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Intervals of special sign regular matrices

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We consider classes of n-by-n sign regular matrices, i.e. of matrices with the property that all their minors of fixed order k have one specified sign or are allowed also to vanish, $k = 1, \ldots, n$. If the sign is nonpositive for all k, such a matrix is called totally nonpositive. The application of the Cauchon algorithm to nonsingular totally nonpositive matrices is investigated and a new determinantal test for these matrices is derived. Also matrix intervals with respect to the checkerboard ordering are considered. This order is obtained from the usual entry-wise ordering on the set of the n-by-n matrices by reversing the inequality sign for each entry in a checkerboard fashion. For some classes of sign regular matrices, it is shown that if the two bound matrices of such a matrix interval are both in the same class then all matrices lying between these two bound matrices are in the same class, too.

Keywords: sign regular matrix; totally nonnegative matrix; totally nonpositive matrix; Cauchon algorithm; checkerboard ordering; matrix interval

AMS Subject Classification: 15A48

1. Introduction

A real matrix is called *sign regular* and *strictly sign regular* if all its minors of the same order have the same sign or vanish and are nonzero and have the same sign, respectively. Sign regular matrices have found a wide variety of applications in approximation theory, computer-aided geometric design,[1] numerical mathematics and other fields. If the sign of all minors of any order is nonnegative (nonpositive), then the matrix is called *totally nonnegative* (*totally nonpositive*). Totally nonnegative matrices arise in a variety of ways in mathematics and its applications. For background information, the reader is referred to the monographs.[2,3]

In [4], we apply the Cauchon algorithm [5,6] to totally nonnegative matrices and prove a long-standing conjecture posed by the second author on intervals of nonsingular totally nonnegative matrices. The underlying ordering is the checkerboard ordering which is obtained from the usual entry-wise ordering in the set of the square real matrices of fixed order by reversing the inequality sign for each entry in a checkerboard fashion. In this

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paper, we continue our study of the Cauchon algorithm and apply it to several classes of sign regular matrices: Firstly, to the nonsingular totally nonpositive matrices for which we derive a new characterization using the matrix obtained by the Cauchon algorithm and an efficient determinantal test; we also show that all matrices lying between two nonsingular totally nonpositive matrices (with respect to the checkerboard ordering) have also this property (termed *interval property* henceforth). Secondly, we prove that some other classes of nonsingular sign regular matrices possess the interval property, too.

The organization of our paper is as follows. In Section 2, we introduce our notation and give some auxiliary results which we use in the subsequent sections. In Section 3, we recall from [5,6] the Cauchon algorithm and its inverse, the Restoration algorithm, on which our proofs heavily rely. In Section 4, we apply the Cauchon algorithm to the nonsingular totally nonpositive matrices and derive a new characterization and a determinantal test for these matrices. In Section 5, we give a representation of the entries of the matrix that is obtained by the Cauchon algorithm when it is applied to a nonsingular totally nonpositive matrix. In Section 6, we prove the interval property for, e.g. the nonsingular totally nonpositive matrices and the nonsingular almost strictly sign regular matrices, a class between the sign regular and the strictly sign regular matrices.

2. Notation and auxilary results

2.1. *Notation*

We now introduce the notation used in our paper. For κ, n, we denote by $Q_{\kappa,n}$ the set of all strictly increasing sequences of κ integers chosen from $\{1, 2, \ldots, n\}$. We use the set theoretic symbols $\cup$ and $\setminus$ to denote somewhat not precisely but intuitively the union and the difference, respectively, of two index sequences, where we consider the resulting sequences as strictly increasing ordered. Let A be a real $n \times n$ matrix. For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\kappa)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_\kappa) \in Q_{\kappa,n}$, we denote by $A[\alpha|\beta]$ the $\kappa \times \kappa$ submatrix of A contained in the rows indexed by $\alpha_1, \alpha_2, \ldots, \alpha_\kappa$ and columns indexed by $\beta_1, \beta_2, \ldots, \beta_\kappa$. We suppress the brackets when we enumerate the indices explicitly. We set $\alpha_{\hat{\alpha}_i} := (\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_\kappa)$ for some $i \in \{1, \ldots, \kappa\}$. If both α and β are formed from consecutive indices, we call the minor $\det A[\alpha \mid \beta]$ *contiguous*. Let $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ be a signature sequence, i.e. $\epsilon \in \{1, -1\}^n$. The matrix A is called *strictly sign regular* (abbreviated SSR henceforth) and *sign regular* (abbreviated SR) *with signature* ϵ if $0 < \epsilon_\kappa \det A[\alpha|\beta]$ and $0 \leq \epsilon_\kappa \det A[\alpha|\beta]$, respectively, for all $\alpha, \beta \in Q_{\kappa,n}$, $\kappa = 1, 2, \ldots, n$. If A is SSR (SR) with signature $\epsilon = (1, 1, \ldots, 1)$, then A is called *totally positive* (abbreviated TP) (*totally nonnegative* (abbreviated TN)). If A is SSR (SR) with signature $\epsilon = (-1, -1, \ldots, -1)$, then A is called *totally negative* (abbreviated $t.n.$) (*totally nonpositive* (abbreviated $t.n.p.$)). If A is in a certain class of SR matrices and in addition also nonsingular then we affix Ns to the name of the class. We reserve throughout the notation $T_n = (t_{ij})$ for the *anti-diagonal* matrix with $t_{ij} := \delta_{n+1-i,j}$, $i, j = 1, \ldots, n$, and call $A^\# := T_n A T_n$ the *converse* matrix of A, see, e.g. [7, p.171], [2, p.34]. We note that if A is $Ns.t.n.p.$ then so is $A^\#$.

We endow $\mathbb{R}^{n,n}$, the set of the real $n \times n$ matrices, with two partial orderings: Firstly, with the usual entry-wise partial ordering ($A = (a_{ij})$, $B = (b_{ij}) \in \mathbb{R}^{n,n}$)

$$A \leq B :\Leftrightarrow \; a_{ij} \leq b_{ij}, i, j = 1, \ldots, n.$$

The strict inequality $A < B$ is also understood entry-wise.

Secondly, with the *checkerboard partial ordering*, which is defined as follows. Let $S := \mathrm{diag}(1, -1, \ldots, (-1)^{n+1})$ and $A^* := SAS$.

Then we define

$$A \leq^* B :\Leftrightarrow A^* \leq B^*.$$

2.2. *Auxiliary results*

In this subsection, we introduce briefly some auxiliary results that will be used later.

Lemma 2.1 [8, Lemma 7], [9, Lemma 2.2] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be NsSR with signature $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$. Then the following statements hold.*

(i) *If $\epsilon_2 = 1$, then*

$$\begin{cases} a_{ii} \neq 0, \quad i = 1, \ldots, n, \\ a_{ij} = 0, \ j < i \ \Rightarrow \ a_{kl} = 0 \ \forall \, l \leq j < i \leq k, \\ a_{ij} = 0, \ i < j \ \Rightarrow \ a_{kl} = 0 \ \forall \, k \leq i < j \leq l, \end{cases}$$

which is called a type-I staircase *matrix.*

(ii) *If $\epsilon_2 = -1$, then*

$$\begin{cases} a_{1n} \neq 0, \ a_{2,n-1} \neq 0, \ \ldots, \ a_{n1} \neq 0, \\ a_{ij} = 0, \ n - i + 1 < j \ \Rightarrow \ a_{kl} = 0 \ \forall \, i \leq k, j \leq l, \\ a_{ij} = 0, \ j < n - i + 1 \ \Rightarrow \ a_{kl} = 0 \ \forall \, k \leq i, l \leq j, \end{cases}$$

which is called a type-II staircase *matrix.*

Following [8], we call a minor *trivial* if it vanishes and its zero value is determined already by the pattern of its zero–nonzero entries. We illustrate this definition by the following example. Let

$$A := \begin{pmatrix} * & * & * \\ 0 & * & 0 \\ 0 & * & * \end{pmatrix},$$

where the asterisk denotes a nonzero entry. Then $\det A[2,3|1,2]$ and $\det A[1,2|1,3]$ are trivial, whereas $\det A$ and $\det A[1,2|2,3]$ are nontrivial minors.

Lemma 2.2 [8, p.4183] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be a staircase matrix and let $\alpha, \beta \in Q_{\kappa,n}$. Then A possesses the following properties.*

(i) *If A is a type-I staircase matrix, then*

$$\det A[\alpha \mid \beta] \textit{ is a nontrivial minor } \Leftrightarrow a_{\alpha_1,\beta_1} \cdot a_{\alpha_2,\beta_2} \cdots a_{\alpha_\kappa,\beta_\kappa} \neq 0.$$

(ii) *If A is a type-II staircase matrix, then*

$$\det A[\alpha \mid \beta] \textit{ is a nontrivial minor } \Leftrightarrow a_{\alpha_1,\beta_\kappa} \cdot a_{\alpha_2,\beta_{\kappa-1}} \cdots a_{\alpha_\kappa,\beta_1} \neq 0.$$

(iii) *A is a type-I staircase matrix if and only if $T_n A$ is a type-II staircase matrix, and*

$$\det A[\alpha \mid \beta] \textit{ is a nontrivial minor } \Leftrightarrow \det (T_n A)[\alpha' \mid \beta] \textit{ is a nontrivial minor,}$$

where $\alpha' \in Q_{\kappa,n}$ *is defined by* $\alpha'_i := n - \alpha_i + 1,\ i = 1, \ldots, \kappa$.

Now we present the definition of an almost strictly sign regular matrix and give a characterization for it in the nonsingular case.

Definition 1 [8, Definition 8] Let $A \in \mathbb{R}^{n,n}$ and $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ be a signature sequence.

(i) If for all the nontrivial minors

$$0 < \epsilon_k \det A[\alpha \mid \beta] \text{ for all } \alpha, \beta \in Q_{k,n},\ k = 1, \ldots, n,$$

holds, then A is called *almost strictly sign regular* (abbreviated $ASSR$) *with signature* ϵ.

(ii) If all the nontrivial minors of A are positive, then A is called *almost totally positive* (ATP).

THEOREM 2.3 [8, Theorem 10] *Let* $A \in \mathbb{R}^{n,n}$ *and* $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ *be a signature sequence. Then* A *is* $NsASSR$ *with signature* ϵ *if and only if* A *is a type-I or type-II staircase matrix, and for all the nontrivial contiguous minors holds*

$$0 < \epsilon_k \det A[\alpha \mid \beta] \ \text{for all } \alpha, \beta \in Q_{k,n},\ k = 1, \ldots, n.$$

LEMMA 2.4 [8, Lemma 9] *Let* $A \in \mathbb{R}^{n,n}$ *be a type-I staircase matrix and* $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ *be a signature sequence. Set*
$r := \min\left\{|j - i| \mid a_{ij} = 0 \text{ for some } i, j \in \{1, \ldots, n\}\right\}$ *and suppose that* $0 < r$. *If for all the nontrivial contiguous minors*

$$0 < \epsilon_k \det A[\alpha \mid \beta] \ \text{for all } \alpha, \beta \in Q_{k,n},\ k = 1, \ldots, n,$$

holds, then

$$\epsilon_2 = \epsilon_1^2,\ \epsilon_3 = \epsilon_1^3,\ \ldots,\ \epsilon_{n-r+1} = \epsilon_1^{n-r+1}.$$

LEMMA 2.5 [10, Proposition 3.2] *If* $A \in \mathbb{R}^{n,n}$ *is* $Ns.t.n.p.$ *with* $a_{11} < 0$, *then* $a_{ij} < 0$ *for all* $i, j = 1, \ldots, n$ *with* $(i, j) \neq (n, n)$.

LEMMA 2.6 [11, Theorem 5] *Let* $A \in \mathbb{R}^{n,n}$ *be nonsingular. Then* A *is* $t.n.p.$ *if and only if the following conditions hold*

$$\left.\begin{array}{l} a_{11}, a_{nn} \le 0;\ a_{n1}, a_{1n} < 0; \\ \det A[\alpha|k+1, \ldots, n] \le 0 \text{ for all } \alpha \in Q_{n-k,n}, \\ \det A[k+1, \ldots, n|\beta] \le 0 \text{ for all } \beta \in Q_{n-k,n}, \\ \det A[k, \ldots, n] < 0, \end{array}\right\} \text{ for } k = 1, \ldots, n-1.$$

3. Cauchon diagrams and the Cauchon algorithm

In this section, we first recall from [5,6] the definition of a Cauchon diagram and of the Cauchon algorithm.[1] Since we are mainly interested in the case of nonsingular matrices, we present the algorithm here only for square matrices. The extension to rectangular matrices will be obvious.

Definition 2 An $n \times n$ *Cauchon diagram* C is an $n \times n$ grid consisting of n^2 squares coloured black and white, where each black square has the property that either every square to its left (in the same row) or every square above it (in the same column) is black.

We denote by $\mathcal{C}_n$ the set of the $n \times n$ Cauchon diagrams. We fix positions in a Cauchon diagram in the following way: For $C \in \mathcal{C}_n$ and $i, j \in \{1, \ldots, n\}$, $(i, j) \in C$ if the square in row i and column j is black. Here we use the usual matrix notation for the (i, j) position in a Cauchon diagram, i.e. the square in $(1, 1)$ position of the Cauchon diagram is in its top left corner.

Definition 3 Let $A \in \mathbb{R}^{n,n}$ and let $C \in \mathcal{C}_n$. We say that A is a *Cauchon matrix associated with the Cauchon diagram* C if for all $(i, j), i, j \in \{1, \ldots, n\}$, we have $a_{ij} = 0$ if and only if $(i, j) \in C$. If A is a Cauchon matrix associated with an unspecified Cauchon diagram, we just say that A is a *Cauchon matrix*.

If A is a Cauchon matrix, then we also say that C is the *Cauchon diagram associated to* A if A is a Cauchon matrix associated with the Cauchon diagram C.

To recall the Cauchon algorithm, we denote by $\leq$ and $\leq_c$ the lexicographic and colexicographic order, respectively, on $\mathbb{N}^2$, i.e.

$$
\begin{aligned}
(g, h) \leq (i, j) &:\Leftrightarrow (g < i) \text{ or } (g = i \text{ and } h \leq j),\\
(g, h) \leq_c (i, j) &:\Leftrightarrow (h < j) \text{ or } (h = j \text{ and } g \leq i).
\end{aligned}
$$

Set $E^\circ := \{1, \ldots, n\}^2 \setminus \{(1, 1)\}$, $E := E^\circ \cup \{(n + 1, 1)\}$.

Let $(s, t) \in E^\circ$. Then $(s, t)^+ := \min \{(i, j) \in E \mid (s, t) \leq (i, j),\ (s, t) \neq (i, j)\}$; here the minimum is taken with respect to the lexicographical order.

Cauchon algorithm Let $A \in R^{n,n}$. As r runs in decreasing order over the set E, we define matrices $A^{(r)} = (a_{ij}^{(r)}) \in R^{n,n}$ as follows:

(1) Set $A^{(n+1,1)} := A$.
(2) For $r = (s, t) \in E^\circ$ define the matrix $A^{(r)} = (a_{ij}^{(r)})$ as follows:

   (a) If $a_{st}^{(r^+)} = 0$, then put $A^{(r)} := A^{(r^+)}$.
   (b) If $a_{st}^{(r^+)} \neq 0$, then put

$$
a_{ij}^{(r)} := \begin{cases} a_{ij}^{(r^+)} - \dfrac{a_{it}^{(r^+)} a_{sj}^{(r^+)}}{a_{st}^{(r^+)}} & \text{for } i < s \text{ and } j < t,\\ a_{ij}^{(r^+)} & \text{otherwise.} \end{cases}
$$

(3) Set $\tilde{A} := A^{(1,2)}$ [2]; $\tilde{A}$ is called *the matrix obtained from A* (*by the Cauchon algorithm*).

The formulae of the Cauchon algorithm allow us to express the entries of $A^{(r)}$ in terms of $A^{(r^+)}$. These expressions also constitute the so-called Restoration algorithm, see, e.g. [5, Section 3], which is the inverse of the Cauchon algorithm.

Restoration algorithm Let $A \in \mathbb{R}^{n,n}$. As r runs (in increasing order) over the set E°, we define matrices $A^{(r)} = (a_{ij}^{(r)}) \in \mathbb{R}^{n,n}$ as follows:

(1) Set $A^{(1,2)} := A$.
(2) For $r = (s,t) \in E^\circ$ define the matrix $A^{(r^+)} = (a_{ij}^{(r^+)})$ as follows:

   (a) If $a_{st}^{(r)} = 0$, then put $A^{(r^+)} := A^{(r)}$.
   (b) If $a_{st}^{(r)} \neq 0$, then put

$$a_{ij}^{(r^+)} := \begin{cases} a_{ij}^{(r)} + \frac{a_{it}^{(r)} a_{sj}^{(r)}}{a_{st}^{(r)}} & \text{for } i < s \text{ and } j < t, \\ a_{ij}^{(r)} & \text{otherwise.} \end{cases}$$

(3) Set $\bar{A} := A^{(n+1,1)}$; $\bar{A}$ is called *the matrix obtained from A* (*by the Restoration algorithm*).

THEOREM 3.1 [5, Theorem 4.1] *Let $A \in \mathbb{R}^{n,n}$ be a nonnegative Cauchon matrix. Then $\bar{A}$ is TN.*

4. Nonsingular totally nonpositive matrices and the Cauchon algorithm

In this section, we apply the Cauchon algorithm to $Ns.t.n.p.$ matrices. Before we present our results, we first recall two propositions from [5] which relate the determinants of some special submatrices of the intermediate matrices during the performance of the Restoration algorithm (or its inverse, the Cauchon algorithm). In the sequel, we use the following notations.

Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ and $\delta = \det A[\alpha|\beta]$ be a minor of A. If $r = (s,t) \in E$, set

$$\delta^{(r)} := \det A^{(r)}[\alpha|\beta].$$

For $i \in \alpha$ and $j \in \beta$, set

$$\delta_{\hat{i},\hat{j}}^{(r)} := \det A^{(r)}[\alpha_{\hat{i}}|\beta_{\hat{j}}].$$

PROPOSITION 4.1 [5, Proposition 3.7] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ and $r = (s,t) \in E^\circ$. Assume that $a_{st} \neq 0$. Let $\delta = \det A[\alpha|\beta]$ with $\alpha, \beta \in Q_{l,n}$ with $(\alpha_l, \beta_l) = r$. Then $\delta^{(r^+)} = \delta_{\hat{s}\hat{t}}^{(r)} a_{st}$ holds.*

PROPOSITION 4.2 [5, Proposition 3.11] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ and $r = (s,t) \in E^\circ$. Let $\delta = \det A[\alpha_1, \ldots, \alpha_l|\beta_1, \ldots, \beta_l]$ be a minor of A with $(\alpha_l, \beta_l) < r$. If $a_{st} = 0$, or if $\alpha_l = s$, or if $t \in \{\beta_1, \ldots, \beta_l\}$, or if $t < \beta_1$, then $\delta^{(r^+)} = \delta^{(r)}$.*

From the last two propositions, we derive a useful representation of the determinant of a nonsingular matrix.

THEOREM 4.3 *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ and assume that $\tilde{a}_{ii} \neq 0$, $i = 1, \ldots, n$. Then it holds that*

$$\det A = \tilde{a}_{11} \cdots \tilde{a}_{nn}. \tag{1}$$

Proof Since $a_{nn}^{(n+1,1)} = a_{nn} = \tilde{a}_{nn} \neq 0$ it follows from Proposition 4.1 that

$$\det A = \det A^{(n+1,1)} = \det A^{(n,n)}[1, \ldots, n-1] \cdot \tilde{a}_{nn}. \tag{2}$$

Furthermore, we have

$$\det A^{(n,n)}[1, \ldots, n-1] = \det A^{(n,1)}[1, \ldots, n-1] \tag{3}$$

because the latter submatrix is obtained from the first one by a sequence of adding a scalar multiple of one column to another column. Now we set $r := (n-1, n)$; then $r^+ = (n, 1)$ and the application of Proposition 4.2 to $\det A[1, \ldots, n-1|1, \ldots, n]$ yields

$$\det A^{(n,1)}[1, \ldots, n-1] = \det A^{(n-1,n)}[1, \ldots, n-1]. \tag{4}$$

By assumption $a_{n-1,n-1}^{(n-1,n)} = \tilde{a}_{n-1,n-1} \neq 0$ holds. Application of Proposition 4.1 to the matrix $A^{(n-1,n)}[1, \ldots, n-1|1, \ldots, n]$ (as matrix A) with $r := (n-1, n-1)$ results in

$$\det A^{(n-1,n)}[1, \ldots, n-1] = \det A^{(n-1,n-1)}[1, \ldots, n-2] \cdot \tilde{a}_{n-1,n-1}. \tag{5}$$

Plugging (5) into (4), the resulting identity into (3), and finally the obtained identity into (2) gives

$$\det A = \det A^{(n-1,n-1)}[1, \ldots, n-2] \cdot \tilde{a}_{n-1,n-1} \cdot \tilde{a}_{nn}.$$

Continuing in this way, we arrive at (1). □

The statement of Theorem 4.3 remains true if $\tilde{a}_{11} = 0$ and $\tilde{a}_{ii} \neq 0$ for $i = 2, \ldots, n$ while it fails if we waive the assumption that $\tilde{a}_{ii} \neq 0$, $i = 2, \ldots, n$. A counterexample is provided by the matrix

$$A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Now we present the changes in the entries and minors of a given $Ns.t.n.p.$ matrix with nonzero entry in position (n, n) during running the Cauchon algorithm. By Lemma 2.5 applied to $A^\#$ all the entries of such a matrix are negative except possibly the entry in position $(1, 1)$. The following theorem gives the changes for the steps $r = (n, n), \ldots, (n, 2)$.

THEOREM 4.4 *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be $Ns.t.n.p.$ with $a_{nn} < 0$. If we apply the Cauchon algorithm to A, then we have the following properties:*

(i) *All entries of $A^{(n,t)}[1, \ldots, n-1]$ are nonnegative for all $t = 2, \ldots, n$.*
(ii) *$A^{(n,t)}[1, \ldots, n-1 \mid 1, \ldots, t-1]$ is TN for all $t = 2, \ldots, n$.*
(iii) *$A^{(n,t)}[1, \ldots, n-1]$ is TN for all $t = 2, \ldots, n$.*
(iv) *$A^{(n,2)}[1, \ldots, n-1]$ is $NsTN$.*
(v) *$A^{(n,2)}$ is a Cauchon matrix.*
(vi) *For $t = 1, \ldots, n$, $\det A^{(n,t)}[\alpha \mid \beta] \leq 0$ for all $\alpha \in Q_{l,n-1}$, $\beta \in Q_{l,n}$ with $\beta_l = n$ and $l = 1, \ldots, n-1$.*

Proof

(i) If $t = n$ then let $r = (n, n)$ and by Proposition 4.1 we have $\det A^{(r^+)}[i, n \mid j, n] = \det A^{(r)}[i \mid j] \cdot a_{nn}$, hence $\det A[i, n \mid j, n] = a_{ij}^{(r)} \cdot a_{nn}$. Since A is $t.n.p.$ and $a_{nn} < 0$ it follows that $0 \le a_{ij}^{(r)}$ for all $i, j = 1, \ldots, n-1$. This proves the case $t = n$. Proposition 4.2 implies that $\det A^{(r)}[i, n \mid j, h] = \det A[i, n \mid j, h] \le 0$ for all $h \le n-1$. In the remaining cases, we proceed by induction and repeat the above arguments and use the fact that $a_{nj} < 0$ for all $j = 1, \ldots, n$.

(ii) We prove this property only for the case $t = n$ since in the other cases we proceed by induction and repeat the arguments.
If $t = n$ then by (i) $A^{(n,n)}[1, \ldots, n-1]$ is a nonnegative matrix. It follows from Proposition 4.1 that

$$\begin{aligned}\det A[\alpha_1, \ldots, \alpha_k, n \mid \beta_1, \ldots, \beta_k, n] &= \det A^{(n+1,1)}[\alpha_1, \ldots, \alpha_k, n \mid \beta_1, \ldots, \beta_k, n] \\ &= \det A^{(n,n)}[\alpha_1, \ldots, \alpha_k \mid \beta_1, \ldots, \beta_k] \cdot a_{nn},\end{aligned}$$

for all $\alpha_k, \beta_k \le n-1$. Since A is $t.n.p.$ and $a_{nn} < 0$, we have

$$0 \le \det A^{(n,n)}[\alpha_1, \ldots, \alpha_k \mid \beta_1, \ldots, \beta_k].$$

Hence $A^{(n,n)}[1, \ldots, n-1]$ is TN. This proves the case $t = n$. For the other cases, we use the fact that $a_{nj} < 0$ for all $j = 1, \ldots, n$ and for $\beta_{k+1} < n$

$$\det A^{(n,n)}[\alpha_1, \ldots, \alpha_k, n \mid \beta_1, \ldots, \beta_k, \beta_{k+1}] = \det A[\alpha_1, \ldots, \alpha_k, n \mid \beta_1, \ldots, \beta_k, \beta_{k+1}]$$

which follows by Proposition 4.2.

(iii) We proceed by induction on t (primary induction) and l (secondary induction), where l is the order of the minors.
The case $t = n$ is a consequence of (ii).
Suppose that $A^{(n,t+1)}[1, \ldots, n-1]$ is TN; we want to show that $A^{(n,t)}[1, \ldots, n-1]$ is TN, i.e. $0 \le \det A^{(n,t)}[\alpha \mid \beta]$ for all $\alpha, \beta \in Q_{l,n-1}$.
The case $l = 1$ is a consequence of (i). So, we assume that $2 \le l$.
If $\beta_l < t$, then the statement follows from (ii).
If $t < \beta_1$ or t is contained in β then by Proposition 4.2, we have

$$\det A^{(n,t+1)}[\alpha \mid \beta] = \det A^{(n,t)}[\alpha \mid \beta]$$

which implies by the induction hypothesis on t that $0 \le \det A^{(n,t)}[\alpha \mid \beta]$. So, it just remains to consider the case where there exists h, $1 \le h \le l-1$, such that $\beta_h < t < \beta_{h+1}$.
In order to prove the statement, in this case we simplify the notation and proceed parallel to the proof given in [5, p.822–823]. We set for $\alpha, \beta \in Q_{l,n}$

$$[\alpha \mid \beta] := \det A^{(n,t)}[\alpha \mid \beta], [\alpha \mid \beta]^+ := \det A^{(n,t+1)}[\alpha \mid \beta],$$

and for $k \in \{1, \ldots, l\}$, $m \in \{1, \ldots, h\}$,

$$\alpha^{(k)} := (\alpha_1, \ldots, \hat{\alpha}_k, \ldots, \alpha_l), \beta^{(m)} := (\beta_1, \ldots, \hat{\beta}_m, \ldots, \beta_{l-1}),$$

where the 'hat' over an entry indicates that this entry has to be discarded from the index sequence (note that the sequences $\alpha^{(k)}$ and $\beta^{(m)}$ have different lengths).

By using Muir's law of extensible minors [12], we have for $k = 1, \ldots, l$

$$[\alpha^{(k)} \mid \beta^{(m)} \cup \{t\}] \cdot [\alpha \mid \beta] = [\alpha^{(k)} \mid \beta^{(m)} \cup \{\beta_l\}] \cdot [\alpha \mid \beta^{(m)} \cup \{\beta_m, t\}] \\ + [\alpha^{(k)} \mid \beta^{(m)} \cup \{\beta_m\}] \cdot [\alpha \mid \beta^{(m)} \cup \{t, \beta_l\}]. \quad (6)$$

It follows from the induction on l that the minors $[\alpha^{(k)} \mid \beta^{(m)} \cup \{t\}]$, $[\alpha^{(k)} \mid \beta^{(m)} \cup \{\beta_l\}]$, $[\alpha^{(k)} \mid \beta^{(m)} \cup \{\beta_m\}]$ are nonnegative. Furthermore, it follows from Proposition 4.2 that $[\alpha \mid \beta^{(m)} \cup \{\beta_m, t\}] = [\alpha \mid \beta^{(m)} \cup \{\beta_m, t\}]^+$ and $[\alpha \mid \beta^{(m)} \cup \{t, \beta_l\}] = [\alpha \mid \beta^{(m)} \cup \{t, \beta_l\}]^+$ and so we deduce by the induction on t that the two minors are nonnegative. Hence all of these inequalities together imply that the left-hand side of (6) is nonnegative. If $0 < [\alpha^{(k)} \mid \beta^{(m)} \cup \{t\}]$ for some k and m, then $0 \leq [\alpha \mid \beta]$, as desired. If for all k, m $[\alpha^{(k)} \mid \beta^{(m)} \cup \{t\}] = 0$ then it follows by Laplace expansion that $[\alpha \mid \beta^{(m)} \cup \{\beta_l, t\}] = 0$. Then by [5, Lemma B.3] we have

$$\det A^{(n,t+1)}[\alpha \mid \beta] = \det A^{(n,t)}[\alpha \mid \beta].$$

Hence we obtain by induction on t that $0 \leq \det A^{(n,t)}[\alpha \mid \beta]$, as desired. This completes the induction step for the proof of (iii).

(iv) By (iii) $A^{(n,2)}[1, \ldots, n-1]$ is TN. Similarly as in the proof of Theorem 4.3 we obtain

$$\det A = \det A^{(n,2)}[1, \ldots, n-1] \cdot a_{nn}.$$

Since A is $Ns.t.n.p.$ and $a_{nn} < 0$ we have that $0 < \det A^{(n,2)}[1, \ldots, n-1]$. Hence $A^{(n,2)}[1, \ldots, n-1]$ is $NsTN$.

(v) Since the entries in the last row and last column of A are negative (and are not changed when running the Cauchon algorithm) and since by (iv) $A^{(n,2)}[1, \ldots, n-1]$ is $NsTN$, $A^{(n,2)}$ is a Cauchon matrix.

(vi) We prove the statement by induction on l and decreasing induction on t.
The case $l = 1$ is a consequence of the negativity of the entries in the last column of $A^{(n,t)}$, $t = 2, \ldots, n$.
If $t = n$ then by Proposition 4.2 we have $\det A^{(n,n)}[\alpha \mid \beta] = \det A[\alpha \mid \beta]$ since $\beta_l = n$.
Suppose that the statement is true for all minors of order less than l (secondary induction) and for all $t+1, \ldots, n$ (primary induction).
If $t < \beta_1$ or $t = \beta_h$ for some $h = 1, \ldots, l$ then by Proposition 4.2 we have $\det A^{(n,t+1)}[\alpha \mid \beta] = \det A^{(n,t)}[\alpha \mid \beta]$, and by the induction hypothesis on t we are done.
If $\beta_h < t < \beta_{h+1}$ for some $h = 1, \ldots, l-1$ then we consider again (6).
The minors $[\alpha^{(k)} \mid \beta^{(m)} \cup \{t\}]$, $[\alpha \mid \beta^{(m)} \cup \{\beta_m, t\}]$, $[\alpha^{(k)} \mid \beta^{(m)} \cup \{\beta_m\}]$ are nonnegative by (iii), $[\alpha^{(k)} \mid \beta^{(m)} \cup \{\beta_l\}]$ is nonpositive by the induction hypothesis on l, $[\alpha \mid \beta^{(m)} \cup \{t, \beta_l\}] = [\alpha \mid \beta^{(m)} \cup \{t, \beta_l\}]^+$ by Proposition 4.2, and by the induction hypothesis on t the latter minor is nonpositive. All of these inequalities yield

$$[\alpha^{(k)} \mid \beta^{(m)} \cup \{t\}] \cdot [\alpha \mid \beta] \leq 0.$$

If $0 < [\alpha^{(k)} \mid \beta^{(m)} \cup \{t\}]$ for some k and m, then we have $[\alpha \mid \beta] \leq 0$, as desired.
If for all k, m $[\alpha^{(k)} \mid \beta^{(m)} \cup \{t\}] = 0$, then proceeding parallel to the last part of (iii) we get

$$\det A^{(n,t+1)}[\alpha \mid \beta] = \det A^{(n,t)}[\alpha \mid \beta].$$

Hence by the induction hypothesis on t, we obtain $\det A^{(n,t)}[\alpha \mid \beta] \leq 0$, as desired. □

By sequentially repeating the steps of the proof of Theorem 4.4, we obtain the following theorem.

THEOREM 4.5 *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be $Ns.t.n.p.$ with $a_{nn} < 0$. Then it holds that*

(i) $A^{(s,t)}[1, \ldots, s-1|1, \ldots, t-1]$ *is TN for all $s, t = 2, \ldots, n$.*
(ii) $A^{(s,2)}[1, \ldots, s-1]$ *is $NsTN$ for all $s = 2, \ldots, n$.*
(iii) $\tilde{A}[1, \ldots, n-1]$ *is a nonnegative matrix.*
(iv) $\tilde{A}$ *is a Cauchon matrix.*

Inspection of the proofs of Theorems 4.4 and 4.5 shows that the nonsingularity assumption is only needed for the nonsingularity statements in Theorems 4.4 (iv) and 4.5 (ii). In the following corollary, we present the weakened version of Theorems 4.5. and 4.4 may be weakened accordingly.

COROLLARY 4.6 *Let $A \in \mathbb{R}^{n,n}$ have all the entries in its bottom row negative and let A be a $t.n.p.$ matrix. Then it holds that*

(i) $A^{(s,t)}[1, \ldots, s-1|1, \ldots, t-1]$ *is TN for all $s, t = 2, \ldots, n$.*
(ii) $\tilde{A}[1, \ldots, n-1]$ *is a nonnegative matrix.*
(iii) $\tilde{A}$ *is a Cauchon matrix.*

In the next section, we will make use of the following proposition and theorem.

PROPOSITION 4.7 *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be $t.n.p.$ with $a_{nn} < 0$. Then A is nonsingular if and only if $0 < \tilde{a}_{ii}$, $i = 1, \ldots, n-1$.*

Proof Let A be $Ns.t.n.p.$ with $a_{nn} < 0$. By Theorem 4.5, $A^{(s,2)}[1, \ldots, s-1]$ is $NsTN$ and therefore possesses only positive principal minors, e.g. [3, Theorem 1.13]. In particular, $0 < a^{(s,2)}_{s-1,s-1} = \tilde{a}_{s-1,s-1}$, $s = 2, \ldots, n$. The converse direction follows from Theorem 4.3. □

The following theorem provides necessary and sufficient conditions for a given matrix whose entries are all negative except possibly the $(1, 1)$ entry which is nonpositive to be $Ns.t.n.p.$ using the Cauchon algorithm.

THEOREM 4.8 *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ have all entries negative except possibly $a_{11} \leq 0$. Then the following two properties are equivalent.*

(i) *A is a $Ns.t.n.p.$ matrix.*
(ii) *$\tilde{A}$ is a Cauchon matrix and $\tilde{A}[1, \ldots, n-1]$ is a nonnegative matrix with positive diagonal entries.*

Proof The implication (i) $\Rightarrow$ (ii) follows by Theorem 4.5 and Proposition 4.7.

(ii) $\Rightarrow$ (i): By Proposition 4.7, A is nonsingular with $\det A < 0$ since $0 < \tilde{a}_{ii}$, $i = 1, \ldots, n-1$, and $a_{nn} < 0$. $\tilde{A}^{(n,n)}$ is the matrix that we obtain after running the Restoration algorithm applied to $\tilde{A}$ with $r = (n, n-1)$. By the definition of the Restoration algorithm, the entries of $\tilde{A}^{(n,n)}[1, \ldots, n-1]$ are nonnegative .[3] Note that if in step 2(b) in the Restoration algorithm $s = n$ or $t = n$ then the negativity of $\tilde{a}_{nj}^{(r)}$ and $\tilde{a}_{nt}^{(r)}$ and of $\tilde{a}_{in}^{(r)}$ and $\tilde{a}_{sn}^{(r)}$, respectively, results in a nonnegative value of the quotient. In the proof of Theorem 3.1 given in [5, Theorem 4.1], it is shown that if $N \in \mathbb{R}^{n,n}$ is a nonnegative Cauchon matrix then

$$0 \le \det N^{(r)}[\alpha|\beta] \quad \text{for all} \;\; \alpha, \beta \in Q_{l,n} \text{ with } (\alpha_l, \beta_l) \le r. \tag{7}$$

Again this result carries over to $\tilde{A}^{(r)}$ provided that $\alpha_l, \beta_l < n$, irrespectively of the negativity of the entries in the last column and row of $\tilde{A}$ as long as $r < (n, n)$.

Now let $2 \le l$, $\alpha, \beta \in Q_{l,n}$ with $\alpha_l = n$ and put $t := \beta_l$, $r := (n, t)$. It follows from Proposition 4.1 that for $\delta = \det \tilde{A}[\alpha|\beta]$

$$\det \tilde{A}^{(r^+)}[\alpha|\beta] = \delta^{(r^+)} = \delta_{\hat{n}\hat{t}}^{(r)} \tilde{a}_{nt} = \delta_{\hat{n}\hat{t}}^{(r)} a_{nt}. \tag{8}$$

By (7), we have $0 \le \delta_{\hat{n}\hat{t}}^{(r)}$, whence by (8) $\det \tilde{A}^{(r^+)}[\alpha|\beta] \le 0$. By $\bar{\tilde{A}} = A$ and by repeated application of Proposition 4.2 we obtain

$$\det A[\alpha|\beta] \le 0 \;\; \text{for all} \;\; \alpha, \beta \in Q_{l,n} \text{ with } \alpha_l = n. \tag{9}$$

Similarly we can prove that

$$\det A[\alpha|l, \ldots, n] \le 0 \;\; \text{for all} \;\; \alpha \in Q_{n-l+1,n}, \; l = 2, \ldots, n. \tag{10}$$

Finally, since the result of the Cauchon algorithm applied to $A[l, \ldots, n]$ coincides with $\tilde{A}[l, \ldots, n]$ Theorem 4.3 implies that

$$\det A[l, \ldots, n] = \tilde{a}_{ll} \cdots \tilde{a}_{nn} < 0, \; l = 1, \ldots, n. \tag{11}$$

By the condition on the sign of the entries of A and (9)–(11) it follows from Lemma 2.6 that A is $Ns.t.n.p.$ $\square$

If $A \in \mathbb{R}^{n,n}$ is $Ns.t.n.p.$ with $a_{nn} = 0$ replace in Theorem 4.8 A by $B := AG$, $B = (b_{ij})$, where $G = (g_{ij}) \in \mathbb{R}^{n,n}$ is the matrix defined by $g_{ii} := 1$, $i = 1, \ldots, n$, $g_{n-1,n} := 1$ and all other entries are 0. Then $b_{nn} < 0$ and A is $Ns.t.n.p.$ if and only if B is $Ns.t.n.p.$, see, e.g. [13, proof of Theorem 3.1]. Hence A is $Ns.t.n.p.$ if and only if $\tilde{B}$ is a Cauchon matrix and $\tilde{B}[1, \ldots, n-1]$ is a nonnegative matrix with positive diagonal entries.

If in the proof of Theorem 4.8 $0 < N$ then (7) holds with the strict inequality. Combining this with a necessary and sufficient condition for a matrix to be $t.n.$ [14, Theorem 6] we obtain by a similar proof the following corollary.

Corollary 4.9 *Let $A \in \mathbb{R}^{n,n}$ and $A < 0$. Then the following properties are equivalent:*

(i) *A is $t.n.$,*
(ii) $0 < \tilde{A}[1, \ldots, n-1]$.

By proceeding similarly as in the proof of Theorem 4.8 and using [15, Proposition 3.1] instead of Lemma 2.6, we obtain the following corollary.

Corollary 4.10 *Let $A = (a_{ij}) \in \mathbb{R}^{n,m}$ (with $n \leq m$) have all its entries negative except possibly $a_{11} \leq 0$. Then the following two properties are equivalent:*

(i) *A is a t.n.p. matrix and $A[1, \ldots, n \mid m-n+1, \ldots, m]$ is nonsingular.*
(ii) *$\tilde{A}$ is a Cauchon matrix, $\tilde{A}[1, \ldots, n-1 \mid 1, \ldots, m-1]$ is a nonnegative matrix, and $\tilde{A}[1, \ldots, n-1 \mid m-n+1, \ldots, m-1]$ has positive diagonal entries.*

We conclude this section with an efficient determinantal test to check whether a given matrix is nonsingular totally nonpositive or not.

We firstly recall from [6] the definition of a lacunary sequence.

Definition 4 Let $C \in \mathcal{C}_n$. We say that a sequence

$$\gamma := ((i_k, j_k), k = 0, 1, \ldots, p) \tag{12}$$

which is strictly increasing in both arguments is a *lacunary sequence with respect to C* if the following conditions hold:

(1) $(i_k, j_k) \notin C, k = 1, \ldots, p$;
(2) $(i, j) \in C$ for $i_p < i \leq n$ and $j_p < j \leq n$.
(3) Let $s \in \{0, \ldots, p-1\}$. Then $(i, j) \in C$ if

  (i) either for all (i, j), $i_s < i < i_{s+1}$ and $j_s < j$,
  (ii) or for all (i, j), $i_s < i < i_{s+1}$ and $j_0 \leq j < j_{s+1}$

  and

  (iii) either for all (i, j), $i_s < i$ and $j_s < j < j_{s+1}$,
  (iv) or for all (i, j), $i < i_{s+1}$, and $j_s < j < j_{s+1}$.

In [6, Proposition 4.1], the conclusion from hypothesis (b) therein depends only on the zero–nonzero values (and not on the positivity) of the involved determinants. Therefore, we obtain the following proposition (which we formulate for later use in the rectangular case).

Proposition 4.11 *Let $A \in \mathbb{R}^{n,m}$ and C be an $n \times m$ Cauchon diagram. For each position in C fix a lacunary sequence γ given by (12) (with respect to C) starting at this position. Assume that for all (i_0, j_0), we have $0 = \det A[i_0, \ldots, i_p | j_0, \ldots, j_p]$ if and only if $(i_0, j_0) \in C$. Then*

$$\det A[i_0, \ldots, i_p | j_0, \ldots, j_p] = \tilde{a}_{i_0, j_0} \cdot \tilde{a}_{i_1, j_1} \cdots \tilde{a}_{i_p, j_p} \tag{13}$$

for all lacunary sequences γ given by (12).

As in [16], we relate to each entry $\tilde{a}_{i_0, j_0}$ of $\tilde{A}$ a sequence γ given by (12). It is sufficient to describe the construction of the sequence from the starting pair (i_0, j_0) to the next pair

(i_1, j_1) with $0 < \tilde{a}_{i_1,j_1}$ ($\delta_{i_1,j_1} < 0$, see below) if $i_1, j_1 < n$ since for a given matrix A the determinantal test is performed by moving row by row from the bottom to the top row. Once we have found the next index pair (i_1, j_1) we append to (i_0, j_0) the sequence starting at (i_1, j_1). By construction, the sequence γ is uniquely determined.

In the sequel let $\delta_{ij} := \det A[i_0, i_1, \ldots, i_p \mid j_0, j_1, \ldots, j_p]$ be the minor of A associated to the sequence γ given by (12) according to (13) which starts at position $(i, j) = (i_0, j_0)$ and which is constructed by the following procedure.

Procedure 4.12 Construction of the sequence γ given by (12) starting at (i_0, j_0) to the next index pair (i_1, j_1) for the n-by-n $Ns.t.n.p.$ matrix A.

> **If** $i_0 = n$ or $j_0 = n$ or $\mathcal{U} := \left\{(i, j) \mid i_0 < i \leq n,\ j_0 < j \leq n, \text{ and } \delta_{ij} < 0\right\}$ is void **then** terminate with $p := 0$;
> **else**
>
> > put (i_1, j_1) as the minimum of $\mathcal{U}$ with respect to the colexicographic order and lexicographic order if $j_0 \leq i_0$ and $i_0 < j_0$, respectively;
>
> **end if**.

After all sequences γ starting in row $i_0 + 1$ are determined it is checked whether the matrix $B := A[i_0 + 1, \ldots, n \mid 1, \ldots, n]$ fulfils conditions (i), (ii) and (iii) of Theorem 4.13 below (with the obvious modifications in the rectangular case). If one of the conditions is violated for any instance, the test can be terminated since A is not $Ns.t.n.p.$

THEOREM 4.13 *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ with all entries are negative except possibly $a_{11} \leq 0$. Then A is $Ns.t.n.p.$ if and only if for all $i, j = 1, \ldots, n$ the quantities δ_{ij} obtained by the sequences that start from positions (i, j) and are constructed by Procedure 4.12 satisfy the following conditions:*

(i) $\delta_{ii} < 0$;
(ii) $\delta_{ij} \leq 0$;
(iii) *if $\delta_{qg} = 0$ for some $q, g \in \{1, \ldots, n\}$, then $\delta_{q,t_1} = 0$ for all $t_1 < g$ if $g < q$ and $\delta_{t_2,g} = 0$ for all $t_2 < q$ if $q < g$.*

Proof Suppose that all entries of A are negative except possibly $a_{11} \leq 0$ and A is $Ns.t.n.p.$ For each sequence $((i_0, j_0), (i_1, j_1), \ldots, (i_p, j_p))$ that is obtained by Procedure 4.12 set

$$a'_{i_0,j_0} := \begin{cases} a_{i_0,j_0} & \text{if } p = 0, \\ \frac{\delta_{i_0,j_0}}{\delta_{i_1,j_1}} & \text{if } 0 < p. \end{cases} \tag{14}$$

By construction a'_{i_0,j_0} is well-defined for each $(i_0, j_0) \in \{1, \ldots, n\}^2$. Define $A' := (a'_{i_0,j_0})_{i_0,j_0=1}^n$.

Claim $A' = \tilde{A}$ and $\delta_{i_0,j_0} = \tilde{a}_{i_0,j_0} \cdot \tilde{a}_{i_1,j_1} \cdots \tilde{a}_{i_p,j_p}$. □

Proof of the claim We proceed by decreasing induction with respect to the lexicographical order on the pairs (i, j), $i, j = 1, \ldots, n$.

If $i = n$ then by the definition $a'_{nj} = a_{nj} = \tilde{a}_{nj}$ for all $j = 1, \ldots, n$. For $j = n$, the claim also holds by the definition. Suppose that we have shown the claim for all pairs (i, j) such that $i = i_0 + 1, \ldots, n$, $j = 1, \ldots, n$ and $i = i_0$, $j = j_0 + 1, \ldots, n$ holds with $j_0 < n$. We want to show the claim for the pair $(i, j) = (i_0, j_0)$. Since A is $Ns.t.n.p.$ then we have by Theorem 4.8 that $\tilde{A}$ is a Cauchon matrix and $\tilde{A}[1, \ldots, n-1]$ is a nonnegative matrix with positive diagonal entries. Since all the entries of A are negative except possibly $a_{11} \leq 0$ we have that $1 \leq p$. Hence by the induction hypothesis, we obtain that the sequence which starts from the position (i_0, j_0) and is constructed by Procedure 4.12 is a lacunary sequence with respect to the Cauchon diagram that is associated with $\tilde{A}$. Moreover, it is easy to see that $\tilde{A}[i_0, \ldots, n \mid j_0, \ldots, n]$ is a Cauchon matrix. We add a sufficiently large positive number t to the $(1, 1)$ entry of the matrix $D := A[i_0, \ldots, n \mid j_0, \ldots, n]$ in order to be able to use Proposition 4.11 and name the resulting matrix D_t. Application of Proposition 4.11 to the matrix D_t (note that $\tilde{D}_t$ is a Cauchon matrix with the $(1, 1)$ entry equal to $\tilde{a}_{i_0, j_0} + t$ and the sequence $((i_0, j_0), \ldots, (i_p, j_p))$ is a lacunary sequence with respect to the Cauchon diagram that is associated with $\tilde{D}_t$) and Laplace expansion yield

$$\begin{aligned}\det A[i_0, i_1, \ldots, i_p \mid j_0, j_1, \ldots, j_p] \;+\; & t \det A[i_1, \ldots, i_p \mid j_1, \ldots, j_p] \\ & = (\tilde{a}_{i_0, j_0} + t) \cdot \tilde{a}_{i_1, j_1} \cdots \tilde{a}_{i_p, j_p}.\end{aligned}$$

By the induction hypothesis it follows that

$$\begin{aligned}\delta_{i_0, j_0} + t\delta_{i_1, j_1} &= (\tilde{a}_{i_0, j_0} + t) \cdot \frac{\delta_{i_1, j_1}}{\delta_{i_2, j_2}} \cdot \frac{\delta_{i_2, j_2}}{\delta_{i_3, j_3}} \cdots \frac{\delta_{i_p, j_p}}{1} \\ &= \tilde{a}_{i_0, j_0} \cdot \delta_{i_1, j_1} + t\delta_{i_1, j_1}.\end{aligned}$$

Hence we obtain that $\tilde{a}_{i_0, j_0} = \frac{\delta_{i_0, j_0}}{\delta_{i_1, j_1}}$. Therefore the claim follows and since $A' = \tilde{A}$ is a Cauchon matrix, $\tilde{A}[1, \ldots, n-1]$ is a nonnegative matrix with positive diagonal entries, and $i_p = n$ or $j_p = n$. Hence (i)–(iii) follow.

Conversely, suppose that (i)–(iii) hold. We want to show that under these conditions the claim holds. Again we proceed by decreasing induction with respect to the lexicographical order on the pairs (i, j), $i, j = 1, \ldots, n$. For $i = n$ or $j = n$, the claim holds trivially. Suppose that we have shown that the claim holds with $j_0 < n$ for all the pairs (i, j) such that $i = i_0 + 1, \ldots, n$, $j = 1, \ldots, n$ and $i = i_0$, $j = j_0 + 1, \ldots, n$. We want to show that the claim holds for the pair $(i, j) = (i_0, j_0)$. By the induction hypothesis and (i)–(iii) $\tilde{A}[i_0, \ldots, n \mid j_0, \ldots, n]$ is a Cauchon matrix. Define D and D_t as in the first implication. Then it is easy to see that $\tilde{D}$ coincides with $\tilde{A}[i_0, \ldots, n \mid j_0, \ldots, n]$ and by the induction hypothesis that $\tilde{D}$ and $\tilde{D}_t$ are Cauchon matrices. Moreover, D_t satisfies the conditions of Proposition 4.11.

The sequence that is constructed by Procedure 4.12 (with the obvious modification for the rectangular case) and starts at the position $(1, 1)$ in D_t is a lacunary sequence with respect to the Cauchon diagram that is associated with $\tilde{D}_t$; it coincides with the sequence that is constructed by Procedure 4.12 which starts at the position (i_0, j_0) in A. By application of Proposition 4.11 to D_t and Laplace expansion, we obtain as in the first implication using the induction hypothesis that $\tilde{a}_{i_0, j_0} = \frac{\delta_{i_0, j_0}}{\delta_{i_1, j_1}}$, whence the claim holds. Therefore, under the conditions (i)–(iii) $\tilde{A}$ is a Cauchon matrix and $\tilde{A}[1, \ldots, n-1]$ is a nonnegative matrix with positive diagonal entries. Hence by Theorem 4.8 A is $Ns.t.n.p.$ □

If we proceed from row $i_\mu + 1$ to row i_μ, we already know the determinantal entries which appear in row $i_\mu + 1$ and therefore we can easily check when $j_\mu < i_\mu$ whether all entries in the row $i_\mu + 1$ to the left of $\tilde{a}_{i_\mu+1, j_\mu+1}$ vanish. To check in the case $i_\mu < j_\mu$ whether all entries in the column $j_\mu + 1$ above $\tilde{a}_{i_\mu+1, j_\mu+1}$ vanish, we have to compute in addition the minors which are associated with the positions $(s, j_\mu+1)$, $s = 1, \ldots, i_\mu$. These minors differ in only one row index. Since a zero column stays a zero column through the performance of the Cauchon algorithm, the sign of altogether n^2 minors have to be checked (which include also trivial minors of order 1). These are significantly fewer than the number of determinants needed by the determinantal tests which are based on [7, Theorem 2.1] or Lemma 2.6. The latter one requires to check $2^{n+1} - n - 2$ minors, see [16, Section 5.1], but is independent of the matrix to be checked in contrast to the test based on Theorem 4.13. If we test a given matrix A for $t.n.$ it suffices to check n^2 fixed determinants (independently of A) for negativity because by Corollary 4.9 we may choose all sequences γ running diagonally.

5. Representation of the entries during running the Cauchon algorithm

In this section, we derive a representation of the entries of $\tilde{A}$ that will be helpful in the last section.

Proposition 5.1 *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be $Ns.t.n.p.$ with $a_{nn} < 0$. Then the entries $\tilde{a}_{kj}$ of the matrix $\tilde{A}$ can be represented as* $(k, j = 1, \ldots, n)$

$$\tilde{a}_{kj} = \frac{\det A[k, \ldots, i+p \mid j, \ldots, j+p]}{\det A[k+1, \ldots, i+p \mid j+1, \ldots, j+p]}, \tag{15}$$

with a suitable $0 \le p \le n-k$, if $j \le k$ and $0 \le p \le n-j$, if $k < j$.

We call p the order of the representation (15).

Proof The proof parallels lengthy the proof of [4, Proposition 2.10] (see [17, Proposition 3.8] for a much more elaborated proof) for the analogous statement for $NsTN$ matrices. That proof makes use only of the fact that certain minors are nonzero but not of their common sign; so we can proceed similarly. Therefore, we restrict ourselves here mainly on the parts which require some extra consideration. As we have seen in Theorem 4.5, after the application of the Cauchon algorithm to the given $Ns.t.n.p.$ matrix A the resulting matrix $\tilde{A}$ is a Cauchon matrix, i.e. when an entry vanishes then all the entries left to it or above it vanish, too. It suffices to consider only the case $j \le i$ since the case $i < j$ can be reduced to the latter case: The entries $\tilde{a}_{ij}$ with $i < j$ are identical to the entries $\tilde{c}_{ji}$, where $\tilde{C} = (\tilde{c}_{ij})$ is the matrix obtained from the transpose $C := A^T$ of A at the end of the Cauchon algorithm. If A is $Ns.t.n.p.$ then $0 < \tilde{a}_{ii}$, $i = 1, \ldots, n-1$, by Proposition 4.7. Therefore, if an entry of $\tilde{A}$ below the main diagonal vanishes then the entries in the same row left to it vanish, too. Theorem 4.4 and the proof of Proposition 4.7 show that this property also holds for the intermediate matrices $A^{(s,2)}$, $s = 2, \ldots, n$.

By decreasing induction on the row index one shows then that each entry $\tilde{a}_{ij}$ has a representation of the form (15) and that a neighbouring entry of $\tilde{a}_{ij}$ in the same row or column can be represented in the form (15) of identical order. If the numerator in (15) is negative then by Lemma 2.6 the denominator is negative, too. In the case that an entry in the lower part of $\tilde{A}$ vanishes the entries left to it vanish, too. In the proof of [4, Proposition 2.10] and [17, Proposition 3.8], it was shown that even these entries can be rewritten as ratios of contiguous minors. In the case of the $Ns.t.n.p.$ matrix A this is accomplished by the following proposition and [10, Proposition 3.3] or [18, Proposition 9]. □

PROPOSITION 5.2 [17, Proposition 2.5] *Let* $A \in \mathbb{R}^{n,m}$ *be* $t.n.p.$ *and let* $\alpha = (i+1, \ldots, i+r)$, $\beta = (j+1, \ldots, j+r)$ *for some* $i \in \{1, \ldots, n-r\}$, $j \in \{1, \ldots, m-r\}$, *and* $2 \le r \le \min\{n, m\} - 1$. *If* $A[\alpha \mid \beta]$ *has rank* $r-1$, *then*

(i) *either the rows* $i+1, \ldots, i+r$ *or the columns* $j+1, \ldots, j+r$ *of* A *are linearly dependent, or*
(ii) $A[1, \ldots, i+r \mid j+1, \ldots, m]$ *or* $A[i+1, \ldots, n \mid 1, \ldots, j+r]$ *has rank* $r-1$.

Proof We follow the proof of [3, Proposition 1.15]. Since $A[\alpha \mid \beta]$ has rank $r-1$ and $2 \le r$ then there exist $p, q \in \{1, \ldots, r\}$ such that $\det A[\alpha_{\widehat{i+p}} \mid \beta_{\widehat{j+q}}] < 0$. Set $B := (b_{kl})$ be the $(n-r+1) \times (m-r+1)$ matrix with

$$b_{kl} := \frac{\det A[\alpha_{\widehat{i+p}} \cup \{k\} \mid \beta_{\widehat{j+q}} \cup \{l\}]}{\det A[\alpha_{\widehat{i+p}} \mid \beta_{\widehat{j+q}}]}, \tag{16}$$

$k \in \{1, \ldots, n\} \setminus \alpha_{\widehat{i+p}}$, $l \in \{1, \ldots, m\} \setminus \beta_{\widehat{j+q}}$,

where the rows and columns are rearranged in increasing order. By Sylvester's Determinant Identity, see, e.g. [3, p.3], B is a TN matrix since A is $t.n.p.$ and $B[i+1 \mid j+1] = 0$ ($b_{i+p,j+q} = 0$ in the notation of (16)) since $A[\alpha \mid \beta]$ has rank $r-1$. Hence by [3, Proposition 1.15] applied to $B[i+1 \mid j+1]$ as submatrix either the row $i+1$ of B or the column $j+1$ of B is zero which implies by [19, Corollary 1, p.84] that (i) holds, or $B[1, \ldots, i+1 \mid j+1, \ldots, m-r+1]$ or $B[i+1, \ldots, n-r+1 \mid 1, \ldots, j+1]$ has rank 0 which implies by [19, Corollary 1, p.84] that (ii) holds. Therefore the claim follows. □

6. Application to interval problems

In this section, we present some results on intervals of matrices with respect to the checkerboard partial ordering. We start with some auxiliary properties.

LEMMA 6.1 [20, Corollary 3.5], [21, Proposition 3.6.6] *Let* $A, B, Z \in \mathbb{R}^{n,n}$, *A and B be nonsingular with* $0 \le A^{-1}, B^{-1}$. *If* $A \le Z \le B$. *Then Z is nonsingular, too, and we have* $B^{-1} \le Z^{-1} \le A^{-1}$.

The next two lemmata provide monotonicity properties of the determinant over intervals of special SR matrices.

LEMMA 6.2 *Let* $2 \le n$, $A, B, Z \in \mathbb{R}^{n,n}$, $A \le^* Z \le^* B$, A, B *be* $t.n.p.$ *Then it holds that*

$$\det B \le \det Z \le \det A,$$

if

(i) $n = 2$;
(ii) $2 < n$, *A is nonsingular and at least one of the following three conditions is fulfilled:*

(a) *B is nonsingular,*
(b) $b_{11} < 0$,
(c) $b_{nn} < 0$.

Proof

(i) is shown by direct computation.
(ii) We proceed by induction on n. Assume that the statement is true for fixed n and let $A, B, Z \in \mathbb{R}^{n+1,n+1}$, A be $Ns.t.n.p.$, B be $t.n.p.$, and $A \leq^* Z \leq^* B$. Assume first that B is nonsingular. Then by Lemma 2.6 $A[2,\ldots,n+1]$, $B[2,\ldots,n+1]$ are $Ns.t.n.p.$ and by the induction hypothesis

$$\det B[2,\ldots,n+1] \leq \det Z[2,\ldots,n+1] \leq \det A[2,\ldots,n+1] < 0. \quad (17)$$

Since $0 \leq (A^*)^{-1}, (B^*)^{-1}$ and $A^* \leq Z^* \leq B^*$, it follows from Lemma 6.1 that

$$(B^*)^{-1}[1] \leq (Z^*)^{-1}[1] \leq (A^*)^{-1}[1],$$

whence

$$\frac{\det B[2,\ldots,n+1]}{\det B} \leq \frac{\det Z[2,\ldots,n+1]}{\det Z} \leq \frac{\det A[2,\ldots,n+1]}{\det A}. \quad (18)$$

From

$$\det B \leq \frac{\det B[2,\ldots,n+1]}{\det Z[2,\ldots,n+1]} \cdot \det Z$$

and (18) we obtain $\det B \leq \det Z$. The remaining inequality follows similarly. If B is singular and $b_{11} < 0$, we first show that $b_{22} < 0$. Suppose that $b_{22} = 0$. Then $\det B[2,3|1,2] = b_{21} \cdot b_{32} \geq 0$ which implies that $b_{21} = 0$ or $b_{32} = 0$ whence $a_{21} = 0$ or $a_{32} = 0$, a contradiction to Lemma 2.5 (note that $a_{11} < 0$). Therefore (17) holds by the induction hypothesis. Set $B(\delta) := B + \delta e_1 e_1^T$ for $0 < \delta < -b_{11}$, where e_1 denotes the first unit vector of $\mathbb{R}^{n+1}$. Laplace expansion of $\det B(\delta)$ along its first row or column shows that $B(\delta)$ is $Ns.t.n.p.$ and the claim follows now from the case that B is nonsingular and letting δ tend to zero. If B is singular and $b_{nn} < 0$ we proceed similarly. □

Lemma 6.3 *Let $A, B, Z \in \mathbb{R}^{n,n}$, A, B be $NsASSR$ with the same signature, and $A \leq^* Z \leq^* B$. Then* $\det Z$ *is intermediate between* $\det A$ *and* $\det B$.

Proof For $\epsilon_2 = 1$ and $\epsilon_{n-1} \cdot \epsilon_n = 1$ we proceed similarly as in the nonsingular case in the proof of Lemma 6.2. Hereby the nonsingularity of $A[2,\ldots,n+1]$ and $B[2,\ldots,n+1]$ is assured by Lemma 2.2 (i). The case $\epsilon_{n-1} \cdot \epsilon_n = -1$ can be reduced to the case $\epsilon_{n-1} \cdot \epsilon_n = 1$ by replacing A, Z, B by $-B, -Z, -A$, respectively. The case $\epsilon_2 = -1$ can be reduced to the case $\epsilon_2 = 1$ by multiplication of A, Z, B by T_n, see Lemma 2.2 (iii). □

The next proposition can be proven using Lemma 6.1.

Proposition 6.4 [22, Theorem 1] *Let $A, B, Z \in \mathbb{R}^{n,n}$ with $A \leq^* Z \leq^* B$. If A, B are SSR with the same signature ϵ, then Z is SSR with signature ϵ.*

Now we are in the position to extend the results of Proposition 6.4 on intervals of SSR matrices and in [23, Theorem 1] on intervals of $NsATP$ matrices to arbitrary $NsASSR$ matrices.

THEOREM 6.5 *Let $A, B, Z \in \mathbb{R}^{n,n}$ with $A \leq^* Z \leq^* B$. If A, B are $NsASSR$ with the same signature $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$, then Z is $NsASSR$ with the signature ϵ.*

Proof By Theorem 2.3 it suffices to consider the nontrivial contiguous minors of Z. Let $\det Z[\alpha|\beta]$ be such a minor of order k. We want to show that $0 < \epsilon_k \det Z[\alpha|\beta]$. We proceed by induction on k. The statement trivially holds for $k = 1$. Suppose that the sign condition is true for $k-1$, we want to show that it is true for k. We have two cases:
Case 1 If A and B are both type-I staircase matrices, then obviously Z is also a type-I staircase matrix. Since $Z[\alpha|\beta]$ is contiguous we have

$$A[\alpha \mid \beta] \leq^* Z[\alpha \mid \beta] \leq^* B[\alpha \mid \beta] \tag{19}$$

or the reverse inequalities. Without loss of generality suppose that (19) holds.
Case 1.1 Suppose that the contiguous minors $\det A[\alpha|\beta]$, $\det B[\alpha|\beta]$ are both nontrivial and therefore are nonsingular. Hence $A[\alpha|\beta]$ and $B[\alpha|\beta]$ are themselves $NsASSR$ (with common signature $(\epsilon_1, \ldots, \epsilon_k)$) and the claim follows by Lemma 6.3.
Case 1.2 Suppose that $\det A[\alpha|\beta]$ or $\det B[\alpha|\beta]$ is trivial. Then Lemma 2.2 (i) implies that $a_{\alpha_1,\beta_1} \cdot a_{\alpha_2,\beta_2} \cdots a_{\alpha_k,\beta_k} = 0$ or $b_{\alpha_1,\beta_1} \cdot b_{\alpha_2,\beta_2} \cdots b_{\alpha_k,\beta_k} = 0$. Let

$$i_0 := \min \left\{ i \in \{1, \ldots, k\} \mid a_{\alpha_i,\beta_i} = 0 \text{ or } b_{\alpha_i,\beta_i} = 0 \right\}.$$

Without loss of generality, we may assume that $1 < i_0$. By (19) we have

$$\det Z[\alpha \mid \beta] = \det Z[\alpha_1, \ldots, \alpha_{i_0-1} \mid \beta_1, \ldots, \beta_{i_0-1}] \cdot \det Z[\alpha_{i_0}, \ldots, \alpha_k|\beta_{i_0}, \ldots, \beta_k]. \tag{20}$$

Since $Z[\alpha|\beta]$ is nontrivial it follows from Lemma 2.2 (i) that $z_{\alpha_1,\beta_1} \cdots z_{\alpha_k,\beta_k} \neq 0$ and $a_{\alpha_1,\beta_1} \cdots a_{\alpha_k,\beta_k} = 0$ or $b_{\alpha_1,\beta_1} \cdots b_{\alpha_k,\beta_k} = 0$ but not both since $z_{\alpha_1,\beta_1} \cdots z_{\alpha_k,\beta_k} \neq 0$ whence both minors on the right-hand side of (20) are nontrivial, too. Lemma 2.4 implies that $\epsilon_j = \epsilon_1^j$, $j = 1, \ldots, k$, and we obtain

$$\begin{aligned}
&\epsilon_k \det Z[\alpha|\beta] = \epsilon_1^k \det Z[\alpha|\beta] \\
= \;& \epsilon_1^{i_0-1} \det Z[\alpha_1, \ldots, \alpha_{i_0-1}|\beta_1, \ldots, \beta_{i_0-1}] \cdot \epsilon_1^{k-i_0+1} \det Z[\alpha_{i_0}, \ldots, \alpha_{i_k}|\beta_{i_0}, \ldots, \beta_k] \\
= \;& \epsilon_{i_0-1} \det Z[\alpha_1, \ldots, \alpha_{i_0-1}|\beta_1, \ldots, \beta_{i_0-1}] \cdot \epsilon_{k-i_0+1} \det Z[\alpha_{i_0}, \ldots, \alpha_{i_k}|\beta_{i_0}, \ldots, \beta_k].
\end{aligned}$$

Both signed minors on the right-hand side of the last equation are positive by the induction hypothesis and it follows that $0 < \epsilon_k \det Z[\alpha|\beta]$, as desired. This completes the proof of Case 1.
Case 2 If A and B are type-II staircase matrices, then obviously Z is also a type-II staircase matrix. By Lemma 2.2 (iii) we can reduce Case 2 to Case 1. □

Using Proposition 5.1 and (18), we obtain by an induction proof similarly as in [4, Proposition 3.3] (see [17, Proposition 4.1] for a much more elaborated proof) the following result.

Proposition 6.6 *Let $A, B, Z \in \mathbb{R}^{n,n}$ with $A \leq^* Z \leq^* B$. If A, B are $Ns.t.n.p.$ with $b_{nn} < 0$, then $\tilde{A} \leq^* \tilde{Z} \leq^* \tilde{B}$.*

Using Propositions 4.7 and 6.6, we get the following theorem from Theorem 4.8 by a proof similar to the one of [4, Theorem 3.6].

Theorem 6.7 *Let $A, B, Z \in \mathbb{R}^{n,n}$ with $A \leq^* Z \leq^* B$. If A, B are $Ns.t.n.p.$ with $b_{nn} < 0$, then Z is $Ns.t.n.p.$*

By passing over to $A^\#$ and back, Theorem 6.7 remains in force if we replace the condition $b_{nn} < 0$ by $b_{11} < 0$. A similar modification applies to the following corollary.

Proceeding similarly as in the proof of the singular case in Lemma 6.2 we obtain the following corollary which provides an extension of the nonsingular case.

Corollary 6.8 *Let $A, B, Z \in \mathbb{R}^{n,n}$ with $A \leq^* Z \leq^* B$, A, B be $t.n.p.$ with $b_{nn} < 0$ and*

(i) *$A[2, \ldots, n]$ nonsingular and $b_{11} < 0$,*

or

(ii) *$A[1, \ldots, n-1]$ nonsingular.*

Then Z is $t.n.p.$

The following remark shows that the negativity of the entries b_{nn} (and b_{11}) in Theorem 6.7 is not necessary.

Remark 1 [17, Remark 4.3] Let $A, B, Z \in \mathbb{R}^{n,n}$ with $A \leq^* Z \leq^* B$ and let A and B be $Ns.t.n.p.$ with $b_{11} = 0$. If $b_{nn} = 0$, then by [18, Proposition 6] there exists a small suitable $0 < \epsilon_0$ such that $B_\epsilon := B - \epsilon e_n e_n^T$ is $Ns.t.n.p.$ for all $0 \leq \epsilon < \epsilon_0$, where e_n denotes the last unit vector of $\mathbb{R}^n$. If $a_{nn} = 0$, then define A_ϵ analogously (with suitable ϵ) otherwise set $A_\epsilon := A$ and by Proposition [18, Proposition 6] A_ϵ and B_ϵ are $Ns.t.n.p.$ matrices. Define Z_ϵ analogously. Hence we have that for $A_\epsilon \leq^* Z_\epsilon \leq^* B_\epsilon$ Theorem 6.7 holds. By [18, Proposition 7] and the definition of $Ns.t.n.p.$ matrices Z is $Ns.t.n.p.$

We conclude this section with the special case of tridiagonal matrices. To recall, $A = (a_{ij}) \in \mathbb{R}^{n,n}$ is called *tridiagonal* if $a_{ij} = 0$ if $1 < |i - j|$, $i, j = 1, \ldots, n$. We need the following two auxiliary results.

Proposition 6.9 [24, Theorem 4.1] *Let $3 \leq n$, $A = (a_{ij}) \in \mathbb{R}^{n,n}$, $0 \leq A$, and A be nonsingular and tridiagonal. Then A is SR if and only if $A[1, \ldots, n-1]$ as well as $A[2, \ldots, n]$ are TN and $A[1, \ldots, n-2]$ as well as $A[2, \ldots, n-1]$ are nonsingular.*

Proposition 6.10 [4, Corollary 3.7] *Let $A, B, Z \in \mathbb{R}^{n,n}$ with $A \leq^* Z \leq^* B$. If A, B are TN and $A[2, \ldots, n]$ or $A[1, \ldots, n-1]$ is nonsingular, then Z is TN.*

THEOREM 6.11 *Let $A, B, Z \in \mathbb{R}^{n,n}$ with $A \leq^* Z \leq^* B$ and A, B be tridiagonal. If A, B are $NsSR$ with the same signature ϵ then Z is $NsSR$ with signature ϵ.*

Proof Without loss of generality, we may assume that $0 \leq A$, otherwise replace A by $-B$ and B by $-A$. Since the statement trivially holds for $n \leq 2$, suppose that $3 \leq n$. It follows from Propositions 6.9 and 6.10 that $Z[1, \ldots, n-1]$ and $Z[2, \ldots, n]$ are TN. Since $A_{n-2} := A[1, \ldots, n-2]$ and $B_{n-2} := B[1, \ldots, n-2]$ are $NsTN$, $0 \leq (A^*_{n-2})^{-1}$, $(B^*_{n-2})^{-1}$, and $A^*_{n-2} \leq Z^*[1, \ldots, n-2] \leq B^*_{n-2}$, Lemma 6.1 implies that $Z[1, \ldots, n-2]$ is nonsingular, too. Similarly it follows that Z, $Z[2, \ldots, n-1]$ are nonsingular. By Proposition 6.9 we obtain that Z is $NsSR$. □

The special case $\epsilon = (1, \ldots, 1, \epsilon_n)$ in Theorem 6.11 follows also from [4, Corollary 3.8] and [11, Theorem 9].

7. Conclusions

We have investigated the application of the Cauchon algorithm to $Ns.t.n.p.$ matrices which has lead us to the interval property of these matrices. We also proved that, e.g. the sets of the $NsASSR$ matrices and the tridiagonal $NsSR$ matrices possess this property, too. In [25], we provide some further signatures for $NsSR$ matrices which allow the interval property. These results together with the results in [4,22,23] on the interval property of some other classes of $NsSR$ matrices evoke the (open) question whether the interval property holds for general $NsSR$ matrices. We mention the following partial answer to this question [26]: All matrices Z with $A \leq^* Z \leq^* B$ are $NsSR$ if all members of a set of matrices $C = (c_{ij})$, $c_{ij} \in \{a_{ij}, b_{ij}\}$, $i, j = 1, \ldots, n$, of cardinality of at most 2^{2n-1} are $NsSR$.

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Notes

1. This algorithm is called in [5] the deleting derivations algorithm and in [6] the Cauchon reduction algorithm.
2. Note that $A^{(k,1)} = A^{(k,2)}$, $k = 1, \ldots, n-1$, and $A^{(2,2)} = A^{(1,2)}$ so that the algorithm could already be terminated when $A^{(2,2)}$ is computed.
3. The negativity of the entries of A and nonpositivity of a_{11} comes into play in the last step, i.e. when applying the Restoration algorithm with $r = (n, n)$.

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