_k of h and g are given by

$$\begin{aligned} \alpha_k &= a_k + a_{m-k}, k = 0, \ldots, n, \\ \beta_k &= a_k - a_{m-k}, k = 0, \ldots, n-1, \end{aligned}$$

and $\alpha_n = 0$ if n is odd.
The following Lemma is an immediate consequence of the definition of h and g.

LEMMA 2. *Let $|s_0| = 1$. Then s_0 is a zero of p if and only if s_0 is a common zero of h and g.*

To check condition (3) of the Zero Exclusion Principle in the case of real coefficient functions (2) we consider the polynomials h and g for $s = e^{j\omega}, \omega \in [0, \pi]$. Then it follows

$$e^{-j\frac{m}{2}\omega} h(e^{j\omega}) = \sum_{k=0}^{n-1} \alpha_k \cos(\frac{m}{2} - k)\omega + \frac{1}{2}\alpha_n =: h^*(\omega), \qquad (8)$$

$$j e^{-j\frac{m}{2}\omega} g(e^{j\omega}) = \sum_{k=0}^{n-1} \beta_k \sin(\frac{m}{2} - k)\omega =: g^*(\omega). \qquad (9)$$

Thus we obtain the representation

$$p(e^{j\omega}) = e^{j\frac{m}{2}\omega}(h^*(\omega) - jg^*(\omega)). \tag{10}$$

To apply the Bernstein expansion we transform the trigonometric polynomials h^* and g^* into algebraic polynomials in the variable t, say, by projecting the upper half of the unit circle onto the interval $[-1,1]$ on the real line [12], see also [13], [14]. This is accomplished by setting $t = \cos\omega$ and using Chebyshev polynomials of the first kind $(n = 0, 1, 2, \ldots)$

$$T_n(t) := \cos(n\omega) = \cos(n \arccos t), \qquad -1 \le t \le 1$$

and second kind $(n = 1, 2, 3, \ldots)$, e.g. [18],

$$U_{n-1}(t) := \begin{cases} \dfrac{1}{n}T_n'(t) = \dfrac{\sin n\omega}{\sin\omega}, & -1 < t < 1, \\ t^{n-1}n, & t = \pm 1, \end{cases}$$

where T_n' denotes the derivative of T_n w. r. t. t.

Case m is even: Then $m/2 = n$ and it follows from (8), (9) that for $\omega \in [0, \pi]$

$$\begin{aligned} h^*(\omega) &= \sum_{k=0}^{n-1} \alpha_k \, T_{n-k}(t) + \frac{1}{2}\alpha_n =: \tilde{h}(t), \\ g^*(\omega) &= \sin\omega \; \tilde{g}(t), \end{aligned}$$

where

$$\tilde{g}(t) := \sum_{k=0}^{n-1} \beta_k U_{n-k-1}(t).$$

The functions $\tilde{h}$ and $\tilde{g}$ are (algebraic) polynomials of maximum degree n and $n-1$, respectively.

Case m is odd: Then $\frac{m}{2} = n - \frac{1}{2}$, $\alpha_n = 0$, and it follows from (8), (9) that for $\omega \in (0, \pi)$

$$\begin{aligned} \cos(\frac{m}{2} - k)\omega &= \cos(n - k - \frac{1}{2})\omega \\ &= \cos\frac{\omega}{2}(\cos(n-k)\omega + (1 - \cos\omega)\frac{\sin(n-k)\omega}{\sin\omega}) \\ &= \cos\frac{\omega}{2}(T_{n-k}(t) + (1-t)U_{n-k-1}(t)), \end{aligned}$$

and similarly,

$$\sin(\frac{m}{2}-k)\omega \quad = \quad \sin\frac{\omega}{2}\ ((1+t)U_{n-k-1}(t)-T_{n-k}(t)).$$

Hence we get similiarly as in the case that m is even for $\omega \in [0,\pi]$

$$h^*(\omega) \quad = \quad \cos\frac{\omega}{2}\tilde{h}(t),$$

where

$$\tilde{h}(t) := \sum_{k=0}^{n-1} \alpha_k (T_{n-k}(t) + (1-t)U_{n-k-1}(t)),$$

and

$$g^*(\omega) \quad = \quad \sin\frac{\omega}{2}\ \tilde{g}(t),$$

where

$$\tilde{g}(t) := \sum_{k=0}^{n-1} \beta_k ((1+t)U_{n-k-1}(t) - T_{n-k}(t)).$$

Both $\tilde{h}$ and $\tilde{g}$ are polynomials of degree n at most.

4. The Procedure

From the representation (10) it follows that the family (1) is robustly stable for $\mathbf{Q}$ if and only if a Schur stable member $p(s,\mathbf{q}^0), \mathbf{q}^0 \in \mathbf{Q}$, exists and

$$0 \notin \{h^*(\omega,\mathbf{q}) - jg^*(\omega,\mathbf{q}) : \omega \in [0,\pi], \mathbf{q} \in \mathbf{Q}\}. \tag{11}$$

Having found a Schur stable member, we will make use of the transformed polynomials $\tilde{h}$ and $\tilde{g}$ in order to check (11). The cases $\omega = 0, \pi$, i.e., $t = \pm 1$, are considered separately first.

It follows from (8), (9) that

$$\begin{aligned} g^*(0,\mathbf{q}) \quad &= \quad 0, \\ h^*(0,\mathbf{q}) \quad &= \quad \sum_{k=0}^{n-1} \alpha_k(\mathbf{q}) + \frac{1}{2}\alpha_n(\mathbf{q}). \end{aligned}$$

Therefore, we have to check whether

$$0 \neq \sum_{k=0}^{n-1} \alpha_k(\mathbf{q}) + \frac{1}{2}\alpha_n(\mathbf{q}) \quad \text{for all } \mathbf{q} \in \mathbf{Q}. \tag{12}$$

If m is even then

$$\begin{aligned} g^*(\pi, \mathbf{q}) &= 0, \\ h^*(\pi, \mathbf{q}) &= \sum_{k=0}^{n-1} (-1)^{n-k} \alpha_k(\mathbf{q}) + \frac{1}{2}\alpha_n(\mathbf{q}), \end{aligned}$$

so that we have to test whether

$$0 \neq \sum_{k=0}^{n-1} (-1)^{n-k} \alpha_k(\mathbf{q}) + \frac{1}{2}\alpha_n(\mathbf{q}) \quad \text{for all } \mathbf{q} \in \mathbf{Q}. \tag{13}$$

If m is odd then

$$\begin{aligned} h^*(\pi, \mathbf{q}) &= 0, \\ g^*(\pi, \mathbf{q}) &= \sum_{k=0}^{n-1} (-1)^{n-k+1} \beta_k(\mathbf{q}), \end{aligned}$$

so that we have to test whether

$$0 \neq \sum_{k=0}^{n-1} (-1)^{n-k+1} \beta_k(\mathbf{q}) \quad \text{for all } \mathbf{q} \in \mathbf{Q}. \tag{14}$$

To show (12) and (13) (resp., (14)) we proceed as follows: To check whether an arbitrary polynomial $f(\mathbf{q})$ has a zero in $\mathbf{Q}$ we expand f into Bernstein polynomials over $\mathbf{Q}$ [1]. Let b_I be the associated Bernstein coefficients. If $0 \notin [\min_{I \in S} b_I, \max_{I \in S} b_I]$, then $f(\mathbf{q})$ has no zero in $\mathbf{Q}$ by the convex hull property (7). Else, if $0 \in [\min_{I \in S_0} b_I, \max_{I \in S_0} b_I]$, then $f(\mathbf{q})$ has a zero $\mathbf{q}$ in $\mathbf{Q}$. In the remaining cases we split $\mathbf{Q}$ to obtain two new patches on which we proceed as before. To continue the process it is advantageous to apply a depth first strategy [1]: If we have the choice between two patches we choose that patch having smallest or greatest Bernstein coefficient. Then the chances are best for finding a subbox on which the polynomial f has not strict sign so that f must have a zero in $\mathbf{Q}$.

We assume for the rest of this section that the test for $t = \pm 1$ did not reveal that the family (1) is not robustly stable for $\mathbf{Q}$. It remains to check the open interval $(-1, 1)$ for common zeros of $\tilde{h}$ and $\tilde{g}$. This interval may be yet tightened using the procedure described in [2]. [2]
In order to use our previous notation we assume that $\dim(\mathbf{Q}) = l - 1$.

[1] after having transformed $\mathbf{Q}$ to the unit box

[2] A tightening might also help to avoid to consider the special cases $t = -1$ or $t = 1$.

For simplicity, we write $\mathbf{x} = (x_1, \ldots, x_l) = (q_1, \ldots, q_{l-1}, t)$. After having transformed $\mathbf{Q} \times [-1, 1]$ to $\mathbf{U}$ we check whether the set

$$\mathcal{P}(\mathbf{U}) := \{(\tilde{h}(\mathbf{x}), \tilde{g}(\mathbf{x})) : \mathbf{x} \in \mathbf{U}\}$$

contains the origin. By expanding $\tilde{h}(\mathbf{x})$ and $\tilde{g}(\mathbf{x})$ simultaneously into their Bernstein forms we obtain a set of points $(b_I(\tilde{h}, \mathbf{U}), b_I(\tilde{g}, \mathbf{U}))$ in the plane, denoted by $b_I(\mathbf{U})$. Then we compute their convex hull which can be done in optimal time using $O(\nu \log \nu)$ operations, e.g.[19],[20], where ν denotes the number of points. As in [17] it can be shown that

$$\text{Conv } \mathcal{P}(\mathbf{U}) \subseteq \text{ Conv} B(\mathbf{U})$$

holds true. By any standard convex hull algorithm, cf. [19],[20], we can check whether the origin belongs to Conv $B(U)$. If it is outside, the family of polynomials (1) is robustly stable. Otherwise an inclusion test given in [1] is performed. If it fails, i.e., it can not be verified that the origin is in the set $\mathcal{P}(U)$, the sweep procedure is applied splitting the domain to obtain two new patches on which we proceed as before. If no patch remains and all inclusion tests have failed the family of polynomials is robustly Schur stable. Otherwise, if an inclusion test is successful the algorithm aborts immediately because we have found an unstable polynomial.

5. Examples

Ex. 1: We consider the following univariate characteristic polynomial p_m of degree $m+1$ taken from [21] arising in the investigation of asymptotic stability properties of numerical methods for delay differential equations:

$$p_m(\lambda;\, a,\, b,\, \Theta) = \Big((a\Theta - m)\, \lambda^{m+1} +\ (m + a\,(1-\Theta))\, \lambda^m + b\Theta\lambda + b\,(1-\Theta)\Big)\,/m$$

where a and b are real parameters and Θ is a real number taken from $[0, 1]$. A necessary condition for Schur stability is that $|A_{m+1}| > |A_0|$, where A_k are compact intervals with

$$\{a_k(\Theta) : \Theta \in [0, 1]\} \subseteq A_k, k = 0, m + 1.$$

A simple calculation gives that for fixed $m \in \{1, 2, \ldots\}$ and $a \in \mathbf{R}$ the necessary condition is violated e.g. for $|b| \geq c$, where c is given by

$$c = \begin{cases} m & \text{if } a \in (0, m), \\ m|\frac{a}{m} - 1| & \text{if } a < 0 \text{ or } a > 2m. \end{cases}$$

In Table I the results of the stability analysis are given for $m = 4$, $a = -3$, and different values of b. The letter A stands for the fact that there exists a stable member but our algorithm reports that the polynomial family is unstable.

Table I. Results for $m = 4, a = -3$

b	result	cause
$\|b\| \geq 7$	unstable	Necessary condition is violated.
$b = 3, 3.2, 3.4, \ldots, 6.8$	unstable	Unstable for $\Theta = 0.5$
$b = -3.2, -3.0, \ldots, 2.8$	stable	
$b = -3.8, \ldots, -3.4$	unstable	A
$b = -6.8, \ldots, -4$	unstable	Unstable for $\Theta = 0.5$

Ex. 2: Because of the lack of larger robust Schur stability problems in literature we choose the following somewhat artificial problem: In [5] the characteristic polynomial associated with the control of the Fiat Dedra engine is considered. This polynomial to be checked for Hurwitz stability is of seventh degree with seven parameters entering quadratically into the transfer function. The parameters vary inside the intervals

$$\begin{array}{ll} q_1 \in [2.1608, 3.4329] & \\ q_2 \in [0.1027, 0.1627] & q_3 \in [0.0357, 0.1139] \\ q_4 \in [0.2539, 0.5607] & q_5 \in [0.0100, 0.0208] \\ q_6 \in [2.0247, 4.4962] & q_7 \in [1.0000, 10.000]. \end{array}$$

By the bilinear mapping we transformed this polynomial family into one having all its zeros in the open unit disc (this causes a considerable blowing up of the original polynomial). Our algorithm reports after 7.1 sec. (using a HP Workstation 9000/755) that this family is robustly stable, where 54 sweeps with a sweep depth of 12 are required.

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