Bounds for the Range of a Bivariate Polynomial over a Triangle
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1. Introduction

The problem of finding bounds for the range of a multivariate polynomial ($K = \mathbb{R}$ or $K = \mathbb{C}$)

$$p(x_1, \ldots, x_l) = \sum_{i_1, \ldots, i_l = 0}^{n} a_{i_1, \ldots, i_l} \, x_1^{i_1} \cdot \ldots \cdot x_l^{i_l}$$

$$\text{with} \quad a_{i_1, \ldots, i_l} \in K$$

over a compact set $\emptyset \neq D \subset K^l$ has attracted the interest of many reseachers working in interval mathematics and related fields. Methods to obtain such bounds range from sampling the polynomial p at certain points (5; 9; 21) to the use of centered forms, e.g. (17). Expansion of the polynomial p into multivariate Bernstein polynomials appeared to be a particularly useful tool. Papers which adopt this approach include

– univariate real case ($l = 1$, $K = \mathbb{R}$):
  D is a compact interval (2; 3; 4; 16; 21; 23; 24; 26), (21) considers also complex coefficients;

– univariate complex case ($l = 1$, $K = \mathbb{C}$):
  D is a rectangle (25) or a disc (10);

– multivariate real case ($l \geq 1$, $K = \mathbb{R}$):
  D is a box (5; 6) or the unit simplex (5).

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In (2; 3; 10; 21; 23; 24; 25; 26) the obtained bounds are improved by raising the degree of the Bernstein expansion. However, it turned out, e.g. (28), that subdivision is superior to degree elevation. Based on the results in (15), subdivision was applied subsequently in (4; 5; 6; 16).

Applications to robust stability problems are given in (6; 7; 19; 22; 29). We also mention a growing interest in the application of interval Bernstein polynomials in Computer Aided Geometric Design, e.g. (13; 27).

In this paper we consider the following

PROBLEM. *Let a bivariate polynomial*

$$p(x, y) = \sum_{\mu,\nu=0}^{n} a_{\mu\nu} x^{\mu} y^{\nu} \quad \text{with} \quad a_{\mu\nu} \in K \tag{1}$$

and a triangle S in the real plane be given. Then we want to find an enclosure for the range $p(S)$ of p over S.

If $K = \mathbb{R}$ (resp., $K = \mathbb{C}$) we simply call p a *real* (resp., a *complex*) *polynomial.*

Compared to rectangles in the plane, triangles have the advantage that far more general geometries can be treated, viz. those which allow a triangulation, cf. Fig. 1.

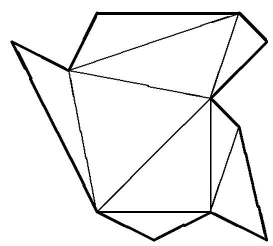

Fig. 1 Triangulation

The organization of our paper is as follows: In the next section we recall from (5) the Bernstein expansion of a bivariate polynomial on the unit square and on the unit triangle. In Sect. 3 we introduce our partition of the unit triangle and apply subdivision. Computation of the Bernstein coefficients, i.e. the coefficients of the Bernstein expansion, on the subregions generated by subdivision is presented in Sect. 4. In Sect. 5 we investigate the convergence of the sequence of enclosures obtained by the Bernstein coefficients. In Sect. 6 we consider the case that we want to find either lower or upper bounds for the range of a real polynomial. In this case we are able to speed up the procedure considerably. In the last section we give related results and discuss the restriction to the bivariate case.

2. Bernstein Expansion of Bivariate Polynomials

In this section we recall from (5) the expansion of a bivariate polynomial on a rectangle and on a triangle. We can confine ourselves to

the *unit square* $S = [0,1]^2$

and to

the *unit triangle* $T = \{(x,y) \in \mathbb{R}^2 \mid x, y \geq 0 \;\wedge\; x + y \leq 1\}$

since any nonempty rectangle and triangle can be mapped by an affine transformation onto S and T, respectively. We write U for S and T if a statement holds true for S and T likewise. For an integer k, we associate with S and T index ranges

$$I_S^{(k)} = \{(i,j) \mid i,j = 0,1,\ldots,k\},$$

$$I_T^{(k)} = \{(i,j) \mid i,j = 0,1,\ldots,k \quad \textit{with} \quad i+j \leq k\}.$$

In the sequel we take $k \geq n$ and $k \geq \max\{\mu + \nu \mid a_{\mu\nu} \neq 0\}$ if we are considering a polynomial on S and T, respectively.

The *Bernstein polynomials* $p_{ij}^{(k)}$ *of kth degree*, $(i,j) \in I_U^{(k)}$, are defined by, cf. (18), p. 51,

$$p_{ij}^{(k)}(x,y) = \begin{cases} \binom{k}{i}\binom{k}{j} x^i (1-x)^{k-i} y^j (1-y)^{k-j} & \text{on } S \\ \\ \binom{k}{i\;\;j} x^i y^j (1-x-y)^{k-i-j} & \text{on } T, \end{cases} \tag{2}$$

where $\displaystyle \binom{k}{i\;\;j} := \binom{k}{i}\binom{k-i}{j}.$

Expanding $x^\mu y^\nu$ into Bernstein polynomials and substituting into (1) results in the *Bernstein expansion of* p *on* U:

$$p(x,y) = \sum_{(i,j) \in I_U^{(k)}} b_{ij}^{(k)} p_{ij}^{(k)}(x,y), \tag{3}$$

where the *Bernstein coefficients* $b_{ij}^{(k)}$ are given by

$$b_{ij}^{(k)} = \sum_{u=0}^{i} \sum_{v=0}^{j} \frac{\binom{i}{u}\binom{j}{v}}{\xi_{uv}^{(k)}} a_{uv} \quad , \quad (i,j) \in I_U^{(k)}, \tag{4}$$

with

$$\xi_{uv}^{(k)} = \begin{cases} \binom{k}{u}\binom{k}{v} & \text{on } S \\ \binom{k}{u\ v} & \text{on } T \end{cases}$$

and with the convention that $a_{uv} = 0$ for $u > k$ or $v > k$.

THEOREM 1. ((5)). *If p is a real polynomial (1) and $p(U) = [\underline{m}\,,\overline{m}]$ then we have for each $k \geq n$*

$$\min_{(i,j)\in I_U^{(k)}} b_{ij}^{(k)} \leq \underline{m} \quad , \quad \overline{m} \leq \max_{(i,j)\in I_U^{(k)}} b_{ij}^{(k)} .$$

Equality holds in the left–hand inequality if and only if

$$\min_{(i,j)\in I_U^{(k)}} b_{ij}^{(k)} \in \begin{cases} \{b_{00}^{(k)}, b_{0k}^{(k)}, b_{k0}^{(k)}, b_{kk}^{(k)}\} & \textit{for } U = S \\ \{b_{00}^{(k)}, b_{0k}^{(k)}, b_{k0}^{(k)}\} & \textit{for } U = T. \end{cases} \tag{5}$$

A similar statement holds for the right–hand inequality.
Furthermore,

$$\max \left\{ \underline{m} - \min_{(i,j)\in I_U^{(k)}} b_{ij}^{(k)} \, , \, \max_{(i,j)\in I_U^{(k)}} b_{ij}^{(k)} - \overline{m} \right\} \leq \frac{c}{k} \, ,$$

where c is a constant not depending on k.

So for a real polynomial the minimum and the maximum of the Bernstein coefficients provide the enclosure wanted. In the case of a complex polynomial we obtain similarly as in (21) the *convex hull property* of the Bernstein coefficients

$$\operatorname{conv} p(U) \subseteq \operatorname{conv} \left\{ b_{ij}^{(k)} \mid (i,j) \in I_U^{(k)} \right\} . \tag{6}$$

Th. 1 and (6) follow immediately from the representation (3) and the fact that the Bernstein polynomials are nonnegative on U and form a partition of unity, e.g. (2).

3. Partition of the Unit Triangle and Subdivision

In partitioning the unit triangle T we are led by the following useful fact (6): When we subdivide a square by successively halving in both coordinate directions and calculate the Bernstein coefficients on a generated subsquare then we obtain as a byproduct of the computation intermediate values of the calculation of the Bernstein coefficients on the neighbouring subsquares. This nice property does not seem fully shared when subdividing T into triangles. This is one of the reasons why we partition T into the square S^1_{11} of edge length 1/2 and the two remaining triangles T^1_1 and T^1_2 at the top left and the bottom right position, cf. Fig. 2.

Then we subdivide the square S^1_{11} into the four subsquares S^2_{11} , S^2_{12} , S^2_{21} , and S^2_{22} of edge length 1/4 and partition the triangles T^1_1 and T^1_2 similarly as we partitioned T, cf. Fig. 3.

Continuing in this way, we obtain a sequence of subsquares

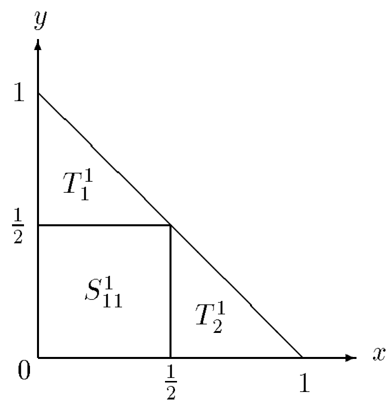

Fig. 2 Partition of the unit triangle T

$$S^m_{r,s} = 2^{-m}[r-1,r] \times [s-1,s]\ ,\ r = 1,\ldots,2^m - 1\ ,\ s = 1,\ldots,2^m - r\ .$$

Only at the right boundary of T we are using triangles, viz. for $t = 1,\ldots,2^m$,

$$T^m_t = \text{conv}\,\{2^{-m}(t-1,2^m-t)\,,2^{-m}(t,2^m-t)\,,2^{-m}(t-1,2^m-t+1)\}.$$

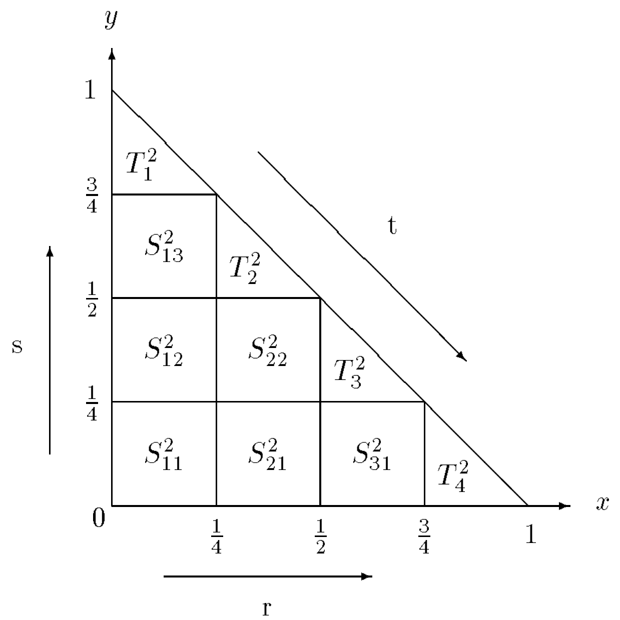

Fig. 3 Subdivision of the unit triangle at level $m = 2$

4. Computation of the Bernstein Coefficients on Subregions

To find an enclosure for the range of the polynomial p over T by Bernstein expansion we have to compute for a fixed subdivision level m the Bernstein coefficients of p on all subregions $S_{r,s}^m$ and T_t^m, i.e. the Bernstein coefficients of the polynomial p when shifted from the respective subregion back to S or T.

For squares we take now $k = n$. The explicit transformation back to S and therefore a calculation involving the original coefficients $a_{\mu\nu}$ is avoided by the following lemma (5).

LEMMA 1. *Let* $m \geq 1$, $r \in \{1, \ldots, 2^m - 1\}$, $s \in \{1, \ldots, 2^m - r\}$. *Then the Bernstein coefficients on the subsquares at subdivision level* $m + 1$ *are given by,* $(i, j) \in I_S^{(n)}$,

$$\begin{aligned} b_{ij}\left(S_{2r-1\,,\,2s-1}^{m+1}\right) &= 2^{-i-j} \sum_{u=0}^{i} \sum_{v=0}^{j} \binom{i}{u} \binom{j}{v} b_{uv}\left(S_{rs}^m\right) , \\ b_{i,n-j}\left(S_{2r-1\,,\,2s}^{m+1}\right) &= 2^{-i-j} \sum_{u=0}^{i} \sum_{v=0}^{j} \binom{i}{u} \binom{j}{v} b_{u,n-v}\left(S_{rs}^m\right) , \end{aligned}$$

$$b_{n-i,j}\left(S^{m+1}_{2r,2s-1}\right) = 2^{-i-j}\sum_{u=0}^{i}\sum_{v=0}^{j}\binom{i}{u}\binom{j}{v} b_{n-u,v}\left(S^m_{rs}\right) ,$$

$$b_{n-i,n-j}\left(S^{m+1}_{2r,2s}\right) = 2^{-i-j}\sum_{u=0}^{i}\sum_{v=0}^{j}\binom{i}{u}\binom{j}{v} b_{n-u,n-v}\left(S^m_{rs}\right) .$$

The Bernstein coefficients on the subsquares at the right boundary of T not covered by Lemma 1 like $S^{m+1}_{2^{m+1}-1,1}$ and $S^{m+1}_{1,2^{m+1}-1}$ can be calculated similarly.

For triangles we choose $k = 2n$. Analogously to Lemma 1, the explicit transformation back to T can be avoided, cf. (8) for similar results:

LEMMA 2. *Let $m \geq 1$ and $t \in \{1, \ldots, 2^m\}$. Then the Bernstein coefficients on the subtriangles at subdivision level $m + 1$ are given by, $(i, j) \in I_T^{(2n)}$,*

$$b_{i,2n-i-j}\left(T^{m+1}_{2t-1}\right) = 2^{-i-j}\sum_{u=0}^{i}\sum_{v=0}^{j}\binom{i}{u}\binom{j}{v} b_{u,2n-u-v}\left(T^m_t\right) ,$$

$$b_{2n-i-j,j}\left(T^{m+1}_{2t}\right) = 2^{-i-j}\sum_{u=0}^{i}\sum_{v=0}^{j}\binom{i}{u}\binom{j}{v} b_{2n-u-v,v}\left(T^m_t\right) .$$

To compute the Bernstein coefficients on subsquares we propose the following procedure:

PROCEDURE 1. *Let $m \geq 1$, $r \in \{1, \ldots, 2^m - 1\}$, $s \in \{1, \ldots, 2^m - r\}$. Start with setting $\beta^{(0)}_{\mu\nu} = b_{\mu\nu}\left(S^m_{rs}\right)$ for all $(\mu, \nu) \in I_S^{(n)}$;*
then for $\kappa = 0, \ldots, n-1$

$$\gamma^{(\kappa)}_{\mu\nu} = \left(\beta^{(\kappa)}_{\mu\nu} + \beta^{(\kappa)}_{\mu,\nu+1}\right)\Big/ 2 , \quad \mu = 0, \ldots, n-\kappa , \quad \nu = 0, \ldots, n-\kappa-1 ,$$

$$\beta^{(\kappa+1)}_{\mu\nu} = \left(\gamma^{(\kappa)}_{\mu\nu} + \gamma^{(\kappa)}_{\mu+1,\nu}\right)\Big/ 2 \quad \text{for all } (\mu, \nu) \in I_S^{(n-\kappa-1)},$$

$$\left.\begin{aligned} \delta^{(\kappa,1)}_{1\nu} &= \gamma^{(\kappa)}_{0\nu} , \\ \delta^{(\kappa,1)}_{2\nu} &= \gamma^{(\kappa)}_{n-\kappa,\nu} , \end{aligned}\right\} \nu = 0, \ldots, n-\kappa-1 ,$$

$$\left.\begin{aligned} \delta^{(\kappa,1)}_{3\mu} &= \left(\beta^{(\kappa)}_{\mu 0} + \beta^{(\kappa)}_{\mu+1,0}\right)\Big/ 2 , \\ \delta^{(\kappa,1)}_{4\mu} &= \left(\beta^{(\kappa)}_{\mu,n-\kappa} + \beta^{(\kappa)}_{\mu+1,n-\kappa}\right)\Big/ 2 , \end{aligned}\right\} \mu = 0, \ldots, n-\kappa-1 ;$$

then for $\sigma = 1, 2, 3, 4$, $\kappa = 0, \ldots, n-2$, and $\tau = 1, \ldots, n-\kappa-1$

$$\delta^{(\kappa,\tau+1)}_{\sigma\nu} = \left(\delta^{(\kappa,\tau)}_{\sigma\nu} + \delta^{(\kappa,\tau)}_{\sigma,\nu+1}\right)\Big/ 2 , \quad \nu = 0, \ldots, n-\kappa-\tau-1 . \qquad (7)$$

LEMMA 3. *The Bernstein coefficients on the subsquares at subdivision level m can be obtained by Procedure 1 from the following relations with $(i,j) \in I_S^{(n)}$:*

$$b_{ij}\left(S^{m+1}_{2r-1,2s-1}\right) = \begin{cases} \delta_{10}^{(i,j-i)} & if \quad i<j \\ \delta_{30}^{(j,i-j)} & if \quad j<i \\ \beta_{00}^{(i)} & if \quad i=j \end{cases}$$

$$b_{ij}\left(S^{m+1}_{2r-1,2s}\right) = \begin{cases} \delta_{1j}^{(i,n-i-j)} & if \quad i+j<n \\ \delta_{40}^{(n-j,i+j-n)} & if \quad n<i+j \\ \beta_{0j}^{(i)} & if \quad i+j=n \end{cases}$$

$$b_{ij}\left(S^{m+1}_{2r,2s-1}\right) = \begin{cases} \delta_{3i}^{(j,n-i-j)} & if \quad i+j<n \\ \delta_{20}^{(n-i,i+j-n)} & if \quad n<i+j \\ \beta_{i0}^{(j)} & if \quad i+j=n \end{cases}$$

$$b_{ij}\left(S^{m+1}_{2r,2s}\right) = \begin{cases} \delta_{4i}^{(n-j,j-i)} & if \quad i<j \\ \delta_{2j}^{(n-i,i-j)} & if \quad j<i \\ \beta_{ii}^{(n-i)} & if \quad i=j\,. \end{cases}$$

Proof. First we show by induction on $\kappa = 0,\ldots,n$ that for all $(\mu,\nu) \in I_S^{(n-\kappa)}$

$$\beta_{\mu\nu}^{(\kappa)} = 4^{-\kappa} \sum_{u,v=0}^{\kappa} \binom{\kappa}{u}\binom{\kappa}{v} b_{u+\mu,v+\nu}(S^m_{rs})\,.$$

Therefore we have for $\nu = 0,\ldots,n-\kappa-1$

$$\begin{aligned} \delta_{1\nu}^{(\kappa,1)} &= \gamma_{0\nu}^{(\kappa)} = \left(\beta_{0\nu}^{(\kappa)} + \beta_{0,\nu+1}^{(\kappa)}\right)\Big/ 2 \\ &= 2^{-2\kappa-1} \sum_{u=0}^{\kappa}\sum_{v=0}^{\kappa+1} \binom{\kappa}{u}\binom{\kappa+1}{v} b_{u,v+\nu}(S^m_{rs}) \end{aligned} \tag{8}$$

with similar relations for $\delta_{\sigma\nu}^{(\kappa,1)}$, $\sigma = 2,3,4$.
By induction on τ it follows from (7) that for $i = 1,\ldots,n-1$, $j = i+1,\ldots,n$

$$\delta_{10}^{(i,j-i)} = 2^{i-j+1} \sum_{h=0}^{j-i-1} \binom{j-i-1}{h} \delta_{1h}^{(i,1)}\,,$$

hence by (8)

$$\begin{aligned}
2^{i+j}\delta_{10}^{(i,j-i)} &= \sum_{h=0}^{j-i-1}\binom{j-i-1}{h}\sum_{u=0}^{i}\sum_{v=0}^{i+1}\binom{i}{u}\binom{i+1}{v} b_{u,v+h}(S_{rs}^m)\\
&= \sum_{u=0}^{i}\sum_{v=0}^{j}\binom{i}{u}\sum_{h=\max\{0,v-i-1\}}^{\min\{v,j-i-1\}}\binom{j-i-1}{h}\binom{i+1}{v-h} b_{u,v}(S_{rs}^m)\\
&= \sum_{u=0}^{i}\sum_{v=0}^{j}\binom{i}{u}\binom{j}{v} b_{u,v}(S_{rs}^m)\,,
\end{aligned}$$

where the last equality holds by the Vandermonde convolution formula, e.g. (20). By Lemma 1 we obtain now

$$\delta_{10}^{(i,j-i)} = b_{ij}\left(S_{2r-1,2s-1}^{m+1}\right)\ .$$

The proof of the other relations is similar.

To compute the Bernstein coefficients on subsquares by Procedure 1 requires $\frac{4}{3}n^3+\frac{5}{2}n^2+\frac{7}{6}n$ additions/multiplications (binary shifts). That is slightly less compared to $\frac{3}{2}n(n+1)^2$ additions/multiplications (binary shifts) when the sweeps based subdivision (6) is used. The Bernstein coefficients on subtriangles can be computed similarly (14). Procedure 1 generalizes to the calculation of the Bernstein coefficients on subboxes generated by subdivision in the l–variate case for general l.

5. Convergence Results

We set $C := \operatorname{conv} p(T)$. The proof of the following theorem is similar to the proof of the enclosure (6) and is therefore suppressed.

THEOREM 2. *Let $m \geq 1$ and let C_m denote the convex hull of the Bernstein coefficients on all subregions at subdivision level m. Then $C \subseteq C_m$.*

Note that if a generating point of C_m is lying on a vertex of the table of Bernstein coefficients on a subregion according to (5) then an improvement of the enclosure C_m at this point is not possible.

We denote by $d(A,B)$ the *Hausdorff distance* of two compact, convex sets $A, B \subset \mathbb{C}$, i.e.

$$d(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}|a-b|\,,\sup_{b\in B}\inf_{a\in A}|a-b|\right\}.$$

The next theorem shows that each elevating of the subdivision level reduces $d(C_m, C)$ by a factor of about $\frac{1}{4}$.

THEOREM 3. *Let $m \geq 1$. Then $d(C_m, C) \leq \gamma 2^{-2m}$, where γ is a constant not depending on m.*

Proof. Let the subdivision level m be fixed. Since $C \subseteq C_m$ by Th. 2 and C_m is a convex polytope we have (∂ denotes the boundary)

$$d(C_m, C) = \max_{a \in \partial C_m} \min_{b \in \partial C} |a - b| \,. \tag{9}$$

Moreover, we see by a geometric argument that the maximum appearing in (9) is attained at a vertex, i.e. at a Bernstein coefficient on a subregion S_{rs}^m or T_t^m. As in the real case (5), the following inequalities are valid for all $r \in \{1, \ldots, 2^m - 1\}$, $s \in \{1, \ldots, 2^m - r\}$, and $(i,j) \in I_S^{(n)}$, and $t \in \{1, \ldots, 2^m\}$, and $(i,j) \in I_T^{(2n)}$, respectively:

$$\left| p_{S_{rs}^m}\left(\tfrac{i}{n}, \tfrac{j}{n}\right) - b_{ij}\left(S_{rs}^m\right) \right| \quad \leq \frac{1}{4} \sum_{\substack{u,v=0 \\ u \geq 2 \,\vee\, v \geq 2}}^{n} \eta_{uv} \left| c_{uv}(S_{rs}^m) \right| ,$$

$$\left| p_{T_t^m}\left(\tfrac{i}{2n}, \tfrac{j}{2n}\right) - b_{ij}\left(T_t^m\right) \right| \leq \frac{1}{4} \Big(\left| c_{11}(T_t^m) \right| + \sum_{\substack{u,v=0 \\ u \geq 2 \,\vee\, v \geq 2}}^{n} \eta_{uv} \left| c_{uv}(T_t^m) \right| \Big) ,$$

where $\eta_{uv} := (\max(u-1,0))^2 + (\max(v-1,0))^2$ and $c_{uv}(S_{rs}^m)$ and $c_{uv}(T_t^m)$ denote the coefficients of the polynomial $p_{S_{rs}^m}$ and $p_{T_t^m}$, respectively, which is the polynomial p shifted to U from the subregion S_{rs}^m and T_t^m , respectively. The claim follows now by estimating $|c_{uv}|$ from above as in (19).

The constant γ can be given explicitly in terms of the coefficients $a_{\mu\nu}$ of p, cf. (14).

The convex hull C may overestimate $p(T)$ by a large amount. Therefore, it may be advantageous to take the union of the approximations (provided by the respective Bernstein coefficients) for the convex hulls of the ranges of p over the subregions generated by subdivision to get a tighter enclosure of $p(T)$.

6. One-sided Bounds in the Real Case

Often only one-sided bounds for the range of a multivariate real polynomial are required, e.g. if a matrix with coefficients depending polynomially on parameters x, y varying inside T is to be tested for nonsingularity. If we are interested in a lower bound (upper bounds can be treated analogously), we apply a depth first strategy, i.e. we continue the subdivision process on a subregion which has smallest Bernstein coefficient. If this Bernstein coefficient, say β, is sharp according to (5) we can remove from the list of subregions to be inspected those subregions which have smallest Bernstein coefficient which is greater or equal to β. If β is additionally the minimum of the Bernstein coefficients on a selection of subregions covering T then the minimum of p on T is attained in β. The process outlined may be speeded up by replacing a complete subdivision of a square into four subsquares by subdivision into two rectangles of the same area, computing the Bernstein coefficients on both rectangles, and comparing with the Bernstein coefficients on other subregions, cf. (29).

7. Conclusions

We conclude this paper with some remarks:

- All rounding errors appearing in the computation of the Bernstein coefficients can be bounded similarly as in (4).

- H. Hong and V. Stahl consider in (12) the Bernstein form $B(X)$ of a real univariate polynomial p, i.e. the interval extension given by the smallest and the largest of its Bernstein coefficients on the interval X. They show that the Bernstein form is *inclusion monotone*, i.e. $X \subseteq Y$ implies $B(X) \subseteq B(Y)$. Their results extends in the multivariate real case to the Bernstein form associated with boxes and in the twodimensional real case to the Bernstein form associated with triangles. Details will be given in (14).

- It seems to be impossible to extend the partition given in Sect. 3 to the three–dimensional case in a reasonable way: Suppose we want to decompose the standard simplex $S = \operatorname{conv}\{(0,0,0), (1,0,0), (0,1,0), (0,0,1)\}$ into cubes and pairwise congruent simplices similar to S, where the cubes are obtained by successively halving similarly as in Sect. 3. From a computational point of view it is reasonable to claim that these polytopes decompose S in such a way that any intersection of two polytopes does not have interior points. Then a tetrahedron remains at the top of S which is again to be decomposed into pairwise congruent simplices similar to S. Therefore, we can assume without loss of generality that we have to decompose S only into simplices. But then (here we follow a personal communication of Prof. Dr. Hertel) S tiles the $\mathbb{R}^3$ and

S must be equidecomposable to a cube. But this is a contradiction to the fact that S and a cube (of the same volume) are not equidecomposable (1; 11).

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