Intervals of Almost Totally Positive Matrices

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Abstract

We consider the class of the totally nonnegative matrices, i.e., the matrices having all their minors nonnegative, and intervals of matrices with respect to the chequerboard partial ordering, which results from the usual entrywise partial ordering if we reverse the inequality sign in all components having odd index sum. For these intervals in 1982 we stated in this journal the following conjecture: If the left and right endpoints of an interval are nonsingular and totally nonnegative then all matrices taken from the interval are nonsingular and totally nonnegative. In this paper we show that this conjecture is true if we restrict ourselves to the subset of the almost totally positive matrices.

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1 Introduction

A real matrix is called totally nonnegative (resp., totally positive) if all its minors are nonnegative (resp., positive). These matrices appear in various branches of mathematics and its applications, e.g., in mechanics [8], statistics [15], combinatorics [7], and computer aided geometric design [4], to name only a few. For a thorough presentation of the properties of these matrices up to 1984 see [1]. Some more recent results can be found in [11]. Some present research focusses on completion problems, cf. [6].

A class of real matrices which is related to this class is the inverse nonnegative matrices; these are nonsingular matrices whose inverses are entrywise nonneg-

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ative. It was shown in [17] that these matrices enjoy a certain interval property: If $\underline{A}$ and $\overline{A}$ are inverse nonnegative and $\underline{A} \leq \overline{A}$ in the usual entrywise partial ordering, then any matrix lying between both matrices is also inverse nonnegative. We have shown in [9] that a related interval property holds true for some classes of the totally nonnegative matrices, where the partial ordering is now the chequerboard partial ordering which results from the usual entrywise partial ordering if we reverse the inequality sign in each component having odd index sum. The subclasses include, e.g., the totally positive matrices and the tridiagonal nonsingular totally nonnegative matrices. We stated therein also the conjecture that the interval property holds true for the entire class of the nonsingular totally nonnegative matrices. In [10] we proved that the conjecture is true if we consider 2^{2n-1} vertex matrices, i.e., matrices having the entries of $\underline{A}$ and $\overline{A}$ as entries (n being the order of the matrices). This paper aims to contribute to settling the above conjecture. We show that the conjecture holds true for the almost totally positive matrices. These matrices are intermediate between the nonsingular totally nonnegative matrices and the totally positive matrices and were introduced in [14].

Now we introduce the notation used in our paper. For $k, n \in \mathbf{N}$, $1 \leq k \leq n$, we denote by $Q_{k,n}$ the set of all strictly increasing sequences of k integers chosen from $\{1, \ldots, n\}$. The set $Q^0_{k,n}$ consists of the sequences from $Q_{k,n}$ which are formed from consecutive numbers. We use the imprecise but intuitive notation $\alpha \setminus \alpha_i$ and $\alpha \setminus (\alpha_i, \alpha_j)$ to denote the sequences in $Q_{k-1,n}$ and $Q_{k-2,n}$ which are obtained from α by discarding its entries α_i and α_i and α_j, respectively. With each $\alpha = (\alpha_i) \in Q_{k,n}$ we associate a number $c(\alpha)$ which will play the role of a measure for the chequeredness of a submatrix:

$$c(\alpha) = \sum_{i=1}^{k-1} \gamma(\alpha_{i+1} - \alpha_i), \quad \text{where} \quad \gamma(\xi) = \begin{cases} 0 \\ 1 \end{cases} \text{if } \xi \text{ is } \begin{matrix} \text{odd} \\ \text{even} \end{matrix},$$

with the convention $c(\xi) = 0$ for $\alpha \in Q_{1,n}$. Let A be a real $n \times n$ matrix. For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k), \beta = (\beta_1, \beta_2, \ldots, \beta_k) \in Q_{k,n}$ we denote by $A[\alpha \,|\, \beta]$ the $k \times k$ submatrix of A contained in the rows indexed by $\alpha_1, \ldots, \alpha_k$ and columns indexed by $\beta_1, \ldots, \beta_k$. We suppress the brackets when we enumerate the indexes explicitly. When $\alpha = \beta$, the principal submatrix $A[\alpha \,|\, \alpha]$ is abbreviated to $A[\alpha]$.

The matrix A is called *totally nonnegative* (resp., *positive*) if $det A[\alpha \,|\, \beta] \geq 0$ (resp., $det A[\alpha \,|\, \beta] > 0$) for all $\alpha, \beta \in Q_{k,n}$, $k = 1, \ldots, n$. Similarly as in [14], A is termed *almost totally positive* if A is nonsingular, totally nonnegative, and possesses the following property: If a minor of A with consecutive rows and columns is zero then one of its diagonal entries is zero. It was proved in [14] that if A is almost totally positive, then this property holds true for any $\alpha, \beta \in Q_{k,n}$, i.e., for all $k = 1, \ldots, n$

$$\forall \alpha, \beta \in Q_{k,n} \ : \ det A[\alpha \,|\, \beta] = 0 \implies \exists i_0 \in \{1, \ldots, n\} \ : \ a_{\alpha_{i_0}, \beta_{i_0}} = 0. \qquad (1)$$

Consequently, for this type of matrix, we know exactly which minors are positive and which are zero. Examples are the Hurwitz matrix [2], [14], [16] and the B–spline collocation matrix [14]. In [12] these matrices are characterised by means of Neville elimination, in terms of the positivity of a reduced number of their minors, and in terms of their factorisation as a product of bidiagonal elementary matrices and of their LU–factorisation.

We will make use of the following properties of totally nonnegative matrices (specified here for our purposes):

Lemma (Cryer [5]) *Let A be a $p \times (k+1)$ matrix with $2 \leq k \leq p$. Assume that the first k as well as the last k columns of A form totally nonnegative matrices. Also assume that for some $\alpha \in Q_{k,p}$ and some m such that $2 \leq m \leq k$, $det A[\alpha \,|\, 1, \ldots, \hat{m}, \ldots, k+1] < 0$* [1]*. Then columns $2, \ldots, \hat{m}, \ldots, k$ of A have rank $k-2$, and column m of A depends linearly upon columns $2, \ldots, \hat{m}, \ldots, k$ of A. If $k = 2$, then the column m of A is zero.*

Theorem (Cryer [5]) *If A is a nonsingular $n \times n$ matrix, then A is totally nonnegative iff for all $k \in \{1, \ldots, n\}$*

$$\forall \alpha \in Q^0_{k,n}, \beta \in Q_{k,n} \ : \ det A[\alpha \,|\, \beta] \geq 0. \qquad (2)$$

An alternative determinantal characterisation is presented in [13].

Shadow Lemma [3] *Let $A = (a_{ij})$ be a nonsingular totally nonnegative matrix of order n and let $a_{i_0, j_0} = 0$. If $i_0 < j_0$ then the submatrix $A[1, 2, \ldots, i_0 - 1, i_0 \,|\, j_0, j_0 + 1, \ldots, n]$ (called the* right shadow*) is zero, and if $i_0 > j_0$ then the submatrix $A[i_0, i_0 + 1, \ldots, n \,|\, 1, 2, \ldots, j_0 - 1, j_0]$ (called the* left shadow*) is zero.*

2 Main Result

We consider the (real) $n \times n$ matrices endowed with the usual (entrywise) partial ordering $\leq$ and with the *chequerboard partial ordering* $\leq^*$: For $A, B \in \mathbf{R}^{n \times n}$, $A = (a_{ij})$, $B = (b_{ij})$,

$$\left.\begin{array}{rcl} A \leq B & \iff & a_{ij} \leq b_{ij}, \\ A \leq^* B & \iff & (-1)^{i+j} a_{ij} \leq (-1)^{i+j} b_{ij}, \end{array}\right\} \; i, j = 1, \ldots, n.$$

[1] The notation $\hat{m}$ means with the exception of m.

For $\downarrow A, \uparrow A \in \mathbf{R}^{n\times n}$ with $\downarrow A \leq^* \uparrow A$, we denote by $[\downarrow A, \uparrow A]^*$ the matrix interval w.r.t. $\leq^*$, i.e.,

$$[\downarrow A, \uparrow A]^* = \{A \in \mathbf{R}^{n\times n} \mid \downarrow A \leq^* A \leq^* \uparrow A\}.$$

Such a *matrix interval* can also be regarded as an *interval matrix*, i.e., a matrix with all entries taken from the set of the compact and nonempty real intervals $[\underline{a}, \overline{a}]$, $\underline{a}, \overline{a} \in \mathbf{R}$ with $\underline{a} \leq \overline{a}$. Then we can represent $\downarrow A$ and $\uparrow A$ in the following way [2]:

$$(\downarrow A)_{ij} = \begin{cases} \underline{a}_{ij} \\ \overline{a}_{ij} \end{cases} \text{if } i+j \text{ is } \begin{matrix} \text{even} \\ \text{odd} \end{matrix}$$

$$(\uparrow A)_{ij} = \begin{cases} \overline{a}_{ij} \\ \underline{a}_{ij} \end{cases} \text{if } i+j \text{ is } \begin{matrix} \text{even} \\ \text{odd} \end{matrix} .$$

To prove the main result, we need two lemmata.

Lemma 1 *Let $\downarrow A, \uparrow A \in \mathbf{R}^{n\times n}$ be nonsingular and totally nonnegative and let $\downarrow A \leq^* \uparrow A$. Then for any $A \in [\downarrow A, \uparrow A]^*$ we have*

$$0 < det A[\alpha] \text{ for all } \alpha \in Q_{k,n} \text{ with } c(\alpha) = 0,\ k = 1, \ldots, n.$$

Proof: From Kuttler's result [17] it follows that $0 < det A$ (cf. [9], [10]). By [1, Cor. 3.8] any principal submatrix of $\downarrow A$ and $\uparrow A$ is nonsingular, so that the above result applies to $\downarrow A[\alpha]$ and $\uparrow A[\alpha]$ for any $\alpha \in Q_{k,n}$ with $c(\alpha) = 0$. ■

Lemma 2 *Let $\downarrow A, \uparrow A$ be nonsingular and totally nonnegative with $\downarrow A \leq^* \uparrow A$. Let $\underline{a}_{i_0,j_0} = 0$. Then for any $A \in [\downarrow A, \uparrow A]^*$ it holds that $a_{ij} = 0$*

either for all
$j_0 \leq j$ *if* $i < i_0$ *and* $j_0 < j$ *if* $i = i_0$, *if* $i_0 < j_0$,
or for all
$j \leq j_0$ *if* $i_0 < i$ *and* $j < j_0$ *if* $i = i_0$, *if* $i_0 > j_0$.

Proof: If $\underline{a}_{i_0,j_0} = 0$ then by Lemma 1 we can conclude that $i_0 \neq j_0$. Assume w.l.o.g. that $1 < i_0 < j_0 < n$. By the Shadow Lemma, $\underline{a}_{i_0,j_0}$ throws a shadow to the right (in $\downarrow A$ or in $\uparrow A$). It follows that the vanishing entries $\overline{a}_{i_0-1,j_0}$, $\underline{a}_{i_0-1,j_0}$, $\overline{a}_{i_0,j_0+1}$, $\underline{a}_{i_0,j_0+1}$ throw a shadow to the right in $\downarrow A$ as well as in $\uparrow A$, from which the statement follows. ■

[2] Note that this notation differs from the one used in [9].

Now we state and prove our main result.

Theorem 1 *Let $\downarrow A, \uparrow A \in \mathbf{R}^{n\times n}$ be almost totally positive with $\downarrow A \leq^* \uparrow A$. Then any $A \in [\downarrow A, \uparrow A]^*$ is almost totally positive.*

Proof: Let $A = (a_{ij}) \in [\downarrow A, \uparrow A]^*$. To show that A is totally nonnegative it is sufficient by Cryer's Theorem to show that (2) holds for all $k = 1, \ldots, n$. We will prove (1), (2) by induction on $k = 1, \ldots, n-1$ (the case $k = n$ is obvious by Lemma 1). The case $k = 1$ is trivial. Assume that $k \geq 2$ and that (1), (2) hold true for all minors of order less than k. To show (1), (2) for all minors of order k we proceed by induction on $l = c(\beta)$. If $l = 0$ then

$$\begin{array}{ll} & \downarrow A[\alpha\,|\,\beta] \leq^* A[\alpha\,|\,\beta] \leq^* \uparrow A[\alpha\,|\,\beta] \\ \text{or} & \\ & \uparrow A[\alpha\,|\,\beta] \leq^* A[\alpha\,|\,\beta] \leq^* \downarrow A[\alpha\,|\,\beta] \end{array}$$

holds. If $\downarrow A[\alpha\,|\,\beta], \uparrow A[\alpha\,|\,\beta]$ are nonsingular, (2) is true by Lemma 1. If $\downarrow A[\alpha\,|\,\beta]$ or $\uparrow A[\alpha\,|\,\beta]$ is singular it follows from the hypothesis of the theorem that there is an $i_0 \in \{1, \ldots, k\}$ with $\underline{a}_{\alpha_{i_0}, \beta_{i_0}} = 0$. Assume w.l.o.g. that $\alpha_{i_0} < \beta_{i_0}$. By Lemma 2 the entries a_{ij} vanish for all $\beta_{i_0} \leq j$ if $i < \alpha_{i_0}$ and for all $\beta_{i_0} < j$ if $i = \alpha_{i_0}$. Then we obtain (assuming w.l.o.g. that $1 < i_0$) that

$$\begin{aligned} det A[\alpha\,|\,\beta] = & \, det A[\alpha_1, \ldots, \alpha_{i_0-1}\,|\,\beta_1, \ldots, \beta_{i_0-1}] \\ & \times det A[\alpha_{i_0}, \ldots, \alpha_k\,|\,\beta_{i_0}, \ldots, \beta_k]. \end{aligned} \tag{3}$$

Now the induction hypothesis applies to (3). To prove (1) let $det A[\alpha\,|\,\beta] = 0$ which implies by Lemma 1 that $\downarrow A[\alpha\,|\,\beta]$ or $\uparrow A[\alpha\,|\,\beta]$ is singular and we again arrive at (3) and the induction hypothesis applies.

Assume now that (1), (2) are shown for all sequences $\beta \in Q_{k,n}$ with $c(\beta) \leq l$. Let $\beta \in Q_{k,n}$ with $c(\beta) = l + 1$ and assume that

$$det A[\alpha\,|\,\beta] < 0. \tag{4}$$

We can choose $\beta' \in Q_{k+1,n}$ with $\beta'_1 = \beta_1, \beta'_{k+1} = \beta_k$ and $c(\beta') = l$. Then we have

$$c(\beta'_1, \ldots, \beta'_k) \leq l \;\text{ and }\; c(\beta'_2, \ldots, \beta'_{k+1}) \leq l. \tag{5}$$

By the induction hypothesis the submatrices $A[\alpha\,|\,\beta'_1, \ldots, \beta'_k]$ and $A[\alpha\,|\,\beta'_2, \ldots, \beta'_{k+1}]$ are totally nonnegative. Now we apply Cryer's Lemma. We treat the case $k = 2$ first. It follows that $a_{\alpha_i, \beta'_2} = 0, i = 1, 2$, and by Lemma 2, $a_{\alpha_1, \beta_2} = 0$ or $a_{\alpha_2, \beta_1} = 0$. But then $det A[\alpha\,|\,\beta] \geq 0$, a contradiction to (4).

Now let $k \geq 3$. By Cryer's Lemma, there is a column of $A[\alpha\,|\,\beta']$ indexed by β'_m, $2 \leq m \leq k$, which depends linearly upon columns $\beta'_2, \ldots, \hat{\beta}'_m, \ldots, \beta'_k$ of $A[\alpha\,|\,\beta']$, whence

$$det A[\alpha_2, \ldots, \alpha_k\,|\,\beta'_2, \ldots, \beta'_k] = 0. \tag{6}$$

It follows from the induction hypothesis that there is an $i_0 \in \{2, \ldots, k\}$ such that $a_{\alpha_{i_0}, \beta'_{i_0}} = 0$. W.l.o.g. assume that $\alpha_{i_0} < \beta'_{i_0}$. Then by Lemma 2 (note that $\beta'_i \leq \beta_i, i = 2, \ldots, k$) we arrive at (3). Its right-hand side is nonnegative by the induction hypothesis, in contradiction to (4). Finally, assume that $det A[\alpha\,|\,\beta] = 0$. We apply an identity given in [15, p.8] (cf. [1, formula (1.39)]) to the submatrix $A[\alpha\,|\,\beta']$ which yields, with the notation $\alpha^{(1)} = \alpha \setminus \alpha_1$,

$$\begin{aligned} & det A[\alpha^{(1)}\,|\,\beta' \setminus (\beta_1, \beta_k)]\; det A[\alpha\,|\,\beta] \\ & \quad = det A[\alpha^{(1)}\,|\,\beta \setminus \beta_1]\; det A[\alpha\,|\,\beta' \setminus \beta_k] \\ & \quad + det A[\alpha^{(1)}\,|\,\beta \setminus \beta_k]\; det A[\alpha\,|\,\beta' \setminus \beta_1]. \end{aligned} \tag{7}$$

- Case (i): $det A[\alpha\,|\,\beta' \setminus \beta_1]\, det A[\alpha\,|\,\beta' \setminus \beta_k] = 0$
  By (5) and the induction hypothesis, a diagonal entry of one of the two submatrices must vanish. Then either we have found a vanishing diagonal entry of the matrix $A[\alpha\,|\,\beta]$ or we can factorise $det A[\alpha\,|\,\beta]$ as in (3). But then the induction hypothesis applies.
- Case (ii): $det A[\alpha\,|\,\beta' \setminus \beta_i] > 0,\ i = 1, k$
  By the induction hypothesis $det A[\alpha^{(1)}\,|\,\beta \setminus \beta_i] \geq 0$ for $i = 1, k$. Since the left-hand side of (7) is zero, by our assumption, we have $det A[\alpha^{(1)}\,|\,\beta \setminus \beta_1] = 0$. Since the diagonal entries of this submatrix are diagonal entries of $A[\alpha\,|\,\beta]$ we are done by the induction hypothesis. ■

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