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Total nonnegativity of matrices related to polynomial roots and poles of rational functions

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ABSTRACT

In this paper totally nonnegative (positive) matrices are considered which are matrices having all their minors nonnegative (positive); the almost totally positive matrices form a class between the totally nonnegative matrices and the totally positive ones. An efficient determinantal test based on the Cauchon algorithm for checking a given matrix for falling in one of these three classes of matrices is applied to matrices which are related to roots of polynomials and poles of rational functions, specifically the Hankel matrix associated with the Laurent series at infinity of a rational function and matrices of Hurwitz type associated with polynomials. In both cases it is concluded from properties of one or two finite sections of the infinite matrix that the infinite matrix itself has these or related properties. Then the results are applied to derive a sufficient condition for the Hurwitz stability of an interval family of polynomials. Finally, interval problems for a subclass of the rational functions, viz. R-functions, are investigated. These problems include invariance of exclusively positive poles and exclusively negative roots in the presence of variation of the coefficients of the polynomials within given intervals.

1. Introduction

In this paper we consider matrices which are related to stability of polynomials and to the localization of the poles and zeros of rational functions. Specifically, in the case of polynomials we focus on matrices of Hurwitz type which are closely related to (Hurwitz) stability of a polynomial, i.e., to the property that all zeros are contained in the open left half of the complex plane. In the case of rational functions we focus on R-functions of negative type, i.e., functions which map the open upper half-plane of the complex plane to the open lower half-plane. For references and properties of this important class of functions the reader is referred to the survey given in [14]. In the polynomial as well as in the rational case we are interested in

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interval problems which arise when the polynomial coefficients are due to uncertainty caused by, e.g., data uncertainties but can be bounded in intervals. For background material from control theory and practical applications see [4,5]. It turns out that certain properties concerning the zeros and the poles remain in force through all the coefficient intervals if up to four polynomials of the entire family have certain properties. Typically, the coefficients of these polynomials alternate in attaining the endpoints of the coefficient intervals. This up-and-down behavior corresponds to a checkerboard pattern of the entries of the associated matrices.

The underlying property of all the matrices considered in this paper is that all their minors are nonnegative. Such matrices are called *totally nonnegative*. For properties of these matrices the reader is referred to the monographs [6,18]. In [2] we derive an efficient determinantal test based on the Cauchon algorithm [12,17] for checking a given matrix for total nonnegativity and related properties. In this paper we apply this test to the matrices mentioned above. To solve the related interval problems we make use of a result in [1] by which from the nonsingularity and the total nonnegativity of two matrices we can infer that all matrices lying between these two matrices are nonsingular and totally nonnegative, too. Here 'between' is meant in the sense of the checkerboard ordering, see above.

The organization of our paper is as follows. In the next section we first introduce the notation and the definitions and recall then some properties of the totally nonnegative matrices which we will use in our paper. We also briefly recall the Cauchon algorithm and characterizations of two subclasses of the totally nonnegative matrices. In Section 3 we show that from properties of finite sections of an infinite Hankel matrix or matrix of Hurwitz type we may conclude that the infinite matrix itself possesses these or related properties. We also derive a sufficient condition for the stability of an interval family of polynomials. In Section 4 we present interval problems related to R-functions.

2. Notation and auxiliary results

For nonnegative integers k, n we denote by $Q_{k,n}$ the set of all strictly increasing sequences of k integers chosen from $\{1, 2, \dots, n\}$. For $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k) \in Q_{k,n}$ and $\beta = (\beta_1, \beta_2, \dots, \beta_l) \in Q_{l,m}$, we denote by $A[\alpha|\beta]$ the $k \times l$ submatrix of A contained in the rows indexed by $\alpha_1, \alpha_2, \dots, \alpha_k$ and columns indexed by $\beta_1, \beta_2, \dots, \beta_l$. We suppress the parentheses when we enumerate the indices implicitly. When $\alpha = \beta$, the principal submatrix $A[\alpha|\alpha]$ is abbreviated to $A[\alpha]$; if $\alpha = \beta = (1, \dots, k)$ the submatrix is called a *leading* principal submatrix. We denote by $\mid \alpha \mid$ the number of members of α. If $k = l$ and $A[\alpha|\beta]$ is formed from consecutive rows and columns of A then it is called *contiguous* and its determinant is termed a *contiguous* minor. A matrix $A \in \mathbb{R}^{m,n}$ is called *totally positive* (abbreviated *TP* henceforth) and *totally nonnegative* (abbreviated *TN*) if $\det A[\alpha|\beta] > 0$ and $\det A[\alpha|\beta] \geq 0$, respectively, for all $\alpha \in Q_{k,m}$, $\beta \in Q_{k,n}$. If $A \in \mathbb{R}^{n,n}$ is *TN* and in addition nonsingular we write A is *NsTN*. In [9] Gasca et al. define the following class of matrices intermediate between the totally nonnegative and the totally positive matrices. If $A \in \mathbb{R}^{m,n}$ is *TN* it is said to be *almost totally positive* (abbreviated *ATP*) if it satisfies the following two conditions:

(i) Any contiguous minor of A is positive if and only if the diagonal entries of the corresponding submatrix are positive.
(ii) In the case that A has a zero row or column, the subsequent rows or columns also are zero, respectively.

It was proven in [9] that if A is *ATP* then (i) holds for any minor of A (not only for the contiguous ones). If A is *ATP* and in addition it is nonsingular then we write A is *NsATP*. These matrices were also introduced independently in [11]. For further properties see [10].

We endow $\mathbb{R}^{m,n}$ with two partial orderings: Firstly, with the usual entry-wise ordering ($A = (a_{ij})$, $B = (b_{ij}) \in \mathbb{R}^{m,n}$)

$$A \leq B \Leftrightarrow a_{ij} \leq b_{ij}, \ \ i = 1, \dots, m, \ j = 1, \dots, n.$$

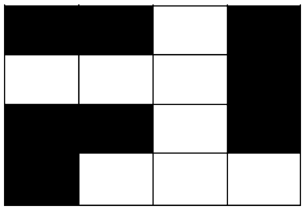

Fig. 1. Example of a 4×4 Cauchon diagram.

Secondly, with the *checkerboard partial ordering*, which is defined as follows: Let $\Delta_r := diag(1, -1, \ldots, (-1)^{r+1})$ and $A^* := \Delta_m A \Delta_n$.

Then we define

$$A \leq^* B \Leftrightarrow A^* \leq B^*.$$

These definitions extend verbatim to infinite matrices. An infinite matrix is said to have rank r if all its minors of order greater than r are zero and there exists at least one nonzero minor of order r. If in addition all its minors up to order r (inclusive) are positive then the matrix is said to be TP_r of rank r.

The following theorem gives an important property of the nonsingular totally nonnegative matrices.

Lemma 2.1. *[6, Corollary 3.8], [18, Theorem 1.13] All principal minors of an NsTN matrix are positive.*

For reference in Section 4 we state a monotonicity property of the determinant and an interval property of the TN matrices.

Proposition 2.2. *[1, Lemma 3.2] Let $A, B, Z \in \mathbb{R}^{n,n}$, A be NsTN, B be TN and $A \leq^* Z \leq^* B$. Then $\det A \leq \det Z \leq \det B$.*

Theorem 2.3. *[1, Theorem 3.6] Let $A, B, Z \in \mathbb{R}^{n,n}$ with $A \leq^* Z \leq^* B$. If A and B are NsTN, then Z is NsTN.*

A direct consequence of the above theorem is the following corollary.

Corollary 2.4. *[1, Remark 3.5] Let $A, B, Z \in \mathbb{R}^{n,n}$ with $A \leq^* Z \leq^* B$. If A and B are TP, then Z is TP.*

A basis tool for obtaining our main results is the Cauchon algorithm [12,17] which we will introduce next. First we recall the definition of a Cauchon diagram and a Cauchon matrix.

Definition 2.5. An m-by-n *Cauchon diagram* C is an $m \times n$ grid consisting of mn squares colored black and white, where each black square has the property that either every square to its left (in the same row) or every square above it (in the same column) is black. We identify the squares of C with coordinates and say $(i, j) \in C$ and $(i, j) \notin C$ if the square in position (i, j) is black and white, respectively, $i = 1, \ldots, m$, $j = 1, \ldots, n$.

An example of a 4×4 Cauchon diagram is given in Fig. 1.

Definition 2.6. Let $A \in \mathbb{R}^{m,n}$ and let C be an m-by-n Cauchon diagram. We say that A is a *Cauchon matrix associated with the Cauchon diagram C* if for all (i, j), $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n\}$ we have $a_{ij} = 0$ if and only if the square (i, j) in C is black. If A is a Cauchon matrix associated with an unspecified Cauchon diagram, we just say that A is a Cauchon matrix.

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In [12] and [17] the Cauchon algorithm is presented which is employed to check whether a given matrix A is totally positive or totally nonnegative. The algorithm starts with A and produces a related matrix $\tilde{A}$. From the sign of the entries of $\tilde{A}$ and its zero-nonzero pattern we can decide whether A is totally positive or totally nonnegative. A compressed form of the algorithm is given in [2].

To recall the Cauchon algorithm and a procedure based on it, we denote by $\leq$ and $\leq_c$ the lexicographic and colexicographic order, respectively, on $\mathbb{N}^2$, i.e.,

$$(g,h) \leq (i,j) :\Leftrightarrow \ (g < i) \text{ or } (g = i \text{ and } h \leq j),$$
$$(g,h) \leq_c (i,j) :\Leftrightarrow \ (h < j) \text{ or } (h = j \text{ and } g \leq i).$$

Set $E^\circ := \{1,\dots,m\} \times \{1,\dots,n\} \setminus \{(1,1)\}$, $E := E^\circ \cup \{(m+1,1)\}$.

Let $(s,t) \in E^\circ$. Then $(s,t)^+ := \mathit{min}\,\{(i,j) \in E \mid (s,t) \leq (i,j),\ (s,t) \neq (i,j)\}$; here the minimum is taken with respect to the lexicographical order.

Cauchon algorithm: Let $A \in R^{m,n}$. As r runs in decreasing order over the set E, we define matrices $A^{(r)} = (a_{ij}^{(r)}) \in R^{m,n}$ as follows.

1. Set $A^{(m+1,1)} := A$.
2. For $r = (s,t) \in E^\circ$ define the matrix $A^{(r)} = (a_{ij}^{(r)})$ as follows.
   (a) If $a_{st}^{(r^+)} = 0$ then put $A^{(r)} := A^{(r^+)}$.
   (b) If $a_{st}^{(r^+)} \neq 0$ then put

$$a_{ij}^{(r)} := \begin{cases} a_{ij}^{(r^+)} - \dfrac{a_{it}^{(r^+)} a_{sj}^{(r^+)}}{a_{st}^{(r^+)}} & \text{for } i < s \text{ and } j < t, \\ a_{ij}^{(r^+)} & \text{otherwise.} \end{cases}$$

3. Set $\tilde{A} := A^{(1,2)}$[2]; $\tilde{A}$ is called *the matrix obtained from A (by the Cauchon algorithm)*.

One of the efficient methods to check whether a given matrix is TN or TP is by using the Cauchon algorithm. The following theorem provides necessary and sufficient conditions for a given matrix to be TP, TN, or $NsTN$.

Theorem 2.7. *[12, Theorem B4], [17, Theorems 2.6 and 2.7], [1, Proposition 2.8] The following statements hold.*

(i) *$A \in \mathbb{R}^{m,n}$ is TP (TN) if and only if $\tilde{A}$ is an entry-wise positive (nonnegative) Cauchon matrix.*
(ii) *If $A \in \mathbb{R}^{n,n}$ is TN then A is nonsingular if and only if $0 < \tilde{a}_{ii}$, $i = 1,\dots,n$.*

The following definition presents a special type of finite sequences which play a fundamental role in characterizing and testing TN matrices.

Definition 2.8. Let C be an m-by-n Cauchon diagram. We say that a sequence

$$\gamma := ((i_k, j_k), k = 0,1,\dots,p) \tag{1}$$

which is strictly increasing in both arguments is a *lacunary sequence with respect to C* if the following conditions hold:

[2] Note that $A^{(k,1)} = A^{(k,2)}$, $k = 1,\dots,m-1$, and $A^{(2,2)} = A^{(1,2)}$ so that the algorithm could already be terminated when $A^{(2,2)}$ is computed.

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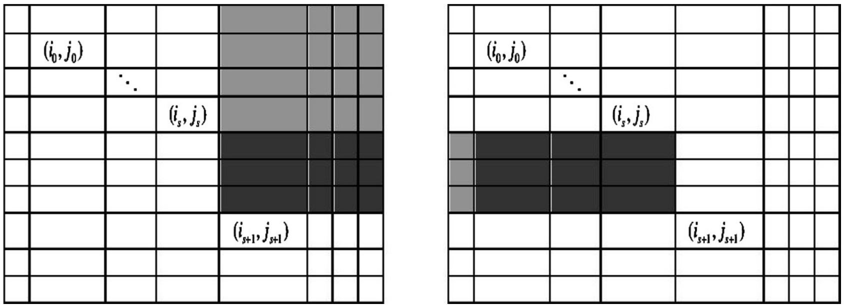

Fig. 2. Condition (iii)(a) of Definition 2.8.

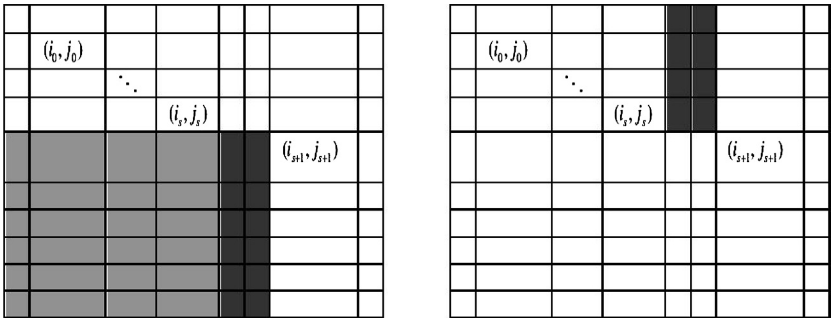

Fig. 3. Condition (iii)(b) of Definition 2.8.

(i) $(i_k, j_k) \notin C$, $k = 1, \ldots, p$.
(ii) $(i, j) \in C$ for $i_p < i \leq m$ and $j_p < j \leq n$.
(iii) Let $s \in \{0, \ldots, p-1\}$. Then $(i, j) \in C$ if
  (a) either for all (i, j), $i_s < i < i_{s+1}$ and $j_s < j$,
  or for all (i, j), $i_s < i < i_{s+1}$ and $j_0 \leq j < j_{s+1}$
  and
  (b) either for all (i, j), $i_s < i$ and $j_s < j < j_{s+1}$
  or for all (i, j), $i < i_{s+1}$, and $j_s < j < j_{s+1}$.

Condition (iii) of Definition 2.8 is illustrated by Figs. 2, 3; here the collection of black squares determined by Condition (a) or (b) (displayed in dark gray) is enlarged by taking into account that the underlying diagram is a Cauchon diagram (displayed in light gray). In Fig. 1, the sequence $((1, 1), (2, 3), (4, 4))$ is a lacunary sequence while the sequence $((1, 1), (2, 2), (4, 4))$ is not.

Definition 2.9. We say that a sequence γ given by (1) is *diagonal* if

$$(i_{k+1}, j_{k+1}) = (i_k + 1, j_k + 1), \quad k = 0, 1, \ldots, p-1.$$

The following proposition gives the relationship between the minors associated with lacunary sequences and entries of $\tilde{A}$.

Proposition 2.10. *[17, Proposition 4.1] Let $A \in R^{m,n}$ be TN and γ given by (1) be a lacunary sequence with respect to the Cauchon diagram that is associated with $\tilde{A}$. Then*

$$\det A[i_0, \ldots, i_p | j_0, \ldots, j_p] = \tilde{a}_{i_0, j_0} \cdot \tilde{a}_{i_1, j_1} \cdots \tilde{a}_{i_p, j_p}. \tag{2}$$

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In [17, Section 3] an algorithm is presented which constructs for a given Cauchon diagram C and any square of C a lacunary sequence (with respect to C) starting at this square. In [2] we design a procedure by which for each entry $\tilde{a}_{i_0,j_0}$ of $\tilde{A}$ a sequence γ given by (1) starting at position (i_0, j_0) is constructed (without explicit reference to a Cauchon diagram).

It is sufficient to describe the construction of the sequence from the starting pair (i_0, j_0) to the next pair (i_1, j_1) with $0 < \tilde{a}_{i_1,j_1}$ since for a given matrix A the determinantal test in [2] is performed by moving row by row from the bottom to the top row. Once we have found the next index pair (i_1, j_1) we append to (i_0, j_0) the sequence starting at (i_1, j_1). By construction, the sequence γ is uniquely determined (but not necessarily lacunary). If conditions (i) and (ii) of Theorem 2.13 below are fulfilled then all these sequences are lacunary.

In the following δ_{ij} denotes the minor of A associated by (2) to the sequence starting at position (i, j) which is constructed by the following procedure.

Remark 2.11. In [2] we present Procedure 2.12 and Theorem 2.13 below in terms of $\tilde{a}_{ij}$ but here we use δ_{ij}. The reason is the following. First note that $\tilde{A}[i_0, \ldots, m \mid j_0, \ldots, n]$ coincides with the matrix which is obtained by application of the Cauchon algorithm to $A[i_0, \ldots, m \mid j_0, \ldots, n]$. One can show by decreasing induction on the pairs (i, j) with respect to the lexicographical order and using (2) that the numerator of $\tilde{a}_{i_0,j_0}$ is equal to δ_{i_0,j_0} for all $i_0 = 1, \ldots, m$, $j_0 = 1, \ldots, n$ (if necessary add an artificial sufficiently large positive quantity t to a_{i_0,j_0} as in the proof of Theorem 3.1 in order to be able to apply Proposition 2.10). Therefore, the quantity δ_{ij} determines whether $\tilde{a}_{ij}$ takes on a zero or nonzero values.

Procedure 2.12. [2, Procedure 5.2] Construction of the sequence γ given by (1) starting at (i_0, j_0) to the next index pair (i_1, j_1) in the *TN* case:

If $i_0 = m$ or $j_0 = n$ or $S := \{(i, j) \mid i_0 < i \le m,\ j_0 < j \le n, \text{ and } 0 < \delta_{ij}\}$ is void **then** terminate with $p := 0$;
else
  if $\delta_{ij_0} = 0$ for all $i = i_0 + 1, \ldots, m$ **then** put $(i_1, j_1) := \min S$ with respect to the colexicographic order
  else
  put $i' := \min\{k \mid i_0 < k \le m \text{ such that } 0 < \delta_{kj_0}\}$,
    $J := \{l \mid j_0 < l \le n \text{ such that } 0 < \delta_{i',l}\}$;
   if J is not void **then** put $(i_1, j_1) := (i', \min J)$
   else put $(i_1, j_1) := \min S$ with respect to the lexicographic order;
   end if
  end if
end if.

Theorem 2.13. *[2, Theorem 5.4] Let $A \in \mathbb{R}^{m,n}$. Then A is TN if and only if for all $i = 1, \ldots, m$, $j = 1, \ldots, n$ the quantities δ_{ij} obtained by the sequences that start from positions (i, j) and are constructed by Procedure 2.12 satisfy the following conditions:*

(i) $0 \le \delta_{ij}$;
(ii) *if $\delta_{i'j'} = 0$ for some $i' \in \{1, \ldots, m\}$, $j' \in \{1, \ldots, n\}$, then $\delta_{i',t_1} = 0$ for all $t_1 < j'$ or $\delta_{t_2,j'} = 0$ for all $t_2 < i'$.*

If we proceed from row $i_\mu + 1$ to row i_μ we already know the determinantal entries which appear in row $i_\mu + 1$ and therefore we can easily check when $j_\mu < i_\mu$ whether all entries in the row $i_\mu + 1$ to the left of $\tilde{a}_{i_\mu+1,j_\mu+1}$ vanish. To check in the case $i_\mu < j_\mu$ whether all entries in the column $j_\mu + 1$ above $\tilde{a}_{i_\mu+1,j_\mu+1}$

vanish we have to compute in addition the minors which are associated with the positions $(s, j_\mu + 1)$, $s = 1, \ldots, i_\mu$. These minors differ in only one row index. Since a zero column stays a zero column through the performance of the Cauchon algorithm, the sign of altogether $n \cdot m$ minors have to be checked (which also include trivial minors of order 1). If in Theorem 2.13, $A \in \mathbb{R}^{n,n}$ and $0 < \delta_{ii}$ for all $i = 1, \ldots, n$, then A is *NsTN* by Theorem 2.7 (ii).

The following theorem gives an efficient criterion for checking a nonsingular matrix for almost total positivity.

Theorem 2.14. *[2, Theorem 5.10] Let $A \in R^{n,n}$ be TN. Then the following two properties are equivalent:*

(i) *A is NsATP.*
(ii) *The matrix $\tilde{A}$ (obtained from A by the Cauchon algorithm) has a positive main diagonal and the same zero–nonzero pattern of its entries as A.*

We conclude this section with the following theorem which extends the above theorem to the general case. The proof uses the same arguments that have been employed in the proof of Theorem 5.10 in [2]. Note that zero rows and columns as in condition (ii) in the definition of an *ATP* matrix stay zero rows and columns, respectively, during the performance of the Cauchon algorithm. Also, sequences γ that are constructed according to Procedure 2.12 coincide with the sequences constructed when zero rows and columns are deleted.

Theorem 2.15. *Let $A \in R^{m,n}$ be TN. Then the following two properties are equivalent:*

(i) *A is ATP.*
(ii) *The matrix $\tilde{A}$ (obtained from A by the Cauchon algorithm) has the same zero–nonzero pattern of its entries as A.*

3. Total nonnegativity of Hankel and Hurwitz matrices and stability of polynomials

In this section we present some new characterizations and give for known results uniform and short proofs based on the application of the Cauchon algorithm and Theorems 2.7, 2.13, 2.14.

We start with the infinite *Hankel matrix*

$$S = (s_{i+j})_{i,j=0}^{\infty}, \tag{3}$$

where s_i, $i = 0, 1, \ldots,$ are given real numbers. The following theorem characterizes totally nonnegative Hankel matrices (for the equivalence of $(i) \Leftrightarrow (ii)$ see Theorem 4.4 and the references on p. 125 in [18]).

Theorem 3.1. *Let S be a real infinite Hankel matrix of rank n. Let furthermore $1 < n$ and $0 \leq \det S[1, \ldots, n|2, \ldots, n+1]$. Put $A := S[1, \ldots, n]$ and $B := A[1, \ldots, n-1 \mid 2, \ldots, n]$. Then the following three statements are equivalent:*

(i) *The matrices A and B are positive definite.*
(ii) *The Hankel matrix A is TP.*
(iii) *S is TN.*

Proof. $(i) \Rightarrow (ii)$ We first note that any principal minor of A and B is positive since A and B are positive definite matrices. According to Theorem 2.7 (i) it is sufficient to show that $\tilde{A}$ is entry-wise positive. The entries in the last row and column of $\tilde{A}$ coincide with the respective entries in the last row and column of A and are positive since they appear on the main diagonal of A or B. Now we turn to the remaining entries of $\tilde{A}$ and assume that there are entries which are not positive. Let (i_0, j_0) be the maximum element of the set $\{1, \ldots, n\}^2$ with respect to the lexicographical order such that $\tilde{a}_{i_0,j_0} \leq 0$. We add to $\tilde{a}_{i_0,j_0}$ a sufficiently large positive number t to make $\tilde{a}_{i_0,j_0} + t$ positive and call D the matrix which is obtained from $A[i_0, \ldots, n|j_0, \ldots, n]$ in this way. Then $\tilde{D}$ is entry-wise positive and the sequence γ given by (1) starting at position (i_0, j_0) found by Procedure 2.12 is diagonal. Note that d_{11} is identical with the entry in position $(1, 1)$ of the matrix which is obtained by the application of Cauchon algorithm to the matrix which results from $A[i_0, \ldots, n|j_0, \ldots, n]$ when adding t to a_{i_0,j_0}. Use of Proposition 2.10 and application of Laplace's expansion to the minor of D associated to γ by (2) yields

$$\det A[i_0, \ldots, i_p|j_0, \ldots, j_p] + t \det A[i_1, \ldots, i_p|j_1, \ldots, j_p] =$$
$$\tilde{a}_{i_0,j_0} \cdot \tilde{a}_{i_1,j_1} \cdots \tilde{a}_{i_p,j_p} + t\tilde{a}_{i_1,j_1} \cdots \tilde{a}_{i_p,j_p}.$$

A further application of Proposition 2.10 gives

$$\det A[i_1, \ldots, i_p|j_1, \ldots, j_p] = \tilde{a}_{i_1,j_1} \cdots \tilde{a}_{i_p,j_p},$$

whence

$$\det A[i_0, \ldots, i_p|j_0, \ldots, j_p] = \tilde{a}_{i_0,j_0} \cdot \tilde{a}_{i_1,j_1} \cdots \tilde{a}_{i_p,j_p}.$$

By the special pattern of the entries of A the minor on the left-hand side is a principal minor of A or B hence it is positive. On the other hand, the product of the right-hand side is nonpositive by our assumption and we have arrived at a contradiction.

$(ii) \Rightarrow (iii)$ Let A be *TP* and let $A_\mu := S[1, \ldots, \mu]$, $\mu = n, n + 1, \ldots$.

Claim. *A_μ is TN and $S[1, \ldots, n - 1 \mid \mu - n + 2, \ldots, \mu]$ is nonsingular for each $\mu = n, n + 1, \ldots$.*

Proof of the Claim. The proof proceeds by induction. For $\mu = n$ the claim holds since $A = A_n$ is *TP*. Suppose the claim holds for μ. We want to show that the claim holds for $\mu + 1$. First of all we prove that $S[1, \ldots, n - 1 \mid \mu - n + 3, \ldots, \mu + 1]$ is nonsingular. If it is singular, then let l be the smallest integer less than or equal to $n - 1$ such that $S[1, \ldots, l \mid \mu - n + 3, \ldots, \mu - n + l + 2]$ is singular. By the special pattern of S we have

$$S[1, \ldots, l \mid \mu - n + 3, \ldots, \mu - n + l + 2] = S[2, \ldots, l + 1 \mid \mu - n + 2, \ldots, \mu - n + l + 1],$$

where the latter submatrix is a submatrix in A_μ. Since by the induction hypothesis A_μ is *TN* we conclude by [18, Proposition 1.15] that either $S[2, \ldots, l + 1 \mid 1, \ldots, l]$ or $S[1, \ldots, l \mid \mu - n + 2, \ldots, \mu - n + l + 1]$ is singular. In either case we have a contradiction since A is *TP* and by the induction hypothesis $S[1, \ldots, n-1 \mid \mu - n + 2, \ldots, \mu]$ is *NsTN*. If $\det S[1, \ldots, n \mid 2, \ldots, n + 1]$ is positive then by proceeding as above we show that $S[1, \ldots, n \mid \mu - n + 2, \ldots, \mu + 1]$ is nonsingular for each $\mu = n, n + 1, \ldots$ provided that A_μ is *TN*. It remains to prove that $A_{\mu+1}$ is *TN*. We distinguish the following two cases:

Case 1: $\det S[1, \ldots, n \mid 2, \ldots, n + 1]$ is zero.

We first show that the $(n+1)$th row can be written as a linear combination of the 2nd, 3rd, $\ldots$, and nth rows. By [7, Theorem 7, p. 205] the $(n+1)$th row can be written as a linear combination of the 1st, 2nd, $\ldots$, and nth rows, i.e., if $\mathbf{R}^1, \mathbf{R}^2, \ldots, \mathbf{R}^{n+1}$ represent the first $n+1$ rows of S, then there exist $r_1, r_2, \ldots, r_n \in \mathbb{R}$ such that

$$\mathbf{R}^{n+1} = \sum_{i=1}^{n} r_i \mathbf{R}^i. \tag{4}$$

By (4), symmetry of S, and determinantal properties we obtain

$$\begin{aligned}\det S[1,\ldots,n \mid 2,\ldots,n+1] &= \det S[2,\ldots,n+1 \mid 1,\ldots,n]\\ &= (-1)^{n-1} r_1 \det S[1,\ldots,n]\end{aligned}$$

and consequently $r_1 = 0$ since $0 < \det S[1,\ldots,n]$ and $\det S[1,\ldots,n \mid 2,\ldots,n+1] = 0$. Therefore by (4), $\mathbf{R}^{n+1}$ can be written as a linear combination of $\mathbf{R}^2, \ldots, \mathbf{R}^n$. Let C be the square Cauchon diagram of order $\mu+1$ that is defined by $(i,j) \in C$ if and only if $(i,j) \in \{1,\ldots,\mu-n+1\}^2$ or $i = \mu-n+2$, $j = 1,\ldots,\mu-n+1$ or $i = 1,\ldots,\mu-n+1$, $j = \mu-n+2$ and let E be the matrix that is obtained from $A_{\mu+1}$ by reversing the order of its rows and columns, i.e., $A_{\mu+1}$ is read from bottom right instead of the top left. We want to show that E is TN and associated to the Cauchon diagram C. For each position $(i_0,j_0) \in \{1,\ldots,\mu+1\}^2$ fix a lacunary sequence given by (1) with respect to C as follows: If $\mu-n+2 \le i_0$ or $\mu-n+2 \le j_0$, then $i_{k+1} := i_k + 1$ and $j_{k+1} := j_k + 1$ for $k = 0,1,\ldots,p-1$. If $i_0 \le \mu-n+1$ and $j_0 \le \mu-n+1$, then set $(i_1,j_1) := (\mu-n+2, \mu-n+2)$ and $i_{k+1} := i_k+1$, $j_{k+1} := j_k+1$ for $k = 1,\ldots,p-1$. Then it is easy to see that all these sequences are lacunary with respect to C. The minors that are associated with the lacunary sequences that start from (i_0,j_0) such that $(i_0,j_0) \notin C$ are positive since A is TP and A_μ is TN and hence $S[1,\ldots,n-1 \mid v-n+2,\ldots,v]$ is $NsTN$ for each $v = n, n+1,\ldots,\mu+1$. The minors associated with the lacunary sequences that start from the positions (i_0,j_0) such that $i = \mu-n+2$ and $j = 1,\ldots,\mu-n+1$ or $i = 1,\ldots,\mu-n+1$ and $j = \mu-n+2$ are zero since the $(n+1)$th row (column) can be written as a linear combination of the 2nd, 3rd, …, nth rows (columns) of S. The minors associated with the lacunary sequences that start from the positions (i_0,j_0) such that $(i_0,j_0) \in \{1,\ldots,\mu-n+1\}^2$ are zero since such minors have order $n+1$ and consequently are zero because the rank of S is n. Thus by [17, Theorem 4.4] E and hence $A_{\mu+1}$ are TN.

Case 2: $\det S[1,\ldots,n \mid 2,\ldots,n+1]$ is positive.

The proof is analogous to Case 1 but with a different Cauchon diagram C. In this case we let C be the square Cauchon diagram of order $\mu+1$ that is defined by $(i,j) \in C$ if and only if $(i,j) \in \{1,\ldots,\mu-n+1\}^2$.

$(iii) \Rightarrow (i)$ Since S is TN and has rank n we may conclude by [14, Theorem 1.2] that A is $NsTN$ and because A is symmetric it is positive definite by Lemma 2.1. The entry $B[1|1] = s_1$ must be positive because otherwise it would follow by $0 < s_2$ that $\det S[1,2|2,3] < 0$ contradicting the fact that S is TN. So we may assume that $3 \le n$. Suppose that B is singular and let r be the smallest integer such that $\det B[1,\ldots,r] = 0$. Then it follows that $2 \le r \le n-1$. By application of [18, Proposition 1.15] to $C := S[1,\ldots,n+1]$ at least one of the following holds. Either the rows $1,\ldots,r$ or the columns $2,\ldots,r+1$ of C are linearly dependent or one of the matrices $C[1,\ldots,n+1|1,\ldots,r+1]$ and $C[1,\ldots,r|2,\ldots,n+1]$ has rank $r-1$. Since A is nonsingular only the last case could be possible. However, this matrix possesses $C[1,\ldots,r|3,\ldots,r+2]$ as a submatrix which is identical with the principal submatrix $A[2,\ldots,r+1]$. Since its determinant is positive we have arrived at a contradiction. Hence B is nonsingular and because it is furthermore symmetric and TN it is positive definite. □

Let p and q be polynomials with real coefficients

$$p(z) := a_0 z^n + a_1 z^{n-1} + \ldots + a_n, \quad a_0 \neq 0, \tag{5}$$

$$q(z) := b_0 z^n + b_1 z^{n-1} + \ldots + b_n. \tag{6}$$

Define the (in)finite matrices of Hurwitz type as follows depending on the case whether b_0 vanishes or not. We set $a_k := 0$ and $b_k := 0$ for $k > n$.

If $\deg q < \deg p$, that is, if $b_0 = 0$, then:

$$H(p,q) := \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & \dots \\ 0 & b_1 & b_2 & b_3 & b_4 & b_5 & \dots \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & \dots \\ 0 & 0 & b_1 & b_2 & b_3 & b_4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{7}$$

$$H_{2n}(p,q) := H[2,\dots,2n+1]; \tag{8}$$

if $\deg q = \deg p$, that is, $b_0 \neq 0$, then

$$H(p,q) := \begin{bmatrix} b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & \dots \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & \dots \\ 0 & b_0 & b_1 & b_2 & b_3 & b_4 & \dots \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{9}$$

$$H_{2n}(p,q) := H[2,\dots,2n+2]. \tag{10}$$

The matrices $H(p,q)$ and $H_{2n}(p,q)$ are called *infinite* and *finite matrices of Hurwitz type*, respectively.

Theorem 3.2. *Let $H(p,q)$ and $H_{2n}(p,q)$ be given as in (7)–(10), respectively. Then the following three statements are equivalent:*

(i) *The matrix $H_{2n}(p,q)$ is NsTN.*
(ii) *The matrix $H_{2n}(p,q)$ is ATP and $0 < a_i$, b_i, $i = 1,\dots,n$.*
(iii) *The matrix $H(p,q)$ is ATP and $0 < a_i$, b_i, $i = 1,\dots,n$.*

Proof. $(i) \Rightarrow (ii)$ By Theorem 2.7, $\tilde{H}_{2n}(p,q)$ is a nonnegative Cauchon matrix with positive diagonal entries. The entries $a_1,\dots,a_n$, $b_1,\dots,b_n$ appear on the main diagonal of $H_{2n}(p,q)$ and are therefore positive by Lemma 2.1. If $b_0 \neq 0$ then the assumption $H_{2n}(p,q)$ is *NsTN* shows that b_0 cannot be negative.

In order to show that the matrix $H_{2n}(p,q)$ is *ATP* it is sufficient by Theorem 2.14 to show that the zero–nonzero pattern of $H_{2n}(p,q) = (h_{ij})$ and $\tilde{H}_{2n}(p,q)$ coincide, i.e., the minors associated with lacunary sequences that are constructed by Procedure 2.12 and start at positions (i,j) such that $h_{ij} = 0$ are zero while those that start at positions (i,j) such that $h_{ij} \neq 0$ are positive.

By application of Procedure 2.12 we find for each entry of $\tilde{H}_{2n}(p,q)$ a lacunary sequence starting from the position of this entry. Any minor associated with a lacunary sequence that starts from a position (i,j) with $h_{ij} = 0$ vanishes since this minor contains a zero row or column. Any minor associated with a lacunary sequence that starts from a position (i,i) or (i,j) such that $j < i$ and $h_{ij} \neq 0$ is a principal minor of $H_{2n}(p,q)$ possibly multiplied by a_0 (b_0). While the minors that are associated with the lacunary sequences that start from positions (i,j) such that $i < j$ and $h_{ij} \neq 0$ are principal minors of $H_{2n}(p,q)$ possibly multiplied by some integer power of a_n. Therefore all minors associated with the lacunary sequences that start from positions (i,j) with $h_{ij} \neq 0$ are positive by Lemma 2.1. Hence $H_{2n}(p,q)$ and $\tilde{H}_{2n}(p,q)$ have the same zero–nonzero pattern.

$(ii) \Rightarrow (iii)$ Without loss of generality we consider only the case $b_0 = 0$. Let $A_\nu := H(p,q)[1,\dots,\nu]$, $\nu \geq 2n+1$. Then A_ν tends to $H(p,q)$ as ν tends to infinity and any submatrix of $H(p,q)$ appears as a submatrix of a suitably chosen A_{ν_0}. We want to show that A_ν is *ATP* for each $\nu \geq 2n+1$. The contiguous minors of A_{2n+1} coincide with minors of $H_{2n}(p,q)$ possibly multiplied by $a_0 > 0$ or vanish. Hence A_{2n+1} is *ATP* since $H_{2n}(p,q)$ is *ATP* by assumption. Suppose now that A_ν is *ATP*. Any contiguous minor of $A_{\nu+1}$

appears as a minor of A_ν possibly multiplied by a_n or vanishes. Hence $A_{\nu+1}$ is ATP and by induction we obtain that $H(p,q)$ is ATP.

$(iii) \Rightarrow (i)$ Since $H_{2n}(p,q)$ is a principal submatrix of $H(p,q)$ with positive diagonal entries the result follows. □

Theorem 3.3. *The matrix $H_{2n}(p,q)$ is ATP and $0 < a_i,\ b_i,\ i = 1,\dots,n$, if and only if all its leading principal minors are positive.*

Proof. If $H_{2n}(p,q)$ is ATP and $0 < a_i,\ b_i,\ i = 1,\dots,n$, then all its principal minors are positive by Theorem 3.2. The converse is a special case of [13, Theorem 2.1], see also [18, Theorem 4.6]. □

We note that the more general result in [13], can also be proven by using the Cauchon algorithm. However, since our proof is not considerably shorter than the proofs in [13] and [18] we will not give it here.

By identification of the coefficients b_i with the odd indexed and the coefficients a_i with the even indexed coefficients of a polynomial, for simplicity p given by (5) say, we obtain from (8) the *Hurwitz matrix $H(p) = (h_{ij})$ associated with p* defined by

$$h_{ij} := \begin{cases} a_{2j-i} & \text{for } 0 \le 2j - i \le n, \\ 0 & \text{otherwise,} \end{cases}$$

such that

$$H(p) = \begin{bmatrix} a_1 & a_3 & a_5 & a_7 & \dots & 0 & 0 \\ a_0 & a_2 & a_4 & a_6 & \dots & 0 & 0 \\ 0 & a_1 & a_3 & a_5 & \dots & 0 & 0 \\ 0 & a_0 & a_2 & a_4 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & a_{n-1} & 0 \\ 0 & 0 & 0 & 0 & \dots & a_{n-2} & a_n \end{bmatrix}. \tag{11}$$

Definition 3.4. A polynomial is called *(Hurwitz) stable* if all its zeros are inside the open left half of the complex plane.

By Hurwitz's Theorem, see, e.g., [18, p. 117], p is stable if all leading principal minors of $H(p)$ and a_0 are positive. The following theorem provides a converse statement.

Theorem 3.5. *[15, Theorem 2] Let p be given as in (5) with $0 < a_0$. If p is stable, then $H(p)$ is $NsATP$.*

Proof. We proceed by induction on n. The statement is obviously true for $n = 2$. Suppose that f given by (5) is stable. Consider the polynomial q of degree $n - 1$, defined by

$$q(z) := a_1 z^{n-1} + (a_2 - a_3 \frac{a_0}{a_1}) z^{n-2} + a_3 z^{n-3} + (a_4 - a_5 \frac{a_0}{a_1}) z^{n-4} + \dots .$$

Then q is the polynomial which corresponds to the second and third rows of the Routh scheme associated to p, see, e.g., [7, Chapter XV, §3], since p is stable it follows by the Routh's Criterion, see [7, p. 180], that q is stable, too, and $0 < a_1$. Hence by the induction hypothesis $H(q)$ is $NsATP$. By Theorem 2.7 $\tilde{H}(q)$ is a nonnegative Cauchon matrix with positive diagonal entries and by Theorem 2.14 $H(q)$ and $\tilde{H}(q)$ have the same zero–nonzero pattern. Let $H_1(p)$ be the matrix that is obtained from $H(p)$ by subtracting from each even indexed row the preceding row multiplied by $\frac{a_0}{a_1}$ and storing the resulting row in this even indexed row. It is easy to see that $H(q) = H_1(p)[2,\dots,n]$ and therefore, $H_1(p)[2,\dots,n]$ and $\tilde{H}_1(p)[2,\dots,n]$ have the same zero–nonzero pattern; note that $\tilde{H}_1(p)[2,\dots,n]$ coincides with the matrix which is obtained by application of the Cauchon algorithm to $H_1(p)[2,\dots,n]$. Since $0 < a_1$ it follows that $H_1(p)$ and hence $H(p)$ are nonsingular.

The minors of $H_1(p)$ that are associated with the sequences which are constructed by Procedure 2.12 and are starting at the positions $(1,l)$, $l = 2,\dots,\lceil\frac{n}{2}\rceil$, are equal to the minors that are associated with the lacunary sequences that are starting from the positions $(3,l+1)$ multiplied by a_n. The minor that is associated with the sequence that starts at the position $(1,1)$ is equal to $\det H_1(p)$ and hence positive. The minors that are associated with the sequences that are starting from the other positions of the first row and column are zero since the corresponding submatrices have a zero row or column. Hence $\tilde{H}_1(p)$ is a nonnegative Cauchon matrix with positive diagonal entries. Moreover, $H_1(p)$ and $\tilde{H}_1(p)$ have the same zero–nonzero pattern. Thus by Theorem 2.14 $H_1(p)$ is $NsATP$.

We access the entries of $H(p)$ through the entries of $H_1(p)$; by adding to each even indexed row in $H_1(p)$ the preceding row multiplied by $\frac{a_0}{a_1}$. Hence by determinantal properties we obtain that $H(p)$ is $NsTN$. By application of Procedure 2.12 to $H(p)$ we find for each position (k,l), $k,l = 1,\dots,n$, a lacunary sequence starting from there. By the special pattern of the entries of $H(p) = (h_{kl})$ the minors associated with the lacunary sequences which start from positions (k,l) such that $h_{kl} \neq 0$ are principal minors in $H(p)$, or principal minors multiplied by some integral power of a_n, or principal minors multiplied by a_0. Hence by Lemma 2.1 they are positive. The minors that are associated with the lacunary sequences which start from positions (k,l) such that $h_{kl} = 0$ are zero since the corresponding submatrices contain a zero row or column. Hence $H(p)$ and $\tilde{H}(p)$ have the same zero–nonzero pattern. Whence it follows by Theorem 2.14 that $H(p)$ is $NsATP$. $\square$

We now turn to the case when the coefficients of a polynomial are not exactly known (due to, e.g., data uncertainties) but can be bounded,

$$a_k \in [\underline{a}_k, \overline{a}_k], \quad k = 0,1,\dots,n. \tag{12}$$

We want to know whether all polynomials p given by (5) satisfying (12) are stable. Intervals of stable polynomials are investigated in [16], see also [4], [5]; also the fact that the Hadamard product of two stable polynomials is stable is proved in [8].

Let $H_{\mathcal{I}}$ be the following $(n+1)\times n$ matrix whose entries are composed from the endpoints of the intervals given by (12) ($\underline{a}_k := 0$ and $\overline{a}_k := 0$ for $k > n$)

$$H_{\mathcal{I}} := \begin{bmatrix} \underline{a}_1 & \overline{a}_3 & \underline{a}_5 & \overline{a}_7 & \underline{a}_9 & \overline{a}_{11} & \dots \\ \overline{a}_0 & \underline{a}_2 & \overline{a}_4 & \underline{a}_6 & \overline{a}_8 & \underline{a}_{10} & \dots \\ 0 & \overline{a}_1 & \underline{a}_3 & \overline{a}_5 & \underline{a}_7 & \overline{a}_9 & \dots \\ 0 & \underline{a}_0 & \overline{a}_2 & \underline{a}_4 & \overline{a}_6 & \underline{a}_8 & \dots \\ 0 & 0 & \underline{a}_1 & \overline{a}_3 & \underline{a}_5 & \overline{a}_7 & \dots \\ 0 & 0 & \overline{a}_0 & \underline{a}_2 & \overline{a}_4 & \underline{a}_6 & \dots \\ 0 & 0 & 0 & \overline{a}_1 & \underline{a}_3 & \overline{a}_5 & \dots \\ 0 & 0 & 0 & \underline{a}_0 & \overline{a}_2 & \underline{a}_4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}. \tag{13}$$

Let $H_1 := H_{\mathcal{I}}[1,\dots,n-1]$ and $H_2 := H_{\mathcal{I}}[3,\dots,n+1|2,\dots,n]$. Assume that $H_{\mathcal{I}}$ is an ATP and $\underline{a}_k > 0$, $k = 0,1,\dots,n$. Then H_1 and H_2 also are ATP since they are submatrices of $H_{\mathcal{I}}$. Moreover, the entries on the main diagonal of H_1 and H_2 are positive. Hence H_1 and H_2 are $NsTN$. Furthermore, we see that $H_1 \leq^* H_2$.

Theorem 3.6. *If H_1 and H_2 are NsTN, then all polynomials p given by (5) satisfying (12) are stable.*

Proof. Let p be any polynomial satisfying (12) and $H(p)$ be its Hurwitz matrix. Then it is easy to see that

$$H_1 \leq^* H(p)[1, \ldots, n-1] \leq^* H_2. \tag{14}$$

By Theorem 2.3 $H(p)[1, \ldots, n-1]$ is $NsTN$, hence p is stable by Hurwitz's Theorem. □

In contrast to Kharitonov's Theorem [16], see also [5, Theorem 5.1], which involves four polynomials whose coefficients are endpoints of the intervals (12), Theorem 3.6 also holds if $\underline{a}_0 = 0$ (but $\underline{a}_1 > 0$), i.e., this theorem holds if the interval polynomial contains polynomials of degree $n-1$. However, our condition is not necessary. A counterexample is provided by the following example.

Example 3.7. Let the family of polynomials be given by

$$\underline{a}_0 = \overline{a}_0 = 1,\ \underline{a}_1 = 3.5,\ \underline{a}_2 = \underline{a}_3 = \overline{a}_1 = 6.5,\ \overline{a}_2 = \overline{a}_3 = 9.5,\ \underline{a}_4 = 1.5, \overline{a}_4 = 4.5.$$

Then all polynomials satisfying (12) are stable, see [3]. However, $\det H_1 < 0$.

The complexity of the stability test based on Theorem 3.6 is $O(n^2)$ for an nth degree polynomial; the test based on checking $H_{\mathcal{I}}$ for being ATP requires slightly less operations.

4. Application to rational functions

In this section we use matrices considered in the preceding section for the study of *interval* problems involving rational functions. Let p and q be polynomials given by (5) and (6), respectively, and let the rational function R be defined by

$$R := q/p. \tag{15}$$

Expand R into its Laurent series at ∞

$$R(z) = s_{-1} + \frac{s_0}{z} + \frac{s_1}{z^2} + \frac{s_2}{z^3} + \ldots . \tag{16}$$

The infinite Hankel matrix (3) formed from the coefficients s_i of this series is denoted by $S(R)$.

Definition 4.1. A rational function R is called an R-function of negative type (respectively, positive type) if it maps the open upper half-plane of the complex plane into the open lower half-plane (respectively, to itself).

R-functions are also called Nevanlinna, Herglotz, and Pick-functions. We will present our results only for R-functions of negative type since the corresponding results for R-functions of positive type can be obtained by replacing the function R by $-R$.

The following theorem plays a fundamental role in our results below.

Theorem 4.2. [14, Theorem 3.18] *Let R be expanded as in (16). Then the following two statements are equivalent:*

(i) *R is an R-function of negative type and has exactly r poles all of which are positive.*
(ii) *The matrix $S(R)$ is TP_r of rank r.*

The above theorem is stated in [14] for $s_{-1} = 0$ but it is also obviously true without this assumption.

Theorem 4.3. *Let R_1, R_2, and R_3 be rational functions with series (16) involving coefficients s_i, t_i, and d_i, $i = -1, 0, 1, \dots$, respectively. Assume that the coefficients satisfy for $i \geq 0$ the following inequalities:*

$$(-1)^i s_i \leq (-1)^i d_i \leq (-1)^i t_i, \quad i = 0, 1, \dots \; . \tag{17}$$

If R_1 and R_2 are R-functions of negative type and both functions have exactly r poles all of which are positive, then R_3 is an R-function of negative type and has exactly r poles all of which are positive.

Proof. By Theorem 4.2 $S(R_1)$ and $S(R_2)$ are TP_r of rank r. Let $S(R_3)[\alpha \mid \beta]$ be any contiguous submatrix of $S(R_3)$ such that $\mid \alpha \mid = \mid \beta \mid = r$. Then as a consequence of (17) the inequalities

$$S(R_1)[\alpha \mid \beta] \leq^* S(R_3)[\alpha \mid \beta] \leq^* S(R_2)[\alpha \mid \beta]$$

or

$$S(R_2)[\alpha \mid \beta] \leq^* S(R_3)[\alpha \mid \beta] \leq^* S(R_1)[\alpha \mid \beta]$$

hold. Hence by Corollary 2.4, $S(R_3)[\alpha \mid \beta]$ is TP, too, and by [6, Corollary 3.1.6] $S(R_3)$ is TP_r. It remains to show that the rank of $S(R_3)$ equals r. By adding a suitable positive number ϵ to s_{2r}, t_{2r}, d_{2r} we can accomplish that the modified leading principal submatrices of $S(R_1)$ and $S(R_2)$ of order $r+1$ become TP. Thus also the modified leading principal submatrix of $S(R_3)$ of order $r+1$ is TP. By using Proposition 2.2 and letting ϵ tend to zero we obtain that $\det S(R_3)[1, \dots, r+1] = 0$ since $\det S(R_1)[1, \dots, r+1] = \det S(R_2)[1, \dots, r+1] = 0$. Repeating this process infinitely many times with each $S(R_3)[1, \dots, r+1 \mid \nu, \dots, \nu+r]$, $\nu = 2, 3, \dots$, we arrive at $\det S(R_3)[1, \dots, r+1 \mid \nu, \dots, \nu+r] = 0$ for all $\nu = 1, 2, \dots$. Hence for each $\nu = 1, 2, \dots$ there exist $c_1^\nu, c_2^\nu, \dots, c_r^\nu \in \mathbb{R}$ such that

$$S(R_3)[r+1 \mid \nu, \dots, \nu+r] = [c_1^\nu \; c_2^\nu \; \dots \; c_r^\nu] S(R_3)[1, \dots, r \mid \nu, \dots, \nu+r] \tag{18}$$

since $S(R_3)[1, \dots, r+1 \mid \nu, \dots, \nu+r]$ has rank r and $\det S(R_3)[1, \dots, r \mid \nu, \dots, \nu+r-1]$ is positive. Therefore by (18)

$$\begin{aligned} S(R_3)[r+1 \mid 2, \dots, r+1] &= [c_1^1 \; c_2^1 \; \dots \; c_r^1] S(R_3)[1, \dots, r \mid 2, \dots, r+1] \\ &= [c_1^2 \; c_2^2 \; \dots \; c_r^2] S(R_3)[1, \dots, r \mid 2, \dots, r+1], \end{aligned}$$

whence

$$([c_1^1 \; c_2^1 \; \dots \; c_r^1] - [c_1^2 \; c_2^2 \; \dots \; c_r^2]) S(R_3)[1, \dots, r \mid 2, \dots, r+1] = 0. \tag{19}$$

Since $S(R_3)[1, \dots, r \mid 2, \dots, r+1]$ is nonsingular we conclude by (19)

$$[c_1^1 \; c_2^1 \; \dots \; c_r^1] = [c_1^2 \; c_2^2 \; \dots \; c_r^2]. \tag{20}$$

Repeating the above steps we obtain $c_j^1 = c_j^\nu$ for each $j = 1, \dots, r$, $\nu = 1, 2, \dots$. Hence the row $r+1$ of $S(R_3)$ can be written as a linear combination of the previous rows and $S(R_3)[1, \dots, r]$ is nonsingular. Thus by [7, Theorem 7, p. 205] $S(R_3)$ has rank r. □

In passing we note that by using similar arguments as in the proof of Theorem 4.3 Markov's Theorem, see, e.g., [7, Theorem 21, p. 242], can be easily proven.

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Theorem 4.4. *Let R_1 and R_2 be as in Theorem 4.3 with coefficients satisfying (17). If R_1 and R_2 are R-functions of negative type and all poles of them are positive, then R_1 and R_2 have the same number of poles.*

Proof. Suppose on the contrary that R_1 has exactly r positive poles and R_2 has exactly k, $k > r$, positive poles. Then by Theorem 4.2, $S(R_1)$ and $S(R_2)$ are TP_r of rank r and TP_k of rank k, respectively. By using Proposition 2.2 we arrive at the contradiction

$$0 < \det S(R_2)[1,\dots,r+1|2,\dots,r+2] \le \det S(R_1)[1,\dots,r+1|2,\dots,r+2] = 0.$$

If R_1 has exactly k, $k > r$ positive poles, and R_2 possesses exactly r positive poles, then we arrive with Proposition 2.2 at the contradiction

$$0 < \det S(R_1)[1,\dots,r+1] \le \det S(R_2)[1,\dots,r+1] = 0.$$

Hence R_1 and R_2 have the same number of positive poles. □

The following two lemmata and theorem provide a further interval property of the rational functions.

Definition 4.5. We call the rational function $R = q/p$ an R^*-*function* if R is an R-function of negative type with only negative zeros and q and p are coprime.

In order to avoid to distinguish the cases when the polynomial degree is even or odd we number the coefficients of a polynomial now in such a way that the coefficient indexed by 0 is the constant term. We affix superscripts to polynomial coefficients for reference to a specific polynomial. Without loss of generality we assume that the leading coefficients of all polynomials appearing in the assumptions of the following statements are positive.

Lemma 4.6. *Let $R_1 = q_1/p$ and $R_2 = q_2/p$ be two R^*-functions, where $\deg q_1 = \deg q_2 = n$ and $\deg p \in \{n-1, n, n+1\}$. Then $R = g_1/p$ is an R^*-function provided that the coefficients of q_1, q_2, g_1 satisfy the following inequalities*

$$(-1)^k a_k^{q_1} \le (-1)^k a_k^{g_1} \le (-1)^k a_k^{q_2}, \qquad k = 0, 1, \dots, n. \tag{21}$$

Proof. From the assumption (21) it follows that

$$q_1(x) \le g_1(x) \le q_2(x) \quad \text{for all} \quad x \le 0.$$

Since q_1 and q_2 have only negative zeros, g_1 and therefore R have only negative zeros, too. By a modification of the Hermite–Biehler Theorem, see, e.g., [14, Theorem 3.4] it is sufficient to show that the zeros of g_1 and p are real, simple, and interlacing, i.e., between any two consecutive zeros of one of the polynomials there is exactly one zero of the other polynomial, and there exists a real number v such that

$$p(v)g_1'(v) - p'(v)g_1(v) < 0. \tag{22}$$

By [14, Theorem 3.4] the zeros of each pair (q_1, p) and (q_2, p) are real, simple, interlacing, and for each real number w and $i = 1, 2$, we have

$$p(w)q_i'(w) - p'(w)q_i(w) < 0. \tag{23}$$

As in the proof of [5, Lemma 5.1], we obtain that the zeros of g_1 and p are real, simple, and interlacing. By setting $w = 0$ in (23) and using (21), we get

$$a_0^p a_1^{g_1} - a_1^p a_0^{g_1} \le a_0^p a_1^{q_1} - a_1^p a_0^{q_1} < 0 \tag{24}$$

Hence (22) is fulfilled. It remains to show that g_1 and p are coprime. Suppose on the contrary that both polynomials have a zero, x_0 say, in common. Then it follows by the interlacing property that $q_1(x_0) < 0 < q_2(x_0)$. We first consider the case when x_0 is greater than the largest zero of q_2. Let x_0' be the next (smaller) zero of p. Then $0 < q_1(x_0') \le q_2(x_0')$, a contradiction because q_2 cannot have a simple zero between x_0' and x_0. If x_0 is smaller than the largest zero of q_2 we distinguish two cases. If p has a zero x_0'', $x_0 < x_0''$, then we arrive likewise by $0 < q_1(x_0'') \le q_2(x_0'')$ at a contradiction. Otherwise we use the fact that $0 < q_2(0)$ to obtain a contradiction. This completes the proof that R is an R^*-function. □

By a similar proof we obtain the dual of Lemma 4.6.

Lemma 4.7. *Let $R_1 = q/p_1$ and $R_2 = q/p_2$ be two R^*-functions, where $\deg q = n$ and $\deg p_1 = \deg p_2 \in \{n-1, n, n+1\}$. Then $R = q/g_2$ is an R^*-function provided that the coefficients of p_1, p_2, g_2 satisfy the following inequalities*

$$(-1)^k a_k^{p_1} \le (-1)^k a_k^{g_2} \le (-1)^k a_k^{p_2}, \qquad k = 0, 1, \ldots, n. \tag{25}$$

By application of the two lemmata, we derive the following theorem.

Theorem 4.8. *Let $R_{11} = q_1/p_1$, $R_{12} = q_1/p_2$, $R_{21} = q_2/p_1$, and $R_{22} = q_2/p_2$ be R^*-functions, where $\deg q_1 = \deg q_2 = n$ and $\deg p_1 = \deg p_2 \in \{n-1, n, n+1\}$. Then $R = g_1/g_2$ is an R^*-function provided that the coefficients of q_i, g_1 and p_i, g_2, $i = 1, 2$, satisfy the inequalities (21) and (25), respectively.*

Proof. Since R_{11} and R_{21} as well as R_{12} and R_{22} are R^*-functions we conclude from Lemma 4.6 that $R_{11}^g = g_1/p_1$ and $R_{12}^g = g_1/p_2$, respectively, are R^*-functions. By application of Lemma 4.7 to R_{11}^g and R_{12}^g we obtain that R is an R^*-function. □

In passing we note that with a matrix of type (13) and two submatrices like H_1 and H_2 we easily obtain using [14, Theorem 3.47] a theorem like Theorem 3.6 on an interval family of R-functions (15) of negative type with exactly n negative poles.

We conclude the paper by relating the coefficients of the representation of a rational function R in form of a Stieltjes continued fraction, see, e.g., [14, Section 1.5], to the entries of the matrix which is obtained by the application of the Cauchon algorithm to finite sections of the infinite Hankel matrix $S(R)$ associated with R.

By [14, formulae (1.113)–(1.114), p. 453] the minors

$$D_i(R) := \det S[1, \ldots, i], \tag{26}$$

$$\hat{D}_i(R) := \det S[1, \ldots, i \mid 2, \ldots, i+1], \quad i = 1, 2, \ldots, r \tag{27}$$

with $D_0(R) := \hat{D}_0(R) := 1$, are connected with the coefficients of the Stieltjes continued fraction expansion through the relations

$$c_{2j} = \frac{-D_j^2(R)}{\hat{D}_{j-1}(R) \cdot \hat{D}_j(R)} \quad c_{2j-1} = \frac{\hat{D}_{j-1}^2(R)}{D_{j-1}(R) \cdot D_j(R)}, \quad j = 1, \ldots, r, \tag{28}$$

where r is the number of poles of the function R.

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In Theorem 3.1 we suppose that S has rank n. A theorem due to Kronecker, see, e.g., [7, p. 207], [14, Theorem 1.3], implies that the rational function

$$R(z) = \frac{s_0}{z} + \frac{s_1}{z^2} + \dots$$

has n poles. Let E be the matrix that is obtained from $S[1, \dots, n+1]$ by reversing the order of its rows and columns, i.e., $S[1, \dots, n+1]$ is read from the bottom right instead of the top left. Then by application of the Cauchon algorithm to E we obtain $\tilde{E}$ which is a nonnegative Cauchon matrix. By Proposition 2.10

$$\begin{aligned} D_i(R) &= \det E[n+2-i, \dots, n+1] \\ &= \tilde{e}_{n+2-i,n+2-i} \cdot \tilde{e}_{n+2-i+1,n+2-i+1} \cdots \tilde{e}_{n+1,n+1}, \\ \hat{D}_i(R) &= \det E[n+2-i, \dots, n+1 \mid n+1-i, \dots, n] \\ &= \tilde{e}_{n+2-i,n+1-i} \cdot \tilde{e}_{n+2-i+1,n+1-i+1} \cdots \tilde{e}_{n+1,n}, \end{aligned}$$

for $i = 1, \dots, n$. Plugging the last expressions into (28) we obtain the relations (29), (30) in the following theorem. It shows that the coefficients c_j of the Stieltjes continued fraction expansion can be recovered from the application of the Cauchon algorithm, i.e., the coefficients c_j can be determined without the need to compute the underlying Hankel determinants $D_i(R)$ and $\hat{D}_i(R)$ defined in (26) and (27), respectively. On the other hand, the Hankel determinants can be computed from the entries of the matrix $\tilde{E}$.

Theorem 4.9. *Let $S(R)$ be a real infinite Hankel matrix associated to the rational function R, let $S(R)$ satisfy all the assumptions of Theorem 3.1 posed on S, and let E be defined as above. If any of the conditions (i)–(iii) of Theorem 3.1 holds for $S(R)$, then the coefficients of the Stieltjes continued fraction expansion corresponding to the function R can be calculated as follows:*

$$c_{2j} = \frac{-\prod_{i=1}^{j} \tilde{e}^2_{n+2-i,n+2-i}}{\tilde{e}_{n+2-j,n+1-j} \prod_{i=1}^{j-1} \tilde{e}^2_{n+2-i,n+1-i}}, \tag{29}$$

$$c_{2j-1} = \frac{\prod_{i=1}^{j-1} \tilde{e}^2_{n+2-i,n+1-i}}{\tilde{e}_{n+2-j,n+2-j} \prod_{i=1}^{j-1} \tilde{e}^2_{n+2-i,n+2-i}}, \tag{30}$$

for $j = 1, \dots, n$.

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