Invariance of Total Positivity of a Matrix under Entry-wise Perturbation and Completion Problems

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ABSTRACT. A totally positive matrix is a matrix having all its minors positive. The largest amount by which the single entries of such a matrix can be perturbed without losing the property of total positivity is given. Also some completion problems for totally positive matrices are investigated.

1. Introduction

In this paper we consider matrices which are totally positive, i.e., all their minors are positive. For the properties of these matrices the reader is referred to the two recent monographs [5] and [11]. In the first part of our paper we are interested in the largest amount by which the single entries of such a matrix can be varied without losing the property of total positivity. This question is answered for a few specified entries in [6], see also [5, Section 9.5]. Similarly, one may ask how much the entries of a totally nonnegative matrix, i.e., a matrix having all its minors nonnegative, can be perturbed without losing the property of total nonnegativity. This problem is solved in [2] for tridiagonal totally nonnegative matrices. A related question is whether all matrices lying between two totally nonnegative matrice with respect to a suitable partial ordering are totally nonnegative, too. The second author conjectured in 1982 [8] that this is true for the nonsingular totally nonnegative matrices and the so-called checkerboard ordering, see also [5, Section 3.2] and [11, Section 3.2]. In [1] we apply the so-called Cauchon algorithm, see, e.g., [3], to settle this conjecture.

In the second part of our paper we solve some completion problems for totally positive matrices. Here we consider the case that in a matrix some entries are specified, while the remaining ones are unspecified and are free to be chosen. All minors contained in the specified part are supposed to be positive. Then the question arises whether values for the unspecified entries can be chosen such that the resulting matrix is totally positive. Solutions of such completion problems can be found in [5, Subsections 9.1-9.4], and [10]. The starting point of our work is the recent paper [9].

The organization of our paper is as follows. In the next section we explain our notation and we collect some auxiliary results in Section 3. In Section 4 we present our main results on the perturbation of entries of totally positive matrices and we show that the derived set of determinantal conditions is minimal. In Section 5 we solve a completion problem for some new patterns of unspecified entries, hereby partially settling two conjectures in [**9**].

2. Notation

We now introduce the notation used in our paper. For κ, n we denote by $Q_{\kappa,n}$ the set of all strictly increasing sequences of κ integers chosen from $\{1, 2, \ldots, n\}$. For $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_\kappa\} \in Q_{\kappa,n}$ the *dispersion* of α is $d(\alpha) = \alpha_\kappa - \alpha_1 - \kappa + 1$. If $d(\alpha) = 0$ then the index set α is called *contiguous*. Let A be a real $n \times m$ matrix. For $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_\kappa\} \in Q_{\kappa,n}, \beta = \{\beta_1, \beta_2, \ldots, \beta_\mu\} \in Q_{\mu,m}$, we denote by $A[\alpha|\beta]$ the $\kappa \times \mu$ submatrix of A contained in the rows indexed by $\alpha_1, \alpha_2, \ldots, \alpha_\kappa$ and columns indexed by $\beta_1, \beta_2, \ldots, \beta_\mu$. We suppress the braces when we enumerate the indices explicitly. In the case that α or β is obtained by taking the union of two index sets we assume that the resulting index set is ordered increasingly. When $\alpha = \beta$, the principal submatrix $A[\alpha|\alpha]$ is abbreviated to $A[\alpha]$. In the special case where $\alpha = \{1, 2, \ldots, \kappa\}$, we refer to the principal submatrix $A[\alpha]$ as a *leading principal submatrix* (and to $\det A[\alpha]$ as a *leading principal minor*). By $A(\alpha|\beta)$ we denote the $(n - \kappa) \times (m - \mu)$ submatrix of A contained in the rows indexed by the elements of $\{1, 2, \ldots, n\}\backslash\{\alpha_1, \alpha_2, \ldots, \alpha_\kappa\}$, and columns indexed by $\{1, 2, \ldots, m\}\backslash\{\beta_1, \beta_2, \ldots, \beta_\mu\}$ (where both sequences are ordered strictly increasingly) with the similar notation $A(\alpha)$ for the complementary principal submatrix.

A minor $\det A[\alpha|\beta]$ is called *row-initial* if $\alpha = \{1, 2, \ldots, \kappa\}$ and $\beta \in Q_{\kappa,m}$ is contiguous, it is termed *column-initial* if $\alpha \in Q_{\kappa,n}$ is contiguous while $\beta = \{1, 2, \ldots, \kappa\}$, and *initial* if it is row-initial or column-initial.

The n-by-n matrix whose only nonzero entry is a one in the $(i, j)^{th}$ position is denoted by E_{ij}. We reserve throughout the notation $T_n = (t_{ij})$ for the permutation matrix with $t_{i,n-i+1} = 1$, $i = 1, \ldots, n$. An n-by-m matrix A is called *totally positive* (abbreviated TP henceforth) and *totally nonnegative* (abbreviated TN) if $\det A[\alpha|\beta] > 0$ and $\det A[\alpha|\beta] \geq 0$, respectively, for all $\alpha, \beta \in Q_{\kappa,n'}$, $\kappa = 1, 2, \ldots, n'$, and $n' := min\{n, m\}$. In passing we note that if an n-by-m matrix A is TP then so are its transpose and $A^\# := T_n A T_m$, see, e.g., [**5**, Theorem 1.4.1]. We will briefly relate our results to TP_k matrices, $k \leq n'$. A is said to be TP_k if all its minors of order less than or equal to k are positive.

We say that a rectangular array is a *partial matrix* if some of its entries are specified, while the remaining, unspecified, entries are free to be chosen. A partial matrix is *partial TP* if each of its fully specified submatrices is TP. A *completion* of a partial matrix is a choice of values for the unspecified entries, resulting in a matrix that agrees with the a given partial matrix in all its specified positions. A pattern P is TP *completable* if every partial TP matrix with pattern P has a TP completion. TP_k *completion* is defined analogously.

We recall from [**9**] the following definitions. We associate with a matrix the directions north, east, south, and west. So the entry in position $(1, 1)$ lies north and west. We call a (possibly rectangular) pattern *jagged* if, whenever a position is unspecified, either all positions north and west of it are unspecified or south and east of it are, and we call a (possibly rectangular) pattern *echelon* if, whenever a

position is unspecied, either all positions north and east of it are unspecied or south and west of it are. Either of these is referred to as *single echelon*, while when both occur, we say *double echelon*. Echelon refers to any of these possibilities.

3. Auxiliary Results

The following proposition shows that it suffices to consider the initial minors if one wants to check a matrix for total positivity.

PROPOSITION 3.1. [**7**, Theorem 4.1]*, see also* [**5**, Theorem 3.1.4] *If all initial minors of a matrix A are positive, then A is TP.*

A fundamental tool for proving in the next section inequalities between ratios of determinants is the following proposition.

PROPOSITION 3.2. [**12**, Theorem 4.2] *Let $\alpha, \alpha', \beta, \beta', \gamma, \gamma', \delta, \delta'$ be subsets of $\{1, 2, \ldots, n\}$ with $\alpha \cup \gamma = \{1, 2, \ldots, p\}$ and $\alpha' \cup \gamma' = \{1, 2, \ldots, p'\}$, $q = |\alpha \cap \gamma|$, $q' = |\alpha' \cap \gamma'|$, and $r := \frac{1}{2}(p - q + p' - q')$. Let η be the unique order preserving map*

$$\eta : (\alpha \backslash \gamma) \cup (\gamma \backslash \alpha) \to \{1, 2, \ldots, p - q\},$$

and let η' be the unique order reversing map

$$\eta' : (\alpha' \backslash \gamma') \cup (\gamma' \backslash \alpha') \to \{p - q + 1, \ldots, 2r\}.$$

Define the subsets α'' and β'' of $\{1, 2, \ldots, 2r\}$ by

$$\begin{aligned} \alpha'' &:= \eta(\alpha \backslash \gamma) \cup \eta'(\gamma' \backslash \alpha'), \\ \beta'' &:= \eta(\beta \backslash \delta) \cup \eta'(\delta' \backslash \beta'). \end{aligned}$$

Then the following two statements are equivalent:

(1) *For each square TN matrix A of order at least n the following relation holds:*

$$\det A[\alpha|\alpha'] \det A[\gamma|\gamma'] \leq \det A[\beta|\beta'] \det A[\delta|\delta'].$$

(2) *The relations $\alpha \cup \gamma = \beta \cup \delta$ and $\alpha' \cup \gamma' = \beta' \cup \delta'$ are fulfilled and the sets α'', β'' satisfy the inequality*

$$(3.1) \qquad \max\{|\omega \cap \beta''|, |\omega \backslash \beta''|\} \leq \max\{|\omega \cap \alpha''|, |\omega \backslash \alpha''|\}$$

for each subset $\omega \subseteq \{1, 2, \ldots, 2r\}$ of even cardinality.

PROPOSITION 3.3. [**13**]*, see also* [**5**, p. 62], [**11**, Theorem 2.6] *The set of the TP n-by-n matrices is dense in the class of TN n-by-n matrices.*

The following propositions are used to solve totally positive completion problems in Section 5.

PROPOSITION 3.4. [**9**, Theorem 5] *Each jagged pattern is TP completable.*

PROPOSITION 3.5. [**5**, Theorem 9.4.4] *Let A be an n-by-m partial TP matrix with only one unspecified entry in the (s, t) position. If $\min\{n, m\} \leq 3$, then A has a TP completion. If $\min\{n, m\} \geq 4$, then any such A has a TP completion if and only if $s + t \leq 4$ or $s + t \geq n + m - 2$.*

We remark that Proposition 3.5 is generalized in [**4**, Theorem 4.5] to the case that the given matrix is partial TP_k, $k \geq 4$.

4. Perturbation of Totally Positive Matrices

In this section we consider the variation of single entries of a TP matrix $A = (a_{ij})$. For simplicity we consider here only the square case ($n = m$). We may restrict the discussion of the off-diagonal entries to the entries which are lying above the main diagonal since the related statements for the entries below the main diagonal follow by consideration of the transposed matrix.

THEOREM 4.1. *Let $A = (a_{ij})$ be a TP matrix and $0 \leq \tau$. Then for $i \leq j$,*

$$(4.1) \qquad A \pm \tau E_{ij} \text{ is } TP \text{ if and only if } \tau < \min S,$$

where in each of the following eight cases S is a set of ratios of minors, where the minor in the denominator is obtained from the minor in the numerator by deleting in A additionally row i and column j. If in an index sequence two indices coincide then the respective matrix has to be removed from the listing. In the following cases only the numerator matrices are listed [1]. The cases $(-)$ and $(+)$ refer to the $--$ and $+$-sign in (4.1). In the case that S is empty put $\min S := \infty$.

(1) $i = 2m$, $j = 2k$

$$(-)\,S : \begin{cases} A, A(n-1, n|1, 2), \ldots, A(n-2k+3, \ldots, n|1, \ldots, 2k-2), \\ A(1, 2|n-1, n), A(1, 2, 3, 4|n-3, n-2, n-1, n), \ldots, \\ A(1, \ldots, 2m-2|n-2m+3, \ldots, n) \end{cases}$$

$$(+)\,S : \begin{cases} A(n|1), A(n-2, n-1, n|1, 2, 3), \ldots, A(n-2k+2, \ldots, n|1, \ldots, 2k-1), \\ A(1|n), A(1, 2, 3|n-2, n-1, n), \ldots, A(1, \ldots, 2m-1|n-2m+2, \ldots, n) \end{cases}$$

(2) $i = 2m$, $j = 2k + 1$

$$(-)\,S : \begin{cases} A(n|1), A(n-2, n-1, n|1, 2, 3), \ldots, A(n-2k+2, \ldots, n|1, \ldots, 2k-1), \\ A(1|n), A(1, 2, 3|n-2, n-1, n), \ldots, A(1, \ldots, 2m-1|n-2m+2, \ldots, n) \end{cases}$$

$$(+)\,S : \begin{cases} A, A(n-1, n|1, 2), \ldots, A(n-2k+1, \ldots, n|1, \ldots, 2k), \\ A(1, 2|n-1, n), A(1, 2, 3, 4|n-3, n-2, n-1, n), \ldots, \\ A(1, \ldots, 2m-2|n-2m+3, \ldots, n) \end{cases}$$

(3) $i = 2m + 1$, $j = 2k$

$$(-)\,S : \begin{cases} A(n|1), A(n-2, n-1, n|1, 2, 3), \ldots, A(n-2k+2, \ldots, n|1, \ldots, 2k-1), \\ A(1|n), A(1, 2, 3|n-2, n-1, n), \ldots, A(1, \ldots, 2m-1|n-2m+2, \ldots, n) \end{cases}$$

$$(+)\,S : \begin{cases} A, A(n-1, n|1, 2), \ldots, A(n-2k+3, \ldots, n|1, \ldots, 2k-2), \\ A(1, 2|n-1, n), A(1, 2, 3, 4|n-3, n-2, n-1, n), \ldots, \\ A(1, \ldots, 2m|n-2m+1, \ldots, n) \end{cases}$$

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[1]E.g., $A(n-1, n|1, 2)$ refers in case 1(-) to the ratio $\frac{\det A(n-1, n|1, 2)}{\det A(2m, n-1, n|1, 2, 2k)}$.

(4) $i = 2m + 1$, $j = 2k + 1$

$$(-) S : \begin{cases} A, A(n-1,n|1,2), \ldots, A(n-2k+1,\ldots,n|1,\ldots,2k), \\ A(1,2|n-1,n), A(1,2,3,4|n-3,n-2,n-1,n), \ldots, \\ A(1,\ldots,2m|n-2m+1,\ldots,n) \end{cases}$$

$$(+) S : \begin{cases} A(n|1), A(n-2,n-1,n|1,2,3), \ldots, A(n-2k+2,\ldots,n|1,\ldots,2k-1), \\ A(1|n), A(1,2,3|n-2,n-1,n), \ldots, A(1,\ldots,2m-1|n-2m+2,\ldots,n). \end{cases}$$

PROOF. The entries in the positions $(1,1)$ and (n,n) can be increased arbitrarily without loosing the property of total positivity because they enter into the top left and bottom right position, respectively, in every submatrix in which they lie. This corresponds to the fact that in the cases $1(+)$ and $4(+)$ the set S is empty for $i = j = 1, n$. In the remaining cases we present the proof here only for $A(\tau) = A - \tau E_{2m,2k}$ (case $1(-)$); the proof of the other perturbations is similar. If $2m = n$, then $2k = n$, too. The only initial minor containing $a_{nn} - \tau$ is $\det A(\tau)$. By expansion of $\det A(\tau)$ along its bottom row we obtain

$$\det A(\tau) = \det A - \tau \det A(n)$$

from which the condition

$$0 \le \tau < \frac{\det A}{\det A(n)}$$

follows. We assume now that $2m < n$. For $\alpha \in Q_{\kappa,n}$ we set

$$\phi(\alpha) := \frac{\det A(\alpha|\alpha)}{\det A(\alpha \cup \{2m\}|\alpha \cup \{2k\})}.$$

We further use the intuitive notation

$$\phi(0) := \frac{\det A}{\det A(2m|2k)}.$$

First we show the inequality

(4.2) $$\phi(0) \le \phi(n).$$

The inequality follows by Proposition 3.2, setting $\alpha := \alpha' := \{1, 2, \ldots, n\}$, $\beta := \beta' := \{1, 2, \ldots, n-1\}$, $\gamma := \beta \backslash \{2m\}$, $\gamma' := \beta' \backslash \{2k\}$, $\delta := \alpha \backslash \{2m\}$, $\delta' := \alpha' \backslash \{2k\}$. Then the assumptions of Proposition 3.2 are fulfilled with $p = p' = n$, $q = q' = n-2$, and therefore $r = 2$, $\alpha'' = \{1, 2\}$, $\beta'' = \{1, 3\}$. For ω the following four sets can be chosen

$$\{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 2, 3, 4\}.$$

In all four cases the inequality (3.1) is fulfilled. Applying (4.2) to $A(n), A(n-1,n), \ldots, A(2m+1,\ldots,n)$, we obtain the chain of inequalities

(4.3) $$\phi(0) \le \phi(n) \le \phi(n-1,n) \le \cdots \le \phi(2m+1,\ldots,n).$$

Now we show that all the row-initial minors of $A(\tau)$ are positive; the proof of the positivity of the column-initial minors is similar. Since by expansion of $\det A(\tau)$ along its $2m^{th}$ row

$$\det A(\tau) = \det A - \tau \det A(2m|2k),$$

we obtain the condition $\tau < \phi(0)$. Similarly for $s = 0, 1, \ldots, n - 2m - 1$,

$$\det A(n-s,\ldots,n)(\tau) =$$
$$\det A(n-s,\ldots,n) - \tau \det A(2m, n-s, \ldots, n|2k, n-s, \ldots, n)$$

is positive if $\tau < \phi(n - s, \ldots, n)$. Therefore by (4.3), all leading principal minors of $A(\tau)\,(\beta_1 = 1)$ are positive if $\tau < \phi(0)$.

Now we consider the row-initial minors $\det A[\alpha|\beta](\tau)$, where $\beta = (\beta_1, \beta_2, \ldots, \beta_s)$ with $\beta_1 > 1$. If β_1 is odd these minors are constant or strictly increasing with respect to τ so that they remain positive under the perturbation. If β_1 is even, we apply the proof in the case $\beta_1 = 1$ to the submatrix $A[\alpha|\beta](\tau)$ and obtain the remaining conditions.

By Proposition 3.1 it follows that $A(\tau)$ is TP if τ is taken as the minimum of S in case $1(-)$. The necessity follows from the fact that all the initial minors are linear functions in τ and that therefore for $\min S \leq \tau$ there is an initial minor which is nonpositive. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

REMARK 4.2. Cases $1(-)$ and $4(-)$ give for $i = j = 1, 2$ the bound $\det A/\det A(i)$ on τ; see [**6**, Theorems 4.2 and 4.3] and [**5**, Theorems 9.5.4 and 9.5.5] for related statements. In case $3(+)$ setting $i = 1$ and $j = 2$ we get the bound $\det A/\det A(1|2)$, see [**6**, Theorem 4.7] and [**5**, Theorem 9.5.8].

The next theorem shows that the set S in Theorem 4.1 is minimal.

THEOREM 4.3. *For an arbitrary TP n-by-n matrix A the set S of determinantal ratios listed in Theorem 4.1 cannot be reduced in each of the eight cases.*

PROOF. We present the proof only for the case 1(-); the proof of the other seven cases is similar.

It suffices to show that the following ratios are not comparable if A runs over the set of the n-by-n TP matrices

$$b \quad := \quad \frac{\det A}{\det A(2m \mid 2k)},$$

$$c_\kappa \quad := \quad \frac{\det A(n - 2\kappa + 3, \ldots, n \mid 1, \ldots, 2\kappa - 2)}{\det A(2m, n - 2\kappa + 3, \ldots, n \mid 1, \ldots, 2\kappa - 2, 2k)}, \quad \kappa = 2, \ldots, k,$$

$$d_\mu \quad := \quad \frac{\det A(1, \ldots, 2\mu - 2 \mid n - 2\mu + 3, \ldots, n)}{\det A(1, \ldots, 2\mu - 2, 2m \mid 2k, n - 2\mu + 3, \ldots, n)}, \quad \mu = 2, \ldots, m.$$

We show here only that the ratios c_κ and d_μ are not comparable; the proof of the other cases is similar (and easier). We first prove that the inequality $c_\kappa \leq d_\mu$ does not hold for all TP n-by-n matrices A. To apply Proposition 3.2, we choose

$$\begin{aligned}
\alpha \quad &:= \quad \{1, \ldots, n - 2\kappa + 2\}, \alpha' := \{2\kappa - 1, \ldots, n\}, \\
\delta \quad &:= \quad \{2\mu - 1, \ldots, n\}, \delta' := \{1, \ldots, n - 2\mu + 2\}, \\
\beta \quad &:= \quad \alpha \setminus \{2m\}, \beta' = \alpha' \setminus \{2k\}, \\
\gamma \quad &:= \quad \delta \setminus \{2m\}, \gamma' = \delta' \setminus \{2k\}.
\end{aligned}$$

Then we have

$$\alpha \cup \gamma = \beta \cup \delta = \alpha' \cup \gamma' = \beta' \cup \delta'$$

by

$$\begin{aligned}
2\mu - 1 \quad &< \quad 2m \leq n - 2\kappa + 2, \\
2\kappa - 1 \quad &< \quad 2k \leq n - 2\mu + 2,
\end{aligned}$$

hence $p = p' = n$;

$$\begin{aligned}
\alpha \cap \gamma \quad &= \quad \{2\mu - 1, \ldots, n - 2\kappa + 2\} \setminus \{2m\}, \\
\alpha' \cap \gamma' \quad &= \quad \{2\kappa - 1, \ldots, n - 2\mu + 2\} \setminus \{2k\},
\end{aligned}$$

hence $q = q' = n - 2\kappa - 2\mu + 3$;

$$\eta : \{1, \ldots, 2\mu - 2, 2m\} \cup \{n - 2\kappa + 3, \ldots, n\} \to \{1, \ldots, 2\kappa + 2\mu - 3\},$$

$$\eta' : \{2k, n - 2\mu + 3, \ldots, n\} \cup \{1, \ldots, 2\kappa - 2\} \to \{2\kappa + 2\mu - 2, \ldots, 4\kappa + 4\mu - 6\},$$

$$
\begin{aligned}
\alpha'' &= \eta(\{1, \ldots, 2\mu - 2, 2m\}) \cup \eta'(\{1, \ldots, 2\kappa - 2\}) \\
&= \{1, \ldots, 2\mu - 1\} \cup \{2\kappa + 4\mu - 3, \ldots, 4\kappa + 4\mu - 6\}, \\
\beta'' &= \eta(\{1, \ldots, 2\mu - 2\}) \cup \eta'(\{1, \ldots, 2\kappa - 2, 2k\}) \\
&= \{1, \ldots, 2\mu - 2\} \cup \{2\kappa + 4\mu - 4, \ldots, 4\kappa + 4\mu - 6\}.
\end{aligned}
$$

Let $w := \{2\kappa + 4\mu - 4, 2\kappa + 4\mu - 3\}$. Then the inequality (3.1) is not fulfilled and by Proposition 3.2 there exists a TN matrix A_1 for which the inequality $c_\kappa > d_\mu$ holds. By interchanging the role of sets $\alpha, \alpha', \gamma, \gamma'$ with the sets $\beta, \beta', \delta, \delta'$, we find by Proposition 3.2 (choosing $w := \{2\mu - 2, 2\mu - 1\}$) a TN matrix A_2 for which the inequality $c_\kappa < d_\mu$ holds. So the ratios c_κ and d_μ are not comparable on the set of the TN matrices. By using Proposition 3.3 we find two TP matrices satisfying the respective inequalities which shows that also on the set of TP matrices the ratios c_κ and d_μ are not comparable. $\qquad\square$

EXAMPLE 4.4. Let A be the Pascal matrix of order 4, i.e.,

$$
A = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 2 & 3 & 4 \\
1 & 3 & 6 & 10 \\
1 & 4 & 10 & 20
\end{pmatrix}.
$$

Then A is TP, see, e.g., [**5**, Example 0.1.6]. In Table 1 we give the largest interval from which τ can be chosen according to Theorem 4.1 such that the matrix $A(\tau) := A + \tau E_{ij}$ is TP, $i, j = 1, \ldots, n$, $i \leq j$. The intervals are given in the (i, j) position of the respective entry. In each case, if τ is chosen as an endpoint of the interval, the respective matrix $A(\tau)$ contains a vanishing minor.

$$
\begin{array}{cccc}
\left(-\frac{1}{4}, \infty\right) & \left(-\frac{1}{6}, \frac{1}{6}\right) & \left(-\frac{1}{4}, \frac{1}{8}\right) & \left(-\frac{1}{3}, \frac{1}{3}\right) \\[2mm]
& \left(-\frac{1}{14}, \frac{1}{4}\right) & \left(-\frac{1}{7}, \frac{1}{11}\right) & \left(-\frac{1}{3}, \frac{1}{3}\right) \\[2mm]
& & \left(-\frac{1}{10}, \frac{1}{2}\right) & \left(-1, \frac{1}{3}\right) \\[2mm]
& & & \left(-1, \infty\right)
\end{array}
$$

TABLE 1. The largest perturbation intervals in Example 4.4.

5. TP Completion Problems

In this section we consider TP completion problems for some new patterns of the unspecified entries.

We recall from [**9**] the definition of the patterns P_1, P_1', and P_2. Let A be an n-by-m matrix. We say that A has a P_1 or P_1' pattern if A has just one unspecified entry, viz. in the $(1, m)$ or $(n, 1)$ positions, respectively. The P_2 pattern has just

two unspecified entries, viz. in positions $(1, m)$ and $(n, 1)$. Since by Proposition 3.1 a matrix is TP if and only if its initial minors are positive, a partial TP matrix with a P_1 pattern has a TP completion if and only if the upper right entry can be chosen so that the upper right minors with contiguous index sets are all positive. A similar condition holds for a P_1' pattern. We introduce two further patterns. We say that A has a P_3 *pattern* if $3 \leq m$ and the unspecified entries are a_{ij},

$i = 1, \ldots, l, \ j = 3, \ldots, k$, and $i = r, \ldots, n, \ j = 1, \ldots, m - 2$,
with $l \in \{1, \ldots, n - 1\}$, $r \in \{l + 1, \ldots, n\}$.

A has a P_4 *pattern* if $2 \leq n$, $4 \leq m$ and the unspecified entries are a_{ij},

$i = 1, \ldots, l, \ j = 1, \ldots, k, \ \text{and} \ i = 1, \ldots, r, \ j = k + 3, \ldots, m, \ \text{and} \ i = p, \ldots, n, \ j = 1, \ldots, h, \ \text{and} \ i = t, \ldots, n, \ j = h + 3, \ldots, m$,
with $1 \leq r \leq l < t \leq p \leq n$, $h, k \in \{1, \ldots, m - 3\}$.

Examples 1 and 2 in [**9**] and 9.1.1 in [**5**] show that the P_1, P_1', and P_2 patterns are not TP completable if $4 \leq min\{n, m\}$. This explains why in the sequel the index ranges will often start at 3.

THEOREM 5.1. *Let A be an n-by-m partial TP matrix with the unspecified entries a_{ij}, $i = 1, \ldots, l, \ j = 3, \ldots, k$, where $l \leq n$, $k \leq m$. Then A is TP completable.*

PROOF. Let $B_{lk} := A[l, \ldots, n \mid 1, 2, k, \ldots, m]$. Then by Proposition 3.5, B_{lk} is TP completable. We enter the value for the unspecified entry a_{lk} into the matrix A and call the resulting matrix A_{lk}. If $l > 1$ let $B_{l-1,k} := A_{lk}[l - 1, \ldots, n \mid 1, 2, k, \ldots, m]$. Then by Proposition 3.5, $B_{l-1,k}$ is TP completable and similarly as for the entry a_{lk} we obtain the n-by-m partial TP matrix $A_{l-1,k}$ which has one unspecified entry less than A_{lk}. Now we continue in this manner until we find values for all the unspecified entries in column k resulting in the partial TP matrix A_{1k}. If $k > 3$ repeat the above process with the partial TP matrix A_{1k} to find values for the unspecified entries in the columns $k - 1, \ldots, 3$. At the end of this process we get the matrix A_{13} which is TP. $\square$

COROLLARY 5.2. *Let A be an n-by-m partial TP matrix with the unspecified entries a_{ij}, $i = r, \ldots, n, \ j = k, \ldots, m - 2$, where $r \leq n$, $k \leq m - 2$. Then A is TP completable.*

PROOF. The matrix $A^{\#}$ is a partial TP matrix with the same pattern of unspecified entries as the one considered in Theorem 5.1, whence $A^{\#}$ is TP completable. Then A is TP completable, too. $\square$

By application of Theorem 5.1 and Corollary 5.2 to A, $A^{\#}$, or A^T it follows that a partial TP matrix A whose pattern is a single echelon pattern is TP completable if and only if it contains no P_1 or P_1' as a subpattern. This settles [**9**, Conjecture 1] in a special case.

THEOREM 5.3. *Let $3 \leq m$ and A be an n-by-m partial TP matrix with the unspecified entries a_{1j}, $j = 3, \ldots, m$, and a_{i1}, $i = l, \ldots, n$, with $l \leq 4$. Then A is TP completable.*

PROOF. If $l = 1$ or 2 then it is easy to find values for the unspecified entries in the positions $(1, 1)$ and $(2, 1)$, respectively, so let $l \in \{3, 4\}$. Let $B_1 := A[2, \ldots, n \mid 1, \ldots, m]$. Then by taking the transpose of B_1 and using Theorem 5.1 (with m

replaced by n), B_1 is TP completable. We enter the values for the unspecified entries a_{i1}, $i = l, \ldots, n$, into the matrix A and call the resulting matrix A_1. The matrix A_1 is a partial TP matrix with the unspecified entries in the first row. By Theorem 5.1 A_1 is TP completable, and so A is TP completable. $\square$

THEOREM 5.4. *Let* $3 \le m$, $4 \le n$, *and* A *be an* n-*by-*m *partial* TP *matrix with the unspecified entries* a_{ij}, $i = 1, \ldots, l$, $j = 3, \ldots, m$, *and* $i = l + 3, \ldots, n$, $j = 1, \ldots, h$, *where* $h \le m$. *Then* A *is* TP *completable.*

PROOF. Let $B_h := [l + 1, \ldots, n \mid h, \ldots, m]$. Then B_h is a partial TP matrix and by Theorem 5.1 (taking the transpose), B_h is TP completable. We enter the values for the unspecified entries of B_h into the matrix A and call the resulting matrix A_h. Repeat the last step to find values for the unspecified entries in the lower part of the columns $h - 1, \ldots, 2$. At the end of this process we get the partial TP matrix A_2 having the unspecified entries in the upper right corner and in the first column below the position $(1, l + 3)$. To find values for the unspecified entries in the first column, we proceed analogously to the proof of Theorem 5.3. Let $C := A_2[l + 1, \ldots, n \mid 1, \ldots, m]$. Then by Theorem 5.1 C is TP completable. Since only the initial minors of A need to be positive, C can be completed independently of the entries of $A[1, \ldots, l \mid 1, 2]$. We enter the values for the unspecified entries of the first column into the matrix A_2 and call the resulting matrix A_1; then A_1 is a partial TP matrix with the unspecified entries in the upper right corner. We apply Theorem 5.1 on A_1 and it follows that A_1 is TP completable. Therefore A is TP completable. $\square$

The following theorem combines the patterns of Theorem 5.1 and Corollary 5.2 and is related to Proposition 3.4.

THEOREM 5.5. *If* $A = (a_{ij})$ *is an* n-*by-*m *partial* TP *matrix with a* P_3 *pattern, then* A *is* TP *completable.*

PROOF. Let A be an n-by-m partial TP matrix with a P_3 pattern. We distinguish two cases.
Case (1): $k = m$. Let $B_1 := A[l + 1, \ldots, n \mid 1, m - 1, m]$. Then by applying Proposition 3.5 successively to $C_\rho := B_1[l + 1, \ldots, \rho \mid 1, m - 1, m]$ we find values for the unspecified entries $a_{\rho 1}$, $\rho = r, \ldots, n$. Therefore B_1 is TP completable. We enter the values for the unspecified entries of B_1 into the matrix A and call the resulting matrix A_1. To find values for the unspecified entries in the second column of A_1 in and below position $(r, 2)$ we similarly apply Proposition 3.5 to the submatrix $B_2 := A_1[l + 1, \ldots, n \mid 1, 2, m - 1, m]$. This completion can be accomplished independently of the entries of $A[1, \ldots, l \mid 1, 2]$, see the proof of Theorem 5.4. We enter the values for the unspecified entries of B_2 into the matrix A and call the resulting matrix A_2. Let $B_3 := A_2[l + 1, \ldots, n \mid 1, 2, \ldots, m]$. Then B_3 is partial TP and by Corollary 5.2 TP completable. We enter the values for the unspecified entries of B_3 into the matrix A_2 and call the resulting matrix A_3. By Theorem 5.1 A_3 is TP completable and hence A is TP completable.
Case (2): $k < m$. We consider first the case $k = m - 1$. If $l + 1 = r$, then we can choose a positive number for the unspecified entry $a_{l, m-1}$ such that the matrix remains partial TP. If $l + 1 < r$, let $B_{l, m-1} := A[l, \ldots, r - 1 \mid 1, 2, m - 1, m]$ and $B'_{l, m-1} := A[l, \ldots, n \mid m - 1, m]$. Then by Proposition 3.5 both submatrices are TP completable. Moreover, we can choose in both matrices a common value for the

unspecified entry $a_{l,m-1}$ because the only nontrivial initial minor of the submatrix $B'_{l,m-1}$ containing this entry, viz. $\det B'_{l,m-1}[l, l+1|m-1, m]$, is also an initial minor of the submatrix $B_{l,m-1}$.

We enter the value for the unspecified entry $a_{l,m-1}$ into the matrix A and call the resulting matrix $A_{l,m-1}$. Repeating this process we find values for the unspecified entries in column $m-1$ and finally obtain the partial TP matrix $A_{1,m-1}$. Let $C := A_{1,m-1}[1, \ldots, r-1 \mid 1, \ldots, m]$. By Theorem 5.1, C is TP completable. We enter the values for the unspecified entries in $A_{1,m-1}$ and call the resulting matrix A', which is a partial TP matrix. By Corollary 5.2, A' is TP completable and we can conclude that A is also TP completable.
If $k < m-1$ we follow the proof in the case $k = m-1$ but we may end the proof already with the definition of the matrix C. $\qquad\square$

THEOREM 5.6. *If $A = (a_{ij})$ is an n-by-m partial TP matrix with a P_4 pattern, then A is TP completable.*

PROOF. Let A be an n-by-m partial TP matrix with a P_4 pattern. Without loss of generality we may assume that $l := r$ and $p = t$. Otherwise let $B_0 := A[r+1, \ldots, p-1|1, \ldots, m]$. Then B_0 has a jagged pattern and can be TP completed by Proposition 3.4 (independently of the entries of $A[1, \ldots, r|k+1, k+2]$ and $A[p, \ldots, n|h+1, h+2]$). In what follows we therefore assume that $l = r$ and $p = t$. Let $B_1 := A[1, \ldots, p-1|k+1, \ldots, m]$; then B_1 is TP completable by Theorem 5.1. We enter the values for the unspecified entries of B_1 into the matrix $A[1, \ldots, p-1|1, \ldots, m]$ and call the resulting matrix B_2. Since B_2 is a partial TP matrix with a jagged pattern it is TP completable by Proposition 3.4. We enter the values for the unspecified entries of B_2 into the matrix A and call the resulting matrix A_1. Proceeding with $A_1^{\#}$ we obtain similarly the TP completion of A_1 and in this way of A, too. $\qquad\square$

THEOREM 5.7. *If $A = (a_{ij})$ is an n-by-m partial TP matrix with the unspecified entries a_{ij} $i = 1, \ldots, l$, $j = 1, \ldots, k$, and $i = 1, \ldots, r$, $j = k+3, \ldots, m$, and $i = t, \ldots, n$, $j = 1, \ldots, m-2$, with $r \leq l < t$, then A is TP completable.*

PROOF. Without loss of generality we may assume that $r = l$, see the proof of Theorem 5.6. Let $C_1 := A[r+1, \ldots, n \mid 1, \ldots, k, m-1, m]$. Then C_1 is TP completable by Corollary 5.2. We enter the values for the unspecified entries of C_1 into the matrix A and call the resulting matrix A_1. Let $C_2 := A_1[r+1, \ldots, n \mid 1, \ldots, k+1, m-1, m]$. Then by an argument similar to that used in the proof of Theorem 5.1, by starting from the unspecified entry $a_{t,k+1}$, and proceeding downwards, we can find values for the unspecified entries in C_2; so C_2 is TP completable. We enter the values for the unspecified entries of C_2 into the matrix A_1 and call the resulting matrix A_2. Similarly, we can find values for the unspecified entries in the column $k+2$, then we enter these values into the matrix A_2 and call the resulting matrix A_3. Let $D := A_3[r+1, \ldots, n \mid 1, \ldots, m]$; then D is a partial TP matrix with the same type of pattern as the one treated in Corollary 5.2, thus D is TP completable. We enter the values for the unspecified entries in D into the matrix A_3 and call the resulting matrix A_4, where A_4 is a partial TP matrix. Let $D_1 := A_4[1, \ldots, n|k+1, \ldots, m]$; then D_1 is TP completable by Theorem 5.1 (with $k = m$). We enter the values for the unspecified entries of D_1 into A_4 and call the resulting matrix A_5 which is partial TP. Since A_5 has a jagged pattern, it is completable by Proposition 3.4, whence A is TP completable. $\qquad\square$

THEOREM 5.8. *If $A = (a_{ij})$ is an n-by-m partial TP matrix with the unspecified entries a_{ij} $i = 1, \ldots, l$, $j = 1, \ldots, k$, and $i = 1, \ldots, r$, $j = k + 3, \ldots, m$, and $i = t, \ldots, n$, $j = h, \ldots, m$, with $r \leq l < t$, and $h < k$, then A is TP completable.*

PROOF. We may assume without loss of generality that $r = l$. Otherwise we proceed as follows: Let $B_1 := A[r + 1, \ldots, t - 1 | h, \ldots, m]$. Then B_1 is partial TP with a jagged pattern, thus TP completable. If $h > 1$ we have to take into account the entries of $A[t, \ldots, n | 1, \ldots, h-1]$ when we want to extend the completion to the left. We proceed element-wise by taking successively the entries $a_{l,h-1}$, $a_{l-1,h-1}$, ..., $a_{r+1,h-1}$, $a_{l,h-2}$, ..., $a_{r+1,h-2}$, ..., $a_{r+1,1}$. For a fixed entry we consider the submatrices which have the chosen entry as the only unspecified entry, viz in position $(1,1)$. For each such submatrix we can find a positive number such that the matrix is TP. Then we take the maximum of all these positive numbers (for the chosen entry).

The matrix $C_1 := A[r + 1, \ldots, n \mid 1, \ldots, k + 1]$ is a partial TP matrix with a jagged pattern and by Proposition 3.4 C_1 is TP completable. We enter the values for the unspecified entries of C_1 into the matrix A and call the resulting matrix A_1. Let $C_2 := A_1[r+1, \ldots, n \mid 1, \ldots, k+2]$, and $C_2' := A_1[1, \ldots, n \mid k+1, k+2]$. Then both submatrices are partial TP matrices with jagged patterns, and so by the argument used in the proof of Theorem 5.5, Case (2), common values for the unspecified entries can be found. We enter the values for the unspecified entries of C_2 into the matrix A_1 and call the resulting matrix A_2. Let $C_3 := A_2[r + 1, \ldots, n \mid 1, \ldots, m]$. Then C_3 is a partial TP matrix with a jagged pattern, thus C_3 is TP completable. We enter the values for the unspecified entries of C_3 into the matrix A_2 and call the resulting matrix A_3. Since A_3 is a partial TP matrix with the same type of pattern as the one considered in the proof of Theorem 5.6, A_3 is TP completable. Hence A is TP completable.   □

Theorems 5.7 and 5.8 represent certain instances of a jagged echelon pattern with no P_1, P_1', or P_2 as a subpattern. Thus both theorems and the following remark settle [**9**, Conjecture 2] in some special cases leaving this conjecture unresolved only in the case of a double echelon pattern.

REMARK 5.9. The following patterns can be proven to be TP completable by using similar methods as in the proofs of the Theorems 5.6, 5.7, and 5.8. The entries a_{ij} are unspecified for

(1) $i = 1, \ldots, l$, $j = 1, \ldots, k$, and $i = r, \ldots, n$, $j = 1, \ldots, m - 2$, with $l < r$;

(2) $i = 1, \ldots, l$, $j = 3, \ldots, m$, and $i = r, \ldots, n$, $j = k, \ldots, m$, with $l < r$;

(3) $i = 1, \ldots, l$, $j = 3, \ldots, m$, and $i = r, \ldots, n$, $j = 1, \ldots, k$, and $i = t, \ldots, n$, $j = k + 3, \ldots, m$, with $l < t < r$;

(4) $i = 1, \ldots, l$, $j = 1, \ldots, k$, and $i = r, \ldots, n$, $j = 1, \ldots, h$, and $i = t, \ldots, n$, $j = h + 3, \ldots, m$, with $l < t < r$ and $h < k$.

REMARK 5.10. By using [**4**, Theorem 5.4] instead of Proposition 3.5 all the results of Section 5 carry over to the case that the given matrix A is partial TP_k, $k \geq 4$.

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