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VERIFIED SOLUTION FOR A SIMPLE TRUSS STRUCTURE WITH UNCERTAIN NODE LOCATIONS

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Abstract. *We consider a structural truss problem where* all *of the physical model parameters are uncertain: not just the material values and applied loads, but also the positions of the nodes are assumed to be inexact but bounded and are represented by intervals. Such uncertainty may typically arise from imprecision during the process of manufacturing or construction, or round-off errors. In this case the application of the finite element method results in a system of linear equations with numerous interval parameters which cannot be solved conventionally. Applying a suitable variable substitution, an iteration method for the solution of a parametric system of linear equations is firstly employed to obtain initial bounds on the node displacements. Thereafter, an interval tightening (pruning) technique is applied, firstly on the element forces and secondly on the node displacements, in order to obtain tight guaranteed enclosures for the interval solutions for the forces and displacements.*

1 INTRODUCTION

Many sources of uncertainty exist in models for the analysis of structural mechanics problems. These include, e.g., measurement imprecision, manufacturing or fabrication imperfections, and round-off errors. An uncertain quantity is often assumed to be unknown but bounded, i.e. lower and upper bounds for this quantity can be provided (without assigning any probability distribution). Therefore, these quantities can be represented by intervals. Interval arithmetic, e.g. [1, 12], provides the means to keep track of such uncertainties throughout the whole computation. Consequently, the result, which is again an interval quantity, is *guaranteed* to contain the exact result.

The numerical method most frequently used in structural mechanics is the finite element method (FEM). Its accuracy is affected by discretisation and rounding errors and model and data uncertainty. In this paper we focus on parametric uncertainty and rounding errors. The source of parametric uncertainty (sometimes also called data uncertainty) is the lack of precise data needed for the analysis. In the FEM, parameters describing the geometry, material, and loads may be uncertain. Parametric uncertainty may result from a lack of knowledge (*epistemic uncertainty* or *reducible uncertainty*), e.g. loads are not exactly known, or an inherent variability (*aleatory uncertainty* or *irreducible uncertainty*) in the parameters, e.g. material parameters are only known to vary within known bounds, cf. [9].

For a decade or more the interval arithmetic approach has been used to handle parameter uncertainty in the application of the FEM to problems in structural mechanics, e.g. [3, 4, 10, 11, 13, 14, 15], to name but a few. Most of these papers consider the case of affine parametric dependency. Typically, more advanced models involve polynomial or rational parameter dependencies, in which case the coefficients of the systems of linear equations to be solved are polynomial or rational functions of the parameters. In [5] we present an approach to solve such systems. Therein we employ a general-purpose fixed-point iteration using interval arithmetic and an efficient method for bounding the range of a multivariate polynomial over a given box based on the expansion of this polynomial into Bernstein polynomials [6, 17]. As an example, we discuss a two-bay two-story frame involving 13 and 37 parameters.

The problem that the lengths of the bars of a truss system are uncertain, due to fabrication errors, is considered in [10]. However, in real-life problems, not only the lengths are uncertain but also the positions of the nodes are not exactly known. To the best of our knowledge, this problem has not been considered so far in the literature.

In this paper we present a simple model with uncertain node locations, consisting of six linear truss elements joined at five nodes. As well as uncertain node coordinates, the material values (Young's modulus and cross-sectional area) and loading forces are also interval parameters. As a consequence of the uncertain node locations, both the element lengths and angles in the problem are also interval values. With a suitable choice of variable substitution for the angles appearing in the system matrix, the resulting parametric system of linear equations is firstly solved by the aforementioned general-purpose parametric fixed-point iteration. However, the tightness of the resulting displacement intervals is not wholly satisfactory. Therefore, two interval pruning techniques are applied to compute and contract the interval enclosures for the element forces and node displacements. These intervals are compared to a tight inner estimation of the true interval solution obtained by a Monte Carlo simulation.

This paper is organised as follows. The next section consists of a brief introduction to interval arithmetic. The model is then presented in detail in Section 3, along with its parameter values. The collection of methods used to solve the problem, including the iteration method for

parametric systems and two interval pruning techniques are described in Section 4. The numerical results may be found in Section 5 and we conclude with some suggestions for continuation of this work.

2   INTERVAL ARITHMETIC

Let $\mathbb{IR}$ denote the set of the compact, nonempty real intervals. The arithmetic operation $\circ \in \{+, -, \cdot, /\}$ on $\mathbb{IR}$ is defined in the following way. If $a = [\underline{a}, \overline{a}], b = [\underline{b}, \overline{b}] \in \mathbb{IR}$, then

$$
\begin{aligned}
a + b &= [\underline{a} + \underline{b}, \overline{a} + \overline{b}], \\
a - b &= [\underline{a} - \overline{b}, \overline{a} - \underline{b}], \\
a \cdot b &= [\min\{\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}\}, \max\{\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}\}], \\
a / b &= [\min\{\underline{a}/\underline{b}, \underline{a}/\overline{b}, \overline{a}/\underline{b}, \overline{a}/\overline{b}\}, \max\{\underline{a}/\underline{b}, \underline{a}/\overline{b}, \overline{a}/\underline{b}, \overline{a}/\overline{b}\}], \text{ if } 0 \notin b.
\end{aligned}
$$

As a consequence of these definitions we obtain the inclusion isotonicity of the interval arithmetic operations: If $a_1, b_1 \in \mathbb{IR}$ with $a_1 \subseteq a$ and $b_1 \subseteq b$ then it holds that

$$a_1 \circ b_1 \subseteq a \circ b \text{ if } a_1 \circ b_1 \text{ is defined.}$$

Note that some relations known to be true in the set $\mathbb{R}$, e.g. the distributive law, are not valid in $\mathbb{IR}$. Here we have the weaker subdistributive law

$$a \cdot (b + c) \subseteq ab + ac \text{ for } a, b, c \in \mathbb{IR}.$$

By $\mathbb{IR}^n$ and $\mathbb{IR}^{n \times n}$ we denote the set of n-vectors and n-by-n matrices with entries in $\mathbb{IR}$, respectively.

Further details on arithmetic with intervals may be found in [1, 12].

3   THE MODEL

We consider the simple mechanical truss structure comprising five nodes connected by six linear elements as depicted in Figure 1; the elements are numbered in circles and the coordinates of the nodes are also given. Two of the nodes, 1 and 2, are fixed; the other three are free-moving. A downward loading force of 50kN is separately applied to both nodes 4 and 5.

Upon loading, we wish to compute the displacements of nodes 3–5, viz. $u_3, v_3, u_4, v_4, u_5, v_5$, and the resultant normal forces in all six elements, $S_1, \ldots, S_6$. Each of these is an interval quantity since the uncertainty in the input data causes uncertainty in the solution. We wish to compute intervals which tightly contain the true ranges of values for each of these variables.

The uncertain paramaters are as follows (see also Table 1):

- The positions of the five nodes of the truss (before loading) are subject to an uncertainty of ± 0.005 in both the x- and y-directions. With metres as the coordinate units, this corresponds to a variation of ± 5mm. Correspondingly, the elements are of uncertain length (depending upon configuration, they may vary upto $\pm 10\sqrt{2}$mm).

- The product of the elements' cross-sectional area with the Young's modulus is subject to an uncertainty of $\pm 5\%$. The nominal value is taken as an IPE 160 steel element ($A = 20.1$cm^2, $E = 2.1 * 10^8$kN/m^2). This results in $EA := [400995, 443205]$. Note that there is a single, global EA parameter.
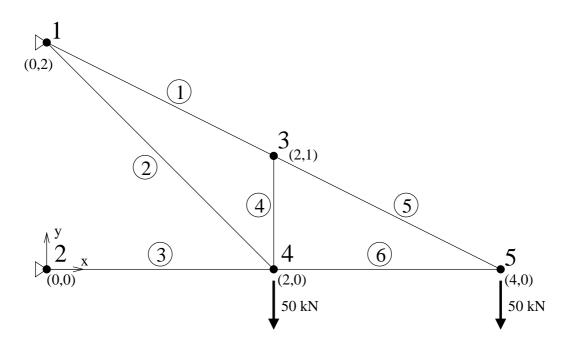
Figure 1: Mechanical Truss Model with Six Elements

- The loading forces applied to all nodes are subject to an uncertainty of ±1kN in both the x- and y-directions. This applies even to nodes which do not have a loading force applied (i.e. node 3).

Table 1: Interval Parameters for the Truss Model

Parameter		Nominal Value	Uncertainty
Young's modulus $*$ area EA		422100 kN	±21105 kN ($\pm5\%$)
Node coordinates	(x_1, y_1)	$(0, 2)$	±0.005 m
	(x_2, y_2)	$(0, 0)$	±0.005 m
	(x_3, y_3)	$(2, 1)$	±0.005 m
	(x_4, y_4)	$(2, 0)$	±0.005 m
	(x_5, y_5)	$(4, 0)$	±0.005 m
Loading forces	F_{x3}, F_{y3}	0 kN, 0 kN	±1 kN
	F_{x4}, F_{y4}	0 kN, -50 kN	±1 kN
	F_{x5}, F_{y5}	0 kN, -50 kN	±1 kN

4 METHODOLOGY

Our solution procedure consists of the following stages:

1. Application of a variable substitution to generate the symbolic system stiffness matrix appearing in the FEM in terms of the interval parameters.

2. Initial enclosures for the node displacements obtained by applying a parametric solver to the interval system.

3. Initial enclosures for the element forces computed from these node displacements.

4. An interval tightening method applied to the element forces.

5. An interval tightening method applied to the node displacements.

4.1 Finite element method

The usual FEM [2, 18] proceeds by the assemblage of a single large system of linear equations. For each structural element in the problem (see Figure 2), an *element stiffness matrix* is created, expressed in terms of $\cos\theta$, $\sin\theta$, EA, and L, the element length.
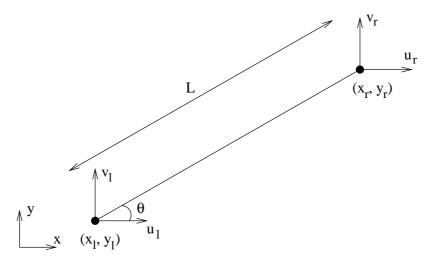

Figure 2: Arrangement of a Single Element Connecting Left-Hand and Right-Hand Nodes

Since the node locations are uncertain, the angles of the various elements are also interval quantities. However, the angles and the element lengths are only implicit interval parameters. Therefore, by means of the following substitutions, we can rearrange each element matrix so that it is expressed only in terms of the explicit interval parameters, viz. EA and the node coordinates, (x_l, y_l) and (x_r, y_r):

$$
\begin{aligned}
\cos\theta &= \frac{x_r - x_l}{L} \\
\sin\theta &= \frac{y_r - y_l}{L} \\
L &= \sqrt{(x_r - x_l)^2 + (y_r - y_l)^2}
\end{aligned}
\tag{1}
$$

This yields the following element stiffness matrix:

$$
k = \frac{EA}{((x_r-x_l)^2+(y_r-y_l)^2)^{\frac{3}{2}}} \cdot
$$
$$
\begin{pmatrix}
(x_r - x_l)^2 & (x_r - x_l)(y_r - y_l) & -(x_r - x_l)^2 & -(x_r - x_l)(y_r - y_l) \\
(x_r - x_l)(y_r - y_l) & (y_r - y_l)^2 & -(x_r - x_l)(y_r - y_l) & -(y_r - y_l)^2 \\
-(x_r - x_l)^2 & -(x_r - x_l)(y_r - y_l) & (x_r - x_l)^2 & (x_r - x_l)(y_r - y_l) \\
-(x_r - x_l)(y_r - y_l) & -(y_r - y_l)^2 & (x_r - x_l)(y_r - y_l) & (y_r - y_l)^2
\end{pmatrix}
$$

The global system stiffness matrix K is assembled in the usual way. If we were to solve the resultant system of equations in the conventional fashion, i.e. by substituting each of the

5

variables by its literal value (in this case, intervals instead of floating-point numbers), we would need to apply a linear system solver (e.g. interval Gaussian elimination [1, 12]), using interval arithmetic where required. However, we will see that in the interval case such an approach is hopeless. Instead, we must store the system matrix in symbolic form.

It is worth mentioning that alternative element stiffness matrices can be obtained by the use of the following alternative transformation:

$$\cos\theta = \frac{1-t^2}{1+t^2}, \quad \sin\theta = \frac{2t}{1+t^2}, \tag{2}$$

where $t = tan\frac{\theta}{2}$. In this case the element stiffness matrix is as follows:

$$k = \frac{EA}{L(1+t^2)^2} \cdot \begin{pmatrix} (1-t^2)^2 & 2t(1-t^2) & -(1-t^2)^2 & -2t(1-t^2) \\ 2t(1-t^2) & 4t^2 & -2t(1-t^2) & -4t^2 \\ -(1-t^2)^2 & -2t(1-t^2) & (1-t^2)^2 & 2t(1-t^2) \\ -2t(1-t^2) & -4t^2 & 2t(1-t^2) & 4t^2 \end{pmatrix}$$

We shall use the former transformation; the relative merits of each are briefly discussed in Section 5.2.

4.2   Parametric system solution

We now have a system of linear equations for the node displacements $u_i, v_i, i = 3, \ldots, 5$:

$$Ku = F,$$

where K is the global system stiffness matrix assembled from the element stiffness matrices k_i, $i = 1, \ldots, 6$, $u = (u_3\ v_3\ u_4\ v_4\ u_5\ v_5)^T$ is the vector of node displacements and $F = (F_{x_3}\ F_{y_3}\ F_{x_4}\ F_{y_4}\ F_{x_5}\ F_{y_5})^T$ is the vector of loading forces.

We will now consider the general case of a system of linear equations with interval parameters. Suppose we have a linear system

$$A(p) \cdot x = b(p), \tag{3}$$

where the coefficients of the $m \times m$ matrix $A(p)$ and the vector $b(p)$ are functions of n parameters varying within given intervals

$$a_{ij}(p) = a_{ij}(p_1, \ldots, p_n), \quad b_i(p) = b_i(p_1, \ldots, p_n), \quad i, j = 1, \ldots, m, \tag{4}$$
$$p \in [p] = ([p_1], \ldots, [p_n])^T. \tag{5}$$

The set of solutions to the above system, called the *parametric solution set*, is

$$\Sigma = \Sigma\left(A(p), b(p), [p]\right) \quad := \quad \{x \in \mathbb{R}^m \mid A(p) \cdot x = b(p) \text{ for some } p \in [p]\}.$$

The set Σ is compact if $A(p)$ is nonsingular for every $p \in [p]$. For a nonempty bounded set $\mathcal{S} \subseteq \mathbb{R}^m$, define its interval hull by $\square\,\mathcal{S} := \cap\{[s] \in \mathbb{IR}^m \mid \mathcal{S} \subseteq [s]\}$. It is generally expensive to obtain Σ or $\square\Sigma$, so instead we seek an interval vector Ω for which it is guaranteed that $\Omega \supseteq \square\Sigma \supseteq \Sigma$.

We apply a general-purpose self-verified method for bounding the solution set of a parametric linear system, which does not assume any particular structure among the parameter dependencies. This method derives from inclusion theory for nonparametric problems, see [16] and

the references therein. In [16, Theorem 4.8] a straightforward generalisation to linear systems with linear parameter dependencies is given. The corresponding theorems can be modified and applied to linear systems involving nonlinear parameter dependencies [14, 15]. The following is a general formulation of the enclosure method for linear systems involving arbitrary parametric dependencies.

Theorem 1. *Consider a parametric linear system defined by Eqs. 3 – 5. Let $R \in \mathbb{R}^{m \times m}$, $[y] \in \mathbb{IR}^m$, $\tilde{x} \in \mathbb{R}^m$ be given and define $[z] \in \mathbb{IR}^m$, $[C] \in \mathbb{IR}^{m \times m}$ by*

$$
\begin{aligned}
[z] &:= \square\{z(p) = R\,(b(p) - A(p)\tilde{x}) \mid p \in [p]\}, \\
[C] &:= \square\{C(p) = I - R \cdot A(p) \mid p \in [p]\},
\end{aligned}
$$

where I denotes the identity matrix. Define $[v] \in \mathbb{IR}^m$ by means of the following Gauss-Seidel iteration

$$
[v_i] := \{[z] + [C] \cdot ([v_1], ..., [v_{i-1}], [y_i], \ldots, [y_m])^T\}_i, \quad 1 \le i \le m.
$$

If $[v] \subsetneq [y]$, then R and every matrix $A(p)$ with $p \in [p]$ are regular, and for every $p \in [p]$ the unique solution $\widehat{x} = A^{-1}(p)b(p)$ of the system defined by Eqs. 3–5 satisfies $\widehat{x} \in \tilde{x} + [v]$.

In our computations we have chosen $R \approx \check{A}^{-1}$ and $\tilde{x} = R \cdot b(\check{p})$, where $\check{p}$ is the midpoint of $[p]$ and $\check{A} = A(\check{p})$.

The above theorem generalises [16, Theorem 4.8] by requiring a sharp enclosure of $C(p) := I - R \cdot A(p)$ for $p \in [p]$, instead of using the interval extension $C([p])$. Examples demonstrating the application of the generalised inclusion theorem can be found in [14, 15]. A detailed description of this algorithm can be found in [14].

4.3 Initial node displacement intervals

After applying the above parametric solver, we have preliminary interval enclosures for the node displacements. By means of the following formula from the FEM [18, p. 98], evaluated using interval arithmetic,

$$
S_i = \frac{EA}{L}(-\cos\theta \ -\sin\theta \ \cos\theta \ \sin\theta)u_i, \tag{6}
$$

where S_i is the resulting normal force (either tension or compression) in element i and $u_i = (u_l \ v_l \ u_r \ v_r)^T$ is the vector of displacements for the ith element's left- and right-hand nodes, preliminary interval enclosures for the element forces can also be obtained.

4.4 Interval tightening (element forces)

At each free-moving node (nodes 3, 4, and 5 in our example), all forces (element forces and loading forces) must be in equilibrium, in both the x- and y-directions. For example, at node 3, the following must hold:

$$
\begin{aligned}
S_1 \cos\theta_1 + F_{x_3} &= S_4 \cos\theta_4 + S_5 \cos\theta_5 & (7) \\
S_1 \sin\theta_1 + F_{y_3} &= S_4 \sin\theta_4 + S_5 \sin\theta_5 & (8)
\end{aligned}
$$

These can be rearranged to give one or more explicit formulae for each element force. Again at node 3, we have:

$$S_1 = \frac{S_4 \cos\theta_4 + S_5 \cos\theta_5 - F_{x3}}{\cos\theta_1}$$

$$S_1 = \frac{S_4 \sin\theta_4 + S_5 \sin\theta_5 - F_{y3}}{\sin\theta_1}$$

$$S_4 = \frac{S_1 \cos\theta_1 - S_5 \cos\theta_5 + F_{x3}}{\cos\theta_4}$$

$$S_4 = \frac{S_1 \sin\theta_1 - S_5 \sin\theta_5 + F_{y3}}{\sin\theta_4}$$

$$S_5 = \frac{S_1 \cos\theta_1 - S_4 \cos\theta_4 + F_{x3}}{\cos\theta_5}$$

$$S_5 = \frac{S_1 \sin\theta_1 - S_4 \sin\theta_4 + F_{y3}}{\sin\theta_5}$$

We apply the following interval tightening (also sometimes known as pruning) technique:

- Using the current values for the element forces, evaluate each of the above formulae in turn, to obtain new interval enclosures for the forces.

- For each element force, the current interval value is intersected with the new computed enclosure(s), yielding a narrower or identical interval.

This procedure is iterated (for all nodes) as desired until the set of resulting set of intervals for the element forces do not contract any further.

4.5 Interval tightening (node displacements)

Using (6) instead of the force equilibrium equations, the above procedure could simply be applied in a similar fashion in order to contract the intervals for the node displacements. While this does indeed achieve a significant contraction of the displacement intervals, their widths are still wider than one would like. This reduced effectiveness is due to the greater number of interval quantities appearing in (6).

We therefore employ a slightly more sophisticated pruning technique. Firstly note that L_i, the length of element i, is related to its normal force S_i by

$$S_i = \frac{EA(L_i - L_{i_0})}{L_{i_0}}, \tag{9}$$

where L_{i_0} is the starting length of element i (i.e. before any loading forces are applied).

For this procedure, we take the new, tight enclosures for the element normal forces obtained above, and from (9) we use interval arithmetic to calculate an interval value for L_i, $i = 1, \ldots, 6$. These L_i interval values will stay fixed.

Now consider element i. Assume for the time being that the displacement of its left-hand node is a known point value, as is the angle θ_i. Given that we know the length, L_i, to within certain bounds, what is the set of possible displacements of the right-hand node (u_r, v_r) which will satisfy this length requirement? As illustrated in Figure 3, this set is bounded by two parallel lines which are perpendicular to the element.
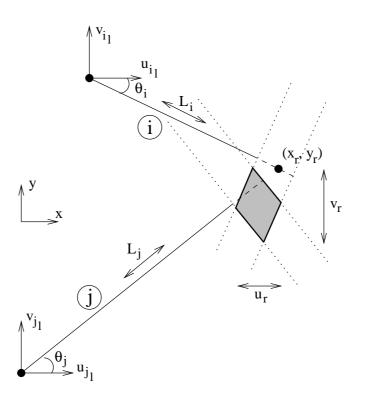
Figure 3: Bounds on the Displacements of a Node Due to Two Elements of Interval Length

Now consider element j, where $i \neq j$, where elements i and j share the same right-hand node. Making the same assumptions about element j, and provided that the two elements are not parallel, the set of possible displacements which will satisfy *both* length requirements is bounded by the intersection of two such pairs of parallel lines, which describes a parallelogram (see Figure 3). By taking the smallest bounding box surrounding this parallelogram, we obtain new bounds for (u_r, v_r).

However, the displacements of the left-hand nodes and the angles of the elements are not point values, which complicates the issue. We thus pursue a combinatorial solution: Let each of these interval values take either their left- or right-endpoint. There are $2^6 = 64$ possible permutations among $\{u_{i_l}, v_{i_l}, \theta_i, u_{j_l}, v_{j_l}, \theta_j\}$. For each such permutation, we can compute the parallelogram intersection and its bounding box. We compute the smallest bounding box containing *all* these parallelograms as a new interval enclosure for (u_r, v_r).

Taking each node in turn, the method thus proceeds as follows:

- Take every possible pair of non-parallel elements which meet at the node, in turn. For example, at node 3 we may consider elements 1 and 4, which meet there, followed by elements 4 and 5 (but not elements 1 and 5, which are parallel, i.e. the intersection of their interval angles θ_1 and θ_5 is non-empty).

- For each such pair, compute an interval enclosure for the displacement of their common node, in the x- and y-directions, as above.

- Take the intersections of these new interval(s) with the current values for the displacement of the node.

Again, this procedure is iterated (for all nodes) as desired until the set of resulting set of displacement intervals do not contract any further.

9

5 RESULTS

In this section the aforementioned methods are applied to the model described in Section 3. We aim to compute intervals for the element forces and node displacements that are guaranteed to contain the true solution, which consists of the set of interval ranges for these quantities when each interval parameter is allowed to vary independently within its domain. Our solution intervals should enclose the true solution as tightly as possible, minimising the overestimation associated with the well-known dependency problem in interval arithmetic, e.g. [12, p. 16–17].

To obtain the initial values for the displacements $u_i, v_i, i = 3, \ldots, 5$, an existing implementation of the parametric system solver for the *Mathematica* environment has been used [14]. The other steps have been implemented in C++; apart from the Monte Carlo method, which is run by way of comparison to estimate the true interval solution to the problem, these are interval methods with interval variables and parameters, using interval arithmetic in place of floating-point arithmetic. The computational results are thus guaranteed, even accounting for rounding errors. The C++ interval library `filib++` [7] is employed.

5.1 Monte Carlo simulation

We wish to firstly compute a tight inner estimation to the true interval solution to the problem, for which the well-known Monte Carlo method is used. This is only done so as to obtain a close approximation to the true result, so as to be able to judge the quality of the guaranteed solution obtained by the other methods.

All starting interval parameters are replaced by point values which are randomly chosen within their domains, and the point problem is solved, using the standard FEM. The result intervals are computed as the interval hulls (see Section 4.2) of the solutions to the point problems. Sufficient (here, 10^6) point problems are run in order to provide a relatively tight inner estimation to the true interval solution.

The inner estimations for the element forces and node displacements obtained by the Monte Carlo simulation are given in Table 4.

5.2 Finite element method

With the variable substitution (1), the FEM yields a system of interval equations. However, the literal intervals appearing in the system stiffness matrix K are of sufficient large width that it is not possible to solve the system using interval Gaussian elimination with partial pivoting, e.g. [8, Sect. 7] and [12, p. 157]. A naive application of the FEM in the interval case will almost always fail or deliver result intervals that are hopelessly wide.

We note that by using the alternative variable substitution (2) the interval entries of K actually become slightly narrower. However this does not suffice for the system to become solvable. Also, this transformation is less suitable for the parametric solution, due to the presence of implicit interval parameters for the element lengths (which depend on the explicit parameters for the node coordinates). This causes the result intervals for the displacements to be wider, since this dependency is not taken into account.

5.3 Parametric solution

The parametric solver from Section 4.2 delivers intervals for the node displacements which are given as the starting values in Table 3. Applying (6), these values are used to generate intervals for the element forces, which are given as the starting values in Table 2. By themselves,

Table 2: Results of Interval Tightening on the Element Forces

Starting Values	Element Force	Value
	S_1	$[-45.95638288, 282.90173927]$
	S_2	$[-48.82304461, 198.42523281]$
	S_3	$[-250.21554428, -57.41859127]$
	S_4	$[-590.63155262, 589.77215288]$
	S_5	$[-388.47735525, 622.81537800]$
	S_6	$[-324.87728856, 113.01160160]$
Iteration 1		
	S_1	$[-45.95638288, 282.90173927]$
	S_2	$[-48.82304461, 198.42523281]$
	S_3	$[-250.21554428, -57.41859127]$
	S_4	$[-94.90232257, 88.58968408]$
	S_5	$[104.66035897, 119.11859164]$
	S_6	$[-291.88018818, 90.55655220]$
Iteration 2		
	S_1	$[101.85101412, 122.03185725]$
	S_2	$[-48.82304461, 198.42523281]$
	S_3	$[-250.21554428, -57.41859127]$
	S_4	$[-75.71394558, 82.79650518]$
	S_5	$[105.02672841, 118.74332277]$
	S_6	$[-107.86134409, -92.32798825]$
Iteration 5		
	S_1	$[105.22056138, 118.56853389]$
	S_2	$[54.29348579, 87.12727064]$
	S_3	$[-169.90212960, -130.39412338]$
	S_4	$[-12.63973687, 7.35954242]$
	S_5	$[107.09448776, 116.62533577]$
	S_6	$[-105.64379816, -94.47996700]$
Iteration 10		
	S_1	$[105.25687341, 118.53136075]$
	S_2	$[56.71628310, 84.77544565]$
	S_3	$[-166.94508270, -133.32673491]$
	S_4	$[-12.61489164, 7.32583063]$
	S_5	$[107.09474113, 116.62507624]$
	S_6	$[-105.62441053, -94.49878139]$

Table 3: Results of Interval Tightening on the Displacements

Starting Values	Node Displacement	Value
	u_3	$[-0.00063546, 0.000579938]$
	v_3	$[-0.00198524, -0.000768579]$
	u_4	$[-0.00111308, -0.000298213]$
	v_4	$[-0.00206827, -0.000683625]$
	u_5	$[-0.00170792, -0.000654659]$
	v_5	$[-0.00768191, -0.00226932]$
Iteration 1		
	u_3	$[-0.00047061, 0.00041743]$
	v_3	$[-0.00198524, -0.00076858]$
	u_4	$[-0.00086044, -0.00057813]$
	v_4	$[-0.00174694, -0.00105952]$
	u_5	$[-0.00166646, -0.00070149]$
	v_5	$[-0.00681014, -0.00331722]$
Iteration 2		
	u_3	$[-0.00030706, 0.00023276]$
	v_3	$[-0.00177271, -0.00103279]$
	u_4	$[-0.00086044, -0.00057813]$
	v_4	$[-0.00174694, -0.00105952]$
	u_5	$[-0.00141118, -0.00098210]$
	v_5	$[-0.00596188, -0.00418864]$
Iteration 5		
	u_3	$[-0.00030707, 0.00023276]$
	v_3	$[-0.00177271, -0.00103279]$
	u_4	$[-0.00086044, -0.00057813]$
	v_4	$[-0.00174694, -0.00105952]$
	u_5	$[-0.00141061, -0.00098265]$
	v_5	$[-0.00585002, -0.00429726]$

Table 4: Comparison of Results

	Inner Estimation (Monte Carlo)	Outer Estimation (Param. Sol. & Tightening)
S_1	$[106.96910595, 116.62162466]$	$[105.25687341, 118.53136075]$
S_2	$[65.13454690, 76.28536251]$	$[56.71628310, 84.77544565]$
S_3	$[-156.77638723, -143.72365775]$	$[-166.94508270, -133.32673491]$
S_4	$[-7.73456140, 2.55284234]$	$[-12.61489164, 7.32583063]$
S_5	$[108.31623710, 115.37942086]$	$[107.09474113, 116.62507624]$
S_6	$[-104.25310839, -95.91022292]$	$[-105.62441053, -94.49878139]$
u_3	$[-0.00008324, 0.00002555]$	$[-0.00030707, 0.00023276]$
v_3	$[-0.00153302, -0.00124259]$	$[-0.00177271, -0.00103279]$
u_4	$[-0.00078219, -0.00064526]$	$[-0.00086044, -0.00057813]$
v_4	$[-0.00153224, -0.00124465]$	$[-0.00174694, -0.00105952]$
u_5	$[-0.00130037, -0.00107663]$	$[-0.00141061, -0.00098265]$
v_5	$[-0.00547419, -0.00459541]$	$[-0.00585002, -0.00429726]$

these result intervals for the displacements are rather wide and thus not completely satisfactory. The resulting intervals for the element forces are much too wide and are unsatisfactory.

The computation time was 14.2 seconds on a PC with an AMD Athlon-64 3GHz processor running the *Mathematica* environment. It should be noted that such a parametric solution rapidly becomes very time-consuming for larger systems.

5.4 Interval tightening (element forces)

The results of applying the interval tightening procedure (Section 4.4) to the element forces obtained above are given in Table 2. The intervals converge rapidly in the first couple of iterations; 10 iterations suffice to achieve convergence to 8 decimal places, for which the computation time is negligible.

The final intervals for the element forces are given in Table 4. Compared to the inner estimates obtained from the Monte Carlo method, we see that the intervals for S_1, S_5, and S_6 are tight. Those for S_2, S_3, and S_4 are not quite so tight, but still acceptable.

5.5 Interval tightening (node displacements)

The results of applying the interval tightening procedure (Section 4.5) to the node displacements obtained by the parametric solution are given in Table 3. Here, 5 iterations suffice to achieve convergence to 8 decimal places. Again, the computation time is negligible.

By comparing with the inner estimates from the Monte Carlo method (see Table 4), we see that the interval enclosures for the node displacements are all of a similar quality, about twice the width of the true solution. This is a noticable improvement on the values obtained from the parametric solver alone.

6 CONCLUSIONS

We have considered a structural truss model for which the node locations, as well as all other parameters, are uncertain. We have performed a suitable variable substitution in order to apply

a parametric solver and have devised interval tightening procedures for both the element forces and node displacements, which deliver a significant improvement to the results. Through the use of interval arithmetic, the result intervals are guaranteed to contain the true solution. The remaining overestimation is due to some lingering occurences of the dependency problem, at least in the current formulation.

Initial investigations have shown that it may be possible to improve the results obtained by the parametric solver, by augmenting the system of equations and the system stiffness matix with additional equations and variables for the element forces (6), adding extra dependencies to the system. However, the resultant impact on the tightening procedure is minimal.

In future, we wish to explore how effectively the method may be applied to truss structures with a greater number of elements and nodes. In such cases it may be necessary to exploit monotonicity arguments so that the parametric solution remains a viable approach.

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