Sign Regular Matrices Having the Interval Property

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Outline

- Interval property of matrices.

- Totally nonnegative matrices.

- The Cauchon Algorithm.

- Sign regular matrices.

- Related recent work.

Notation

$\mathbb{IR}$: set of the compact, nonempty real intervals $[a] = [\underline{a}, \overline{a}]$, $\underline{a} \le \overline{a}$,
$\mathbb{IR}^n$: set of n-vectors with components from $\mathbb{IR}$, *interval vectors*
$\mathbb{IR}^{n\times n}$: set of n-by-n matrices with cmpts. from $\mathbb{IR}$. *interval matrices*

Elements from $\mathbb{IR}^n$ and $\mathbb{IR}^{n\times n}$ may be regarded as vector intervals and matrix intervals, respectively, w.r.t. the usual entrywise partial ordering, e.g.,

$$\begin{aligned} [A] &= \left([a_{ij}]\right)_{i,j=1}^n = \left([\underline{a}_{ij}, \overline{a}_{ij}]\right)_{i,j=1}^n \\ &= [\underline{A}, \overline{A}], \quad \text{where } \underline{A} = \left(\underline{a}_{ij}\right)_{i,j=1}^n, \ \overline{A} = \left(\overline{a}_{ij}\right)_{i,j=1}^n. \end{aligned}$$

A *vertex matrix* of $[A]$ is a matrix $A = (a_{ij})_{i,j=1}^n$ with $a_{ij} \in \{\underline{a}_{ij}, \overline{a}_{ij}\}$, $i, j = 1, \ldots, n$.

Interval property

We say that a class $\mathcal{C}$ of n-by-n matrices possesses the *interval property* if for any n-by-n interval matrix $[A]$ the membership $[A] \subseteq \mathcal{C}$ can be inferred from the membership to $\mathcal{C}$ of a specified set of its vertex matrices.

Examples of matrices possessing the interval property

- M-matrices or, more generally, inverse-nonnegative matrices [Kuttler, 1971], i.e., matrices having an entry-wise nonnegative inverse; here only the bound matrices $\underline{A}$ and $\overline{A}$ are required to be in the class;

- inverse M-matrices [Johnson and Smith, 2002], where all vertex matrices are needed;

- positive definite matrices [Bialas and Garloff, 1984], [Rohn, 1994], where a subset of cardinality 2^{n-1} is required (here only symmetric matrices in $[A]$ are considered).

Def. A real matrix is called *totally nonnegative* and *totally positive* if all its minors are nonnegative and positive, respectively.

S. M. Fallat and C. R. Johnson, *Totally Nonnegative Matrices*, Princeton Univ. Pr., 2011.

A. Pinkus, *Totally Positive Matrices*, Cambridge Univ. Pr., 2010.

Totally nonnegative matrices

A suitable partial order for the totally nonnegative matrices is the *checkerboard order*. For $A, B \in \mathbb{R}^{n \times n}$ define

$$A \leq^* B := (-1)^{i+j} a_{ij} \leq (-1)^{i+j} b_{ij}, \quad i, j = 1, 2, \ldots, n.$$

This partial order is related to the usual entry-wise partial order by

$A \leq^* B \Leftrightarrow A^* \leq B^*$, where $A^* := SAS$, $S := \mathrm{diag}(1, -1, \ldots, (-1)^{n+1})$,

is the *checkerboard transformation*.

A matrix interval $[\underline{A}, \overline{A}]$ with respect to the usual entry-wise partial order can be represented as an interval $[\downarrow A, \uparrow A]^*$ with respect to the checkerboard order, where

$$(\downarrow A)_{ij} := \begin{cases} \underline{a}_{ij} & \text{if } i+j \text{ is even,} \\ \overline{a}_{ij} & \text{if } i+j \text{ is odd,} \end{cases}$$

$$(\uparrow A)_{ij} := \begin{cases} \overline{a}_{ij} & \text{if } i+j \text{ is even,} \\ \underline{a}_{ij} & \text{if } i+j \text{ is odd.} \end{cases}$$

If A is non-singular and totally nonnegative then $0 \leq^* A^{-1}$ and therefore, $0 \leq (A^{-1})^* = (A^*)^{-1}$. Since A^* is inverse nonnegative all results for inverse nonnegative matrices carry over to the totally nonnegative matrices by the checkerboard transformation, e.g., if A and B are non-singular and totally nonnegative then it follows that $A \leq^* B \Rightarrow B^{-1} \leq^* A^{-1}$.

Theorem (M. Adm and JG, 2013): If $\downarrow A$ and $\uparrow A$ are non-singular and totally nonnegative then the whole matrix interval $[\downarrow A, \uparrow A]^*$ is non-singular and totally nonnegative.

This theorem is based on a conjecture (JG, 1982) which has been settled for some subclasses of totally nonnegative matrices, e.g., for the totally positive matrices, i.e., matrices having all their minors positive.

Cauchon Algorithm

We denote by $\leq$ the lexicographic order on $\mathbb{N}^2$, i. e.,

$$(g,h) \leq (i,j) :\Leftrightarrow\ (g < i) \text{ or } (g = i \text{ and } h \leq j).$$

Set $E^\circ := \{1,...,n\}^2 \setminus \{(1,1)\}$, $E := E^\circ \cup \{(n+1,2)\}$.

Let $(s,t) \in E^\circ$. Then $(s,t)^+ := min\,\{(i,j) \in E \mid (s,t) \leq (i,j),\ (s,t) \neq (i,j)\}$.

Algorithm: Let $A \in \mathbb{R}^{n,n}$. As r runs in decreasing order over the set E, we define matrices $A^{(r)} = (a_{ij}^{(r)}) \in \mathbb{R}^{n,n}$ as follows.

1. Set $A^{(n+1,2)} := A$.

2. For $r = (s,t) \in E^{\circ}$:

(a) if $a_{st}^{(r^+)} = 0$ then put $A^{(r)} := A^{(r^+)}$.

(b) if $a_{st}^{(r^+)} \neq 0$ then put

$$a_{ij}^{(r)} := \begin{cases} a_{ij}^{(r^+)} - \dfrac{a_{it}^{(r^+)} a_{sj}^{(r^+)}}{a_{st}^{(r^+)}} & \text{for } i < s \text{ and } j < t, \\ a_{ij}^{(r^+)} & \text{otherwise.} \end{cases}$$

3. Set $\tilde{A} := A^{(1,2)}$ is called *the matrix obtained from A (by the Cauchon Algorithm)*.

If $n = 5$ and A is totally positive, then

$$\tilde{A} = \begin{bmatrix} \frac{[12345]}{[2345]} & \frac{[1234|2345]}{[234|345]} & \frac{[123|345]}{[23|45]} & \frac{[12|45]}{[2|5]} & a_{15} \\ \frac{[2345|1234]}{[345|234]} & \frac{[2345]}{[345]} & \frac{[234|345]}{[34|45]} & \frac{[23|45]}{[3|5]} & a_{25} \\ \frac{[345|123]}{[45|23]} & \frac{[345|234]}{[45|34]} & \frac{[345]}{[45]} & \frac{[34|45]}{[4|5]} & a_{35} \\ \frac{[45|12]}{[5|2]} & \frac{[45|23]}{[5|3]} & \frac{[45|34]}{[5|4]} & \frac{[45]}{[5]} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix}$$

Theorem (Goodearl, Launois, and Lenagan, 2011):

1. A is totally nonnegative iff $0 \leq \tilde{A}$ and for all $i, j = 1, \ldots, n$
   $\tilde{a}_{ij} = 0 \Rightarrow \ \tilde{a}_{ik} = 0 \ \ k = 1, \ldots, j-1,$ or $\tilde{a}_{kj} = 0 \ \ k = 1, \ldots, i-1.$

$$\tilde{A} = \begin{bmatrix} & & & 0 & & \\ & \text{or} & \rightarrow & \vdots & & \\ & \downarrow & & 0 & & \\ 0 & \ldots & 0 & \boxed{0} & & \\ & & & & & \\ & & & & & \end{bmatrix}$$

2. If A is totally nonnegative matrix then A is nonsingular iff $0 < \text{diag}(\tilde{A})$.

Theorem*: Let A, B be nonsingular and totally nonnegative matrices and let $A \leq^* Z \leq^* B$. Then

1. $\tilde{A} \leq^* \tilde{Z} \leq^* \tilde{B}$;

2. Z is nonsingular and totally nonnegative;

3. if A, B possess the same pattern of zero minors then Z has this pattern, too.

*M. Adm and J. Garloff, Intervals of totally nonnegative matrices, *Linear Algebra and its Applications* 439 (Dec. 2013), pp. 3796-3806

The assumption of nonsingularity of certain principal minors cannot be relaxed:

$$A := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \leq^* Z := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \leq^* B := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

totally nonnegative has a negative minor totally nonnegative

Corollary Let $A, B, Z \in \mathbb{R}^{n,n}$ with $A \leq^* Z \leq^* B$. If A, B are totally nonnegative and

$$A[2, \ldots, n] \text{ and } B[2, \ldots, n]$$

or

$$A[1, \ldots, n-1] \text{ and } B[1, \ldots, n-1]$$

are nonsingular, then Z is totally nonnegative, too.

Sign regular matrices

Application of the Cauchon Algorithm to *sign regular matrices*, i.e., matrices whose minors of the same order have the same sign or vanish. The interval property involving only $\downarrow A, \uparrow A$ is established for some subclasses. These include:

- nonsingular totally nonpositive matrices (all minors are nonpositive),
- tridiagonal nonsingular sign regular matrices,
- strictly sign regular matrices (all minors of the same order have the same strict sign),
- almost strictly sign regular matrices.

Open question

Does the interval property involving $\downarrow A, \uparrow A$ hold for general nonsingular sign regular matrices?

It was shown in (JG, 1996) that the interval property holds if we employ a specified set of vertex matrices of cardinality 2^{2n-1}.

Related recent work

- Further investigation of the application of the Cauchon Algorithm to totally nonnegative matrices.
  - a condensed form of the Cauchon Algorithm which reduces the amount of work from $O(n^4)$ to $O(n^3)$,
  - new determinantal criteria,
  - short proofs of properties,
  - new characterizations of subclasses.

  (M. Adm and JG, *Electronic J. Linear Algebra*, 27, pp. 588-610, 2014).

- Invariance of total nonnegativity under perturbation of single entries: Maximum allowable perturbation found for tridiagonal totally nonnegative matrices and for totally positive matrices.

  (M. Adm and JG, *Operators and Matrices*, 8 (1), pp. 129-137, 2012, and *Contemporary Mathematics*, 2015, to appear).

- Completion problems for totally positive (TP) matrices.

  Completion problems for TP matrices, i.e., for a matrix some of its entries are specified, while the remaining entries are unspecified and are free to be chosen. Each of its fully specified submatrices is TP. The question is whether the unspecified entries can be chosen in such a way that the whole matrix is TP.

  (M. Adm and JG, *Contemporary Mathematics*, 2015, to appear).

- Total nonnegativity of matrices related to polynomial roots and poles of rational functions.

  (M. Adm and JG, *Journal of Mathematical Analysis and Applications*, to appear).