PAMM · Proc. Appl. Math. Mech. **10**, 157 – 158 (2010) / **DOI** 10.1002/pamm.201010071

# A Verified Monotonicity-Based Solution of a Simple Finite Element Model with Uncertain Node Locations

**Andrew P. Smith**[1], **Jürgen Garloff**[1,*], and **Horst Werkle**[2]

[1] Faculty of Computer Science, University of Applied Sciences (HTWG) Konstanz, Postfach 100543, D-78405 Konstanz
[2] Faculty of Civil Engineering, University of Applied Sciences (HTWG) Konstanz, Postfach 100543, D-78405 Konstanz

A tight verified solution enclosure is obtained for the node displacements of a simple truss model, whose parameters, including the node locations, are uncertain. The solution is based on a monotonicity analysis of these interval parameters.



## 1 Introduction

Many sources of uncertainty exist in structural mechanics problems, including measurement imprecision, manufacturing imperfections, and round-off errors. These quantities can be represented by intervals, and interval arithmetic, e.g., [1], can be used to track uncertainties throughout the whole computation, yielding an interval result which is guaranteed to contain the exact result. The finite element method (FEM) is frequently used in structural mechanics. However, in the case of a problem where some of the physical model parameters are uncertain, the application of the FEM results in a system of linear equations with numerous interval parameters which cannot be solved conventionally – a naive implementation in interval arithmetic typically delivers result intervals that are excessively large. Various approaches, e.g., [2–4], have been used in order to adapt the interval approach to parameter uncertainty in the application of the FEM to problems in structural mechanics. Such problems exhibit uncertainty in either the material values or the loading forces. In this note, a structural truss problem where *all* of the physical model parameters are uncertain is presented: not just the material values and applied loads, but also the positions of the nodes are assumed to be inexact but bounded and are represented by intervals.

## 2 The model

We consider the simple mechanical truss structure comprising five nodes connected by seven elements as depicted in Figure 1, where the elements are numbered in circles. Two of the nodes, 1 and 2, are fixed; the other three are free-moving. A downward loading force of 50kN is separately applied to both nodes 4 and 5. This is adapted from the model in [6] by the addition of an extra element, making it no longer statically-determinate.

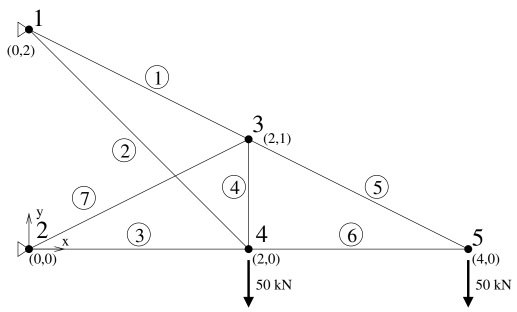


**Fig. 1** A simple mechanical truss model comprising five nodes and seven elements.

The model parameters are given in Table 1. The positions of the five nodes of the truss (before loading) are subject to an uncertainty of $\pm 0.005$m in both the $x$- and $y$-directions. The product of the elements' cross-sectional area with the Young's modulus is subject to an uncertainty of $\pm 5\%$. The nominal value is taken as an IPE 160 steel element ($A = 20.1\text{cm}^2$, $E = 2.1 * 10^8\text{kN/m}^2$). The two non-zero loading force components are subject to an uncertainty of $\pm 1$kN.

## 3 Verified solution

The usual FEM proceeds by the assemblage of a single large system of linear equations $Ku = F$, where $K$ is the system stiffness matrix, $u = (u_3, v_3, \ldots, u_5, v_5)^T$ is the vector of the displacements of nodes 3-5, and $F$ is the corresponding vector

* Corresponding author E-mail: garloff@htwg-konstanz.de, Phone: +49 7531 206627, Fax: +49 7531 206559

**Table 1** Interval parameters for the seven-element truss model.

| Parameter | | Nominal Value | Uncertainty |
|---|---|---|---|
| Young's modulus $*$ area | EA | 422100 kN | $\pm$21105 kN ($\pm$5%) |
| Node coordinates | $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5)$ | $(0, 2), (0, 0), (2, 1), (2, 0), (4, 0)$ | $\pm$0.005 m |
| Loading forces | $F_{y_4}, F_{y_5}$ | $-50$ kN, $-50$ kN | $\pm$1 kN |

**Table 2** Monotonicity information; $+$ or $-$ indicates strictly increasing or decreasing over the whole parameter domain; $(+)$ or $(-)$ indicates strictly increasing or decreasing over a subset of the parameter domain where the max. or min. occurs.

| **Parameter:** | $x_1$ | $y_1$ | $x_2$ | $y_2$ | $x_3$ | $y_3$ | $x_4$ | $y_4$ | $x_5$ | $y_5$ | $F_{y_4}$ | $F_{y_5}$ | $EA$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $u_3$ | $-$ | | $(+)$ | $(-)$ | $-$ | $(-)$ | | $+$ | | $+$ | | $-$ | $-$ |
| $v_3$ | $+$ | $+$ | $+$ | $-$ | | | | | | | $+$ | $+$ | $+$ |
| $u_4$ | | | $+$ | $-$ | | $(+)$ | | | $-$ | | $+$ | $+$ | $+$ |
| $v_4$ | $+$ | $+$ | $+$ | $-$ | | $+$ | | $-$ | $-$ | $-$ | $+$ | $+$ | $+$ |
| $u_5$ | | | $+$ | $-$ | $+$ | | $-$ | | | | $+$ | $+$ | $+$ |
| $v_5$ | $+$ | $(+)$ | | $-$ | $+$ | $+$ | $-$ | | $-$ | $+$ | $+$ | $+$ | $+$ |

**Table 3** Enclosures for the node displacements.

| | **Outer Estimation (Interval Solver)** | **Outer Estimation (Monotonicity)** | **Inner Estimation (Monte Carlo)** |
|---|---|---|---|
| $u_3$ | $[0.00018943, 0.00055964]$ | $[0.00028110, 0.00038895]$ | $[0.00030308, 0.00036239]$ |
| $v_3$ | $[-0.00183366, -0.00058841]$ | $[-0.00181882, -0.00059111]$ | $[-0.00103362, -0.00086732]$ |
| $u_4$ | $[-0.00119331, -0.00037243]$ | $[-0.00097503, -0.00039357]$ | $[-0.00066723, -0.00055739]$ |
| $v_4$ | $[-0.00190384, -0.00063106]$ | $[-0.00111894, -0.00082202]$ | $[-0.00108971, -0.00091329]$ |
| $u_5$ | $[-0.00196654, -0.00071342]$ | $[-0.00150969, -0.00082110]$ | $[-0.00119584, -0.00098396]$ |
| $v_5$ | $[-0.00869767, -0.00360912]$ | $[-0.00577767, -0.00454714]$ | $[-0.00556608, -0.00466507]$ |

of loading forces. Taking the partial derivatives with respect to a chosen parameter $p$ yields

$$K \frac{\partial u}{\partial p} = \left( \frac{\partial F}{\partial p} - \frac{\partial K}{\partial p} u \right),$$

where $p$ may be any single parameter in Table 1. Given suitably tight enclosures for the quantities appearing in the right-hand side of this system of equations, these systems can be solved for the partial derivatives of the node displacements w.r.t. each parameter, in turn. The solution procedure consists of the stages outlined below; a similar scheme was used in [5]. An interval system solver is used throughout to compute the interval hull of the solution set of a system of linear interval equations.

1. Construct the system of interval equations $Ku = F$. The widths of the intervals appearing in the element stiffness matrices can be minimised by elementary analysis.
2. Solve to yield an initial enclosure $u^{\{0\}}$ for the node displacements.
3. Construct systems of interval equations for the partial derivatives of the node displacements w.r.t. each interval parameter, using the current enclosure $u^{\{i\}}$. Solve to yield outer enclosures for the partial derivatives; where zero is excluded, monotonicity is proven.
4. Attempt to minimise/maximise each solution component in turn by restricting the parameter domain for monotone parameters and thusly reconstructing and solving the original system.
5. Iterate 3–4, using both successively tighter solution enclosures and monotonicity information obtained so far, until as many of the solution components as possible are found to be monotone over the restricted parameter domains.

The monotonicity information obtained is displayed in Table 2. Table 3 shows the results obtained from the interval solver, before and after exploiting the monotonicity tests, and a Monte Carlo simulation with $10^6$ runs, for comparison. Exploitation of the monotonicity information can thus be seen to yield a significant tightening of the displacement intervals. A similar procedure can be used for the element forces, which will be detailed in a forthcoming publication.

**Acknowledgements** We gratefully acknowledge support from the State of Baden-Württemberg, Germany.

## References

[1] G. Alefeld and J. Herzberger, Introduction to Interval Computations (Academic Press, New York, 1983).
[2] G. Corliss, C. Foley, and R. B. Kearfott, Formulation for reliable analysis of structural frames, Reliable Computing **13**, 125–147 (2007).
[3] J. Garloff, E. D. Popova, and A. P. Smith, Solving linear systems with polynomial parameter dependency in the reliable analysis of structural frames, Proc. 2nd Int. Conf. Uncertainty in Struct. Dynamics, N. Sims and K. Worden (eds.), Sheffield, UK, 147–156 (2009).
[4] R. L. Muhanna, A. Erdolen, and R. L. Mullen, Geometric uncertainty in truss systems: an interval approach, Proc. 2nd Int. Workshop on Reliable Engineering Computing, R L. Muhanna and R. L. Mullen (eds.), Savannah, Georgia, USA, 475–481 (2006).
[5] E. Popova, Computer-assisted proofs in solving linear parametric problems, Proceedings SCAN'06 (IEEE Computer Society Press, 2006), p. 35.
[6] A. P. Smith, J. Garloff, and H. Werkle, Verified solution for a simple truss structure with uncertain node locations, Proc. 18th Int. Conference on the Application of Computer Science and Mathematics in Architecture and Civil Engineering, K. Gürlebeck and C. Könke (eds.), Weimar, Germany (2009).

 