Application of Bernstein Expansion to the Solution of Control Problems †

Jürgen Garloff

Fachhochschule Konstanz / University of Applied Sciences
FB Informatik
Postfach 10 05 43
D-78405 Konstanz, Germany
garloff@fh-konstanz.de

Abstract. We survey some recent applications of Bernstein expansion to robust stability, viz. checking robust Hurwitz and Schur stability of polynomials with polynomial parameter dependency by testing determinantal criteria and by inspection of the value set. Then we show how Bernstein expansion can be used to solve systems of strict polynomial inequalities.

Keywords. Robust Hurwitz stability, robust Schur stability, polynomial parameter dependency, Bernstein polynomials, polynomial inequalities.

1. Introduction

Bernstein expansion is now a well established tool for bounding the range of a multivariate polynomial over a box; for a nearly exhaustive bibliography, see [1]. In the univariate case, application of interval arithmetic to Bernstein expansion has started with a series of papers by Jon Rokne [2-5]. Fischer proved later in [6] that the use of subdivision provides quadratic convergence. Hong and Stahl [7] showed that the Bernstein form is inclusion monotone.

In this paper some recent applications of Bernstein expansion to control problems are surveyed. The organisation is as follows: In the next section the Bernstein transformation is recalled. In Sect. 3, the applications are presented, viz.

- checking a polynomial family with coefficients depending polynomially on parameters varying in given intervals for stability, in the case of Hurwitz and Schur stability as well as when damping is involved, i. e., the stability domains are the open left half of the complex plane, the open unit disc, and a sector centered around the negative real axis with its vertex at the origin, respectively;
- solving systems of strict polynomial inequalities as they appear, e.g., in robust feedback design.

Control theory provides a wealth of other applications of Bernstein expansion. Typical examples are the determinantal criteria listed in [8], cf. [9, Ch. 17]. Roughly speaking, one can apply Bernstein expansion as soon as the problem at hand can be reduced to strict inequalities (or equations) involving multivariate polynomials. Starting with a box of interest, the algorithm sequentially splits it into subboxes, eliminating infeasible boxes by using bounds for the range of the function f under consideration over each of them, and ending up with a union of boxes that contains all solutions to the problem at hand. This procedure is typical of domain-splitting algorithms, a family to which Bernstein expansion belongs. If f is a multivariate polynomial,

† Extended version of a paper presented at MISC'99 — Workshop on Applications of Interval Analysis to Systems and Control, Feb. 24–26, 1999, Girona, Spain.

such bounds for ranges can easily be calculated by Bernstein expansion. If f is an arbitrary function, not necessarily a polynomial, methods from interval computations [10], [11] can be applied, where the bounds for the range over subboxes are provided by so-called inclusion functions. Examples of application to robust stability and to parameter estimation of such domain-splitting algorithms which use methods from interval computations are given in [12], [13], [14]. In [15] the mean value form, the optimal centered form presented by Baumann [16], and the Taylor form are applied to robust performance problems and compared with Bernstein expansion. The examples presented therein show that in regard to computing time and to tightness of the computed bounds, Bernstein expansion compares favourably with interval methods. It should be noted that the implementation of the Bernstein expansion used in [15] could even be speeded up by exploiting a relationship between the Bernstein coefficients on neighbouring subboxes, cf. Sect. 2.2 below, and by applying a bisection direction rule based on the first partial derivative, cf. Sect. 2.3[1].

The use of determinantal criteria leads to fast algorithms if the number of parameters is small but is prohibitive for larger robust stability problems, where a value set approach, cf. [9, Ch. 7], appears more appropriate. Here interval methods cannot be applied directly, cf. [17], for an extension of standard interval arithmetic. We will see that Bernstein expansion provides tools for testing value sets efficiently. Brief conclusions and directions for future research conclude the paper.

2. Bernstein Expansion

For compactness, we will use multi-indices $I = (i_1, \ldots, i_l)$ and multi-powers $\mathbf{x}^I = x_1^{i_1} x_2^{i_2} \cdot \ldots \cdot x_l^{i_l}$ for $\mathbf{x} \in \mathbf{R}^l$. Inequalities $I \leq N$ for multi-indices are meant componentwise, where $0 \leq i_k, k = 1, \cdots, l$, is implicitly understood. With $I = (i_1, \ldots, i_{r-1}, i_r, i_{r+1}, \ldots, i_l)$ we associate the index $I_{r,k}$ given by $I_{r,k} = (i_1, \ldots, i_{r-1}, i_r + k, i_{r+1}, \ldots, i_l)$, where $0 \leq i_r + k \leq n_r$. With the notation $S = \{I : I \leq N\}$ we can write an l-variate polynomial p in the form

$$p(\mathbf{x}) = \sum_{I \in S} a_I \mathbf{x}^I, \quad \mathbf{x} \in \mathbf{R}^l, \tag{1}$$

and refer to N as the *degree* of p. Also, we write I/N for $(i_1/n_1, \ldots, i_l/n_l)$ and $\binom{N}{I}$ for $\binom{n_1}{i_1} \cdot \ldots \cdot \binom{n_l}{i_l}$.

2.1. Bernstein Transformation of a Polynomial

In this subsection we expand a given l-variate polynomial (1) into Bernstein polynomials to obtain bounds for its range over an l-dimensional box. Without loss of generality we consider the unit box $\mathbf{U} = [0, 1]^l$ since any nonempty box of $\mathbf{R}^l$ can be mapped affinely onto this box.

For $\mathbf{x} = (x_1, \ldots, x_l) \in \mathbf{R}^l$, the Ith *Bernstein polynomial* of degree N is defined as

$$B_{N,I}(\mathbf{x}) = b_{n_1,i_1}(x_1) b_{n_2,i_2}(x_2) \cdot \ldots \cdot b_{n_l,i_l}(x_l),$$

where for $i_j = 0, \ldots, n_j$, $j = 1, \ldots, l$

$$b_{n_j,i_j}(x_j) = \binom{n_j}{i_j} x_j^{i_j} (1 - x_j)^{n_j - i_j}.$$

The transformation of a polynomial from its power form (1) into its *Bernstein form* results in

$$p(\mathbf{x}) = \sum_{I \in S} b_I(\mathbf{U}) B_{N,I}(\mathbf{x}),$$

[1] It should further be mentioned that interval arithmetic is not properly implemented in [15] so that, e.g., rounding errors are not taken into account.

where the *Bernstein coefficients* $b_I(\mathbf{U})$ of p over $\mathbf{U}$ are given by

$$b_I(\mathbf{U}) = \sum_{J \leq I} \frac{\binom{I}{J}}{\binom{N}{J}} a_J, \quad I \in S. \tag{2}$$

We collect the Bernstein coefficients in an array $B(\mathbf{U})$, i.e., $B(\mathbf{U}) = (b_I(\mathbf{U}))_{I \in S}$. A similar notation will be employed for other sets of related coefficients. For an efficient calculation of the Bernstein coefficients, which does not use (2), see [18]. All rounding errors appearing in the computation of the Bernstein coefficients can be taken into account similarly as in [6].

In the following, a special subset of the index set S will be used, comprising those indices that correspond to the indices of the vertices of the array $B(\mathbf{U})$, i.e.,

$$S_0 = \{0, n_1\} \times \ldots \times \{0, n_l\}.$$

We now list some useful properties of the Bernstein coefficients, e.g., [19]. As usual, the convex hull of a set A is denoted by Conv$A$.

LEMMA 1. *Let p be a polynomial (1) of degree N. Then the following properties hold for its Bernstein coefficients $b_I(\mathbf{U})$ (2):*

i) Sharpness of special coefficients:

$$\forall I \in S_0 : b_I(\mathbf{U}) = p(I/N). \tag{3}$$

ii) Convex hull property:

$$Conv\{(\mathbf{x}, p(\mathbf{x})) : \mathbf{x} \in \mathbf{U}\} \subseteq Conv\{(I/N, b_I(\mathbf{U})) : I \in S\}. \tag{4}$$

iii) Range enclosing property:

$$\forall \mathbf{x} \in \mathbf{U} : \min_{I \in S} b_I(\mathbf{U}) \leq p(\mathbf{x}) \leq \max_{I \in S} b_I(\mathbf{U}) \tag{5}$$

with equality in the left (resp., right) inequality if and only if $\min_{I \in S} b_I(\mathbf{U})$ *(resp.,* $\max_{I \in S} b_I(\mathbf{U})$*) is attained at a Bernstein coefficient $b_I(\mathbf{U})$ with $I \in S_0$.*

Property (iii) is immediate from (i) and (ii).

2.2. Sweep Procedure

The bounds obtained by the inequalities (5) can be tightened if the unit box $\mathbf{U}$ is bisected into subboxes and Bernstein expansion is applied to the polynomial p on these subboxes, i.e., to the polynomial shifted from each subbox back to $\mathbf{U}$. A sweep in the rth direction $(1 \leq r \leq l)$ is a bisection perpendicular to this direction and is performed by recursively applying a linear interpolation. Let

$$\mathbf{D} = [\underline{d}_1, \overline{d}_1] \times \ldots \times [\underline{d}_l, \overline{d}_l]$$

be any subbox of $\mathbf{U}$ generated by sweep operations (at the beginning, we have $\mathbf{D} = \mathbf{U}$). Starting with $B^{(0)}(\mathbf{D}) = B(\mathbf{D})$ we set for $k = 1, \ldots, n_r$

$$b_I^{(k)}(\mathbf{D}) = \begin{cases} b_I^{(k-1)}(\mathbf{D}) : i_r < k \\ (b_{I_{r,-1}}^{(k-1)}(\mathbf{D}) + b_I^{(k-1)}(\mathbf{D}))/2 : k \leq i_r. \end{cases}$$

To obtain the new coefficients, this is applied for $i_j = 0, \ldots, n_j$, $j = 1, \ldots, r-1, r+1, \ldots, l$. Then the Bernstein coefficients on $\mathbf{D}_0$, where the subbox $\mathbf{D}_0$ is given by

$$\mathbf{D}_0 = [\underline{d}_1, \overline{d}_1] \times \ldots \times [\underline{d}_r, \hat{d}_r] \times \ldots \times [\underline{d}_l, \overline{d}_l],$$

with $\hat{d}_r$ denoting the midpoint of $[\underline{d}_r, \overline{d}_r]$, are obtained as $B(\mathbf{D}_0) = B^{(n_r)}(\mathbf{D})$. The Bernstein coefficients $B(\mathbf{D}_1)$ on the neighbouring subbox $\mathbf{D}_1$

$$\mathbf{D}_1 = [\underline{d}_1, \overline{d}_1] \times \cdots \times [\hat{d}_r, \overline{d}_r] \times \cdots \times [\underline{d}_l, \overline{d}_r]$$

are obtained as intermediate values in this computation, since for $k = 0, \ldots, n_r$ the following relation holds [20]:

$$b_{i_1,\ldots,n_r-k,\ldots,i_l}(\mathbf{D}_1) = b^{(k)}_{i_1,\ldots,n_r,\ldots,i_l}(\mathbf{D}).$$

By analogy with Computer Aided Geometric Design, we call the arrays of Bernstein coefficients $B(\mathbf{D}_0)$ and $B(\mathbf{D}_1)$ *patches*. A sweep needs $O(\hat{n}^{l+1})$ additions and multiplications, where $\hat{n} = \max\{n_i : i = 1, \ldots, l\}$, cf. [21]. Note that by the sweep procedure the explicit transformation of the subboxes generated by the sweeps back to $\mathbf{U}$ is avoided. Fig. 1 illustrates the sweeping process for $l = 2$ and $r = 1$.

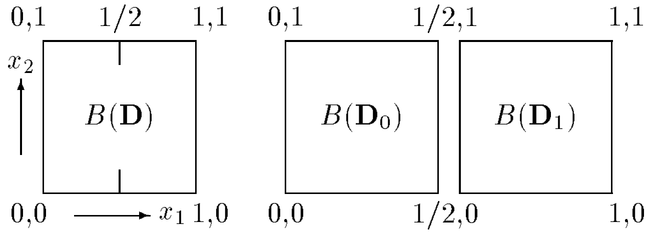

Figure 1. Two new patches are obtained by a sweep in the first direction.

2.3. Selection Procedures

The sweep direction remains to be chosen in the computation of the Bernstein coefficients. Our rule, cf. [21], is based on an upper bound associated with the first partial derivative of a polynomial in Bernstein form and takes advantage of the easy calculation of the partial derivatives of a polynomial in this form, see, e.g., [22]. In an attempt to split the box perpendicular to the direction of maximum variation, the direction for the sweep r_0 is taken as the value of r that maximises

$$\max_{I \leq N_{r,-1}} n_r \left| b_{I_{r,1}}(\mathbf{D}) - b_I(\mathbf{D}) \right| .$$

We conclude this section with a selection rule for the patches. In the applications which we will present in the next section, we will often have to check a polynomial for positivity over a box $\mathbf{D}$. After a sweep we have the choice between two patches $B(\mathbf{D}_0)$ and $B(\mathbf{D}_1)$. In order to find a nonpositive sharp Bernstein coefficient, cf. (3), as soon as possible, we continue on the patch having smallest Bernstein coefficient (*depth first strategy*).

3. Applications

Consider the family of polynomials

$$p(s, \mathbf{q}) = a_0(\mathbf{q}) s^m + \ldots + a_{m-1}(\mathbf{q}) s + a_m(\mathbf{q}) , \tag{6}$$

where the coefficients depend polynomially on the parameters q_i, $i = 1, \ldots, l$, $\mathbf{q} = (q_1, \ldots, q_l)$, i.e., for $k = 0, \ldots, m$,

$$a_k(\mathbf{q}) = \sum_{i_1,\ldots,i_l=0}^{d} a^{(k)}_{i_1\ldots i_l} q_1^{i_1} \cdot \ldots \cdot q_l^{i_l} . \tag{7}$$

The uncertain parameters q_i are known to belong to given intervals

$$q_i \in [\underline{q}_i, \overline{q}_i] \ , i = 1, \ldots, l .$$

We set $\mathbf{Q} = [\underline{q}_1, \overline{q}_1] \times \ldots \times [\underline{q}_l, \overline{q}_l]$. In the following we can assume without loss of generality that the parameter set $\mathbf{Q}$ is the unit box $\mathbf{U}$. We call a polynomial p with real or complex coefficients

$\mathcal{D}$*-stable*, where $\mathcal{D}$ is a set in the complex plane, if all the zeros of p are inside $\mathcal{D}$.

We consider the following *robust* $\mathcal{D}$*-stability problem:* Is the family of polynomials $p(\mathbf{q})$ (robustly) $\mathcal{D}$-stable for $\mathbf{Q}$, i.e., are the polynomials $p(\mathbf{q})$ $\mathcal{D}$-stable for all $\mathbf{q} \in \mathbf{Q}$?

In the sequel we consider Hurwitz stability, i.e., $\mathcal{D}$ is the open left half of the complex plane, Schur stability, i.e., $\mathcal{D}$ is the open unit disc, and damping, i.e., $\mathcal{D}$ is a sector centered around the negative real axis with its vertex at the origin.

3.1. Checking Robust Hurwitz Stability

Assume first that the coefficients $a^{(k)}_{i_1,\ldots i_l}$ in (7) are real and that $a_0(\mathbf{q}) > 0$ for all $\mathbf{q} \in \mathbf{Q}$. To check whether the polynomial family (6) is robustly Hurwitz stable, i.e., whether

$$p(s, \mathbf{q}) \neq 0 \text{ for all } s \in \mathbf{C} \text{ with } Re(s) \geq 0 \text{ and for all } \mathbf{q} \in \mathbf{Q}\,,$$

we can test the Hurwitz determinant associated with this family for positivity over $\mathbf{Q}$. This was done using Bernstein expansion in [20] and [23]. However, this approach is restricted to problems with a moderate number of parameters and to low degree polynomials. Another approach [21], [24] is based on the inspection of the value set of the polynomial family over the imaginary axis. The test that this set does not contain the origin (and therefore that zero crossing is not possible) can again be performed by using Bernstein expansion. The resulting algorithm is capable of solving larger robust stability problems. However, smaller problems should still be solved by the more efficient approach based on testing the Hurwitz determinant.

3.1.1. *Testing the Hurwitz determinant for positivity*

With the polynomial family (6) an m-by-m matrix is associated, the so-called *Hurwitz matrix* $H(p) = (h_{i,k}(p))$ defined by

$$h_{i,k}(p) = a_{2k-i}(\mathbf{q}) \ ,\ i, k = 1, \ldots, m,$$

where by convention

$$a_n(\mathbf{q}) = 0 \text{ if } n < 0 \text{ or } n > m\,. \tag{8}$$

Its determinant, the *Hurwitz determinant,* is denoted by $\tilde{p}$, i.e., $\tilde{p}(\mathbf{q}) = \det H(p(\mathbf{q}))$. The Boundary Crossing Theorem [25], cf. [26, Sect. 4.3], states that, provided that there exists at least one $\mathbf{q} \in \mathbf{Q}$ associated with a stable polynomial, the polynomial family (6) is robustly stable if and only if $\tilde{p}(\mathbf{q})$ is positive over $\mathbf{Q}$. Since $\tilde{p}(\mathbf{q})$ is an l-variate polynomial in $\mathbf{q} = (q_1, \ldots, q_l)$, Bernstein expansion described in Section 2 can be applied. If the minimum of the Bernstein coefficients $b_I(\mathbf{Q}), I \in S$, is positive, it follows by (5) that $p(\mathbf{q})$ is stable for $\mathbf{Q}$. If there exists a nonpositive sharp Bernstein coefficient $b_{I_0}(\mathbf{Q}), I_0 \in S_0$, the family of polynomials (6) is not stable for all $\mathbf{q} \in \mathbf{Q}$ since the Hurwitz determinant associated with the family assumes by (3) nonpositive values on $\mathbf{Q}$. If neither of these two cases apply, then the sweep procedure is performed by splitting $\mathbf{Q}$ to obtain two new patches on which we proceed as before.

3.1.2. *Inspection of the value set*

Assume again that there is a stable member of the family of polynomials (6) for some $\mathbf{q} \in \mathbf{Q}$ and that $a_0(\mathbf{q}) > 0$ for all $\mathbf{q} \in \mathbf{Q}$. Furthermore, suppose that $a_m(\mathbf{q}) > 0$ for all $\mathbf{q} \in \mathbf{Q}$, otherwise the polynomial family would not be robustly stable for $\mathbf{Q}$. To explore *the value set*

$$\{p(j\omega, \mathbf{q}) : \omega \in [0, \infty), \mathbf{q} \in \mathbf{Q}\}$$

of the polynomial family, split the polynomial $p(j\omega, \mathbf{q})$ into its even and odd parts

$$p(j\omega, \mathbf{q}) = p_e(\omega^2, \mathbf{q}) + j\omega p_o(\omega^2, \mathbf{q}),$$

where

$$
\begin{aligned}
p_e(\omega^2,\mathbf{q}) &= \sum_{k=0}^{\mu}(-1)^k a_{m-2k}(\mathbf{q})\omega^{2k},\\
p_o(\omega^2,\mathbf{q}) &= \sum_{k=0}^{\mu}(-1)^k a_{m-2k-1}(\mathbf{q})\omega^{2k},
\end{aligned}
$$

with $\mu=[m/2]$ for the even part and $\mu=[(m-1)/2]$ for the odd part, [ ] denoting the integer part. Then substitute $\sigma=\omega^2$ and consider the pair $(p_e(\sigma,\mathbf{q}),p_o(\sigma,\mathbf{q}))$. Under the above assumptions, the family (6) is robustly stable for $\mathbf{Q}$ if and only if the polynomials $p_e(\sigma,\mathbf{q})$ and $p_o(\sigma,\mathbf{q})$ do not have a positive real zero in common for any $\mathbf{q}\in\mathbf{Q}$. In [24] it is shown how the search for common zeros can be restricted to a compact interval Γ. This is accomplished by extending bounds known from the literature for the positive zeros of a polynomial with constant coefficients to the case of a polynomial family (6). For simplicity, we write $\mathbf{x}=(x_1,\ldots,x_l)=(q_1,\ldots,q_{l-1},\sigma)$. After having transformed $\mathbf{Q}\times\Gamma$ to $\mathbf{U}$ we check whether the set $\mathcal{P}(\mathbf{U})=\{(p_e(\mathbf{x}),p_o(\mathbf{x})):\mathbf{x}\in\mathbf{U}\}$ contains the origin. By expanding $p_e(\mathbf{x})$ and $p_o(\mathbf{x})$ simultaneously into their Bernstein forms, we obtain a set of points $(\,b_I^{(e)}(\mathbf{U}),b_I^{(o)}(\mathbf{U})\,)$ in the plane, denoted by $b_I(\mathbf{U})$. Then we compute its convex hull, which can be done in optimal time using $O(\nu\log\nu)$ operations, see, e.g., [27], where ν denotes the number of points. Then we check whether the origin of the plane is contained in the convex hull, since $\mathrm{Conv}\,\mathcal{P}(\mathbf{U})\subseteq\mathrm{Conv}\,B(\mathbf{U})$ holds true. If the origin is outside, the family of polynomials is robustly stable. Otherwise an inclusion test given in [21] is performed. If it fails, i.e., it cannot be verified that the origin is in the value set, the sweep procedure is applied by splitting the domain to obtain two new patches on which we proceed as before. Here a patch selection rule other than the one described in Subsection 2.3 is used. After a sweep the choice between two resulting patches $B(\mathbf{D}_0)$ and $B(\mathbf{D}_1)$, say, remains. Let $p_i^{(w)}$, $i=1,\ldots,\mu_w$, generate $\mathrm{Conv}\,B(\mathbf{D}_w)$, $w=0,1$, cf. p.131 in [9]. Then that patch is chosen for which the Euclidean distance of the center of gravity of the equally weighted points $p_i^{(w)}$ to the origin, i.e., $(|p_1^{(w)}|+\ldots+|p_{\mu_w}^{(w)}|)/\mu_w$ is minimal. If no patch remains and all inclusion tests have failed then the family of polynomials is robustly stable. Otherwise, if an inclusion test is successful the algorithm terminates immediately because an unstable polynomial has been found.

Example 1: Consider the control of the Fiat Dedra engine [9, Ch. 3 and Sect. 9.5]. The characteristic polynomial is of seventh degree with seven parameters entering quadratically into the transfer function. The parameters vary inside the intervals

$$
\begin{array}{llll}
q_1\in[2.1608,3.4329], & q_2\in[0.1027,0.1627], & q_3\in[0.0357,0.1139], & q_4\in[0.2539,0.5607],\\
q_5\in[0.0100,0.0208], & q_6\in[2.0247,4.4962], & q_7\in[1.0000,10.000]. &
\end{array}
$$

For the coefficient functions see [9, Appendix A], where they fill two pages. The analysis can be restricted to the interval

$$\Gamma=[0.0000,5.4803]$$

for the variable σ. We recall from [21] the results of the application of the above outlined procedure: after 8.3s (on a Hewlett–Packard Workstation 9000/755) and 58 sweeps the algorithm reports that the family of polynomials is robustly stable. An advantage of the approach based on Bernstein expansion is that value sets can be visualised. Fig. 2 shows the Bernstein coefficients of the characteristic polynomial associated with the Fiat Dedra engine which approximates the subset of the value set for a small positive subinterval of Γ. The origin is in the small black hole on the right-hand side.

The case of complex coefficient functions (7)

$$
\begin{aligned}
a_k(\mathbf{q}) &= b_k(\mathbf{q})+jc_k(\mathbf{q})\\
&= \sum_{i_1,\ldots,i_l=0}^{d}(b_{i_1\ldots i_l}^{(k)}+j\;c_{i_1\ldots i_l}^{(k)})q_1^{i_1}\cdots q_l^{i_l},
\end{aligned}
$$

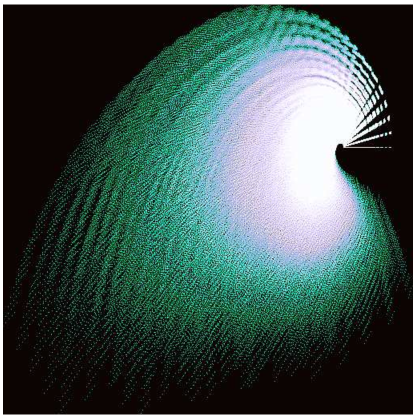

Figure 2. Approximation of a subset of the value set associated with the Fiat Dedra engine

$k = 0, \ldots, m$, can be treated similarly. As in the real case, the polynomial $p(j\omega, \mathbf{q})$ is split into its real and imaginary parts, using

$$p(j\omega, \mathbf{q}) = p_e(\omega, \mathbf{q}) + j p_o(\omega, \mathbf{q}),$$

where

$$\begin{aligned} p_e(\omega, \mathbf{q}) &= b_0(\mathbf{q}) - c_1(\mathbf{q})\omega - b_2(\mathbf{q})\omega^2 + c_3(\mathbf{q})\omega^3 + b_4(\mathbf{q})\omega^4 - c_5(\mathbf{q})\omega^5 - + \ldots \\ p_o(\omega, \mathbf{q}) &= c_0(\mathbf{q}) + b_1(\mathbf{q})\omega - c_2(\mathbf{q})\omega^2 - b_3(\mathbf{q})\omega^3 + c_4(\mathbf{q})\omega^4 + b_5(\mathbf{q})\omega^5 - + \ldots \quad . \end{aligned}$$

Under the assumption that there is a stable member of the polynomial family for some $\mathbf{q} \in \mathbf{Q}$, p is robustly stable for $\mathbf{Q}$ if and only if p_e and p_o do not have a real zero in common. As in the real case, we determine an interval Γ^+ enclosing all possible common positive zeros of p_e and p_o. By passing to $p_e(-\omega)$ and $p_o(-\omega)$ we similarly obtain an interval Γ^- enclosing all possible common negative zeros of p_e and p_o. Then $\Gamma := \Gamma^+ \cup \Gamma^-$ contains all possible common real zeros of p_e and p_o, and we proceed as in the case of real coefficient functions.

Assume now that we have to check the polynomial family p for robust $\mathcal{D}$-stability, where $\mathcal{D}$ is a sector with its vertex at the origin centered around the negative real axis with aperture $\pi - 2\delta, 0 < \delta < \pi/2$. This problem can be reduced to the case of complex coefficient functions, cf. Sect. 6.8. in [9]. If the coefficient functions of p are all real then (again under the assumption that there is a $\mathcal{D}$-stable member) it suffices to check the value set over the imaginary axis of the transformed polynomial family $p(se^{-j\delta}, \mathbf{q}), \mathbf{q} \in \mathbf{Q}$, for zero-exclusion. If the coefficient functions are complex, the value set of the family $p(se^{j\delta}, \mathbf{q}), \mathbf{q} \in \mathbf{Q}$ has to be tested as well.

3.2. Checking Robust Schur Stability

Assume again that the coefficients in (7) are real, that $a_0(\mathbf{q}) > 0$ for all $\mathbf{q} \in \mathbf{Q}$, and that there is a stable member of the polynomial family (6) for some $\mathbf{q} \in \mathbf{Q}$. To check this family for robust Schur stability, i.e., to check that

$$p(s, \mathbf{q}) \neq 0 \text{ for all } s \in \mathbf{C} \quad \text{with } |s| \geq 1 \text{ and for all } \mathbf{q} \in \mathbf{Q},$$

we may use, as in the Hurwitz case, a determinantal criterion as well as inspection of the value set.

3.2.1. *Determinantal criterion*

Using results in [28] we have to verify the following three conditions (i), (ii), (iii) to show robust stability of the polynomial familiy (6)

$$\begin{array}{lll} \text{(i)} & p(1, \mathbf{q}) > 0 & \text{for all } \mathbf{q} \in \mathbf{Q}\,, \\ \text{(ii)} & (-1)^m p(-1, \mathbf{q}) > 0 & \text{for all } \mathbf{q} \in \mathbf{Q}\,, \\ \text{(iii)} & \det J(p(\mathbf{q})) > 0 & \text{for all } \mathbf{q} \in \mathbf{Q}\,. \end{array}$$

Here the $(m-1) \times (m-1)$ matrix $J(p(\mathbf{q}))$ is defined by $J(p(\mathbf{q})) := X(p(\mathbf{q})) - Y(p(\mathbf{q}))$, where $X = (x_{i,k}(\mathbf{q}))$ and $Y = (y_{i,k}(\mathbf{q}))$ are triangular Toeplitz and Hankel matrices, respectively, given by

$$\begin{aligned} x_{i,k}(\mathbf{q}) &= a_{k-i}(\mathbf{q}), \\ y_{i,k}(\mathbf{q}) &= a_{2m-i-k}(\mathbf{q}), \end{aligned}$$

with $i, k = 0, \ldots, m-1$ and the convention (8). Since $p(\pm 1, \mathbf{q})$ and $\det J(p(\mathbf{q}))$ are l-variate polynomials in $\mathbf{q} = (q_1, \ldots q_l)$, the three conditions above can be checked efficiently by Bernstein expansion. For the computation of $\det J(p(\mathbf{q}))$ see [29].

3.2.2. *Inspection of the value set*

Again, the determinantal approach is restricted to problems with a moderate number of parameters and to low degree polynomials. For larger robust stability problems we propose the following approach [30] which relies on the inspection of the value set

$$\mathcal{V} = \{p(e^{j\omega}, \mathbf{q}) : \omega \in [0, \pi], \mathbf{q} \in \mathbf{Q}\}.$$

We consider the family (6) for a fixed $\mathbf{q} \in \mathbf{Q}$ and omit for simplicity the explicit reference to $\mathbf{q}$. We decompose p into its symmetric and antisymmetric parts h and g, respectively, i.e.,

$$p(s) = h(s) + g(s),$$

defined by

$$\begin{aligned} h(s) &= \frac{1}{2}(p(s) + s^m p(1/s)), \\ g(s) &= \frac{1}{2}(p(s) - s^m p(1/s)). \end{aligned}$$

With $n = [\frac{m+1}{2}]$, we have

$$\begin{aligned} h(s) &= \frac{1}{2}\sum_{k=0}^{n-1} \alpha_k (s^k + s^{m-k}) + \frac{\alpha_n}{2} s^n, \\ g(s) &= \frac{1}{2}\sum_{k=0}^{n-1} \beta_k (s^k - s^{m-k}), \end{aligned}$$

where

$$\begin{aligned} \alpha_k &= a_k + a_{m-k}, k = 0, \ldots, n, \\ \beta_k &= a_k - a_{m-k}, k = 0, \ldots, n-1, \end{aligned}$$

and $\alpha_n = 0$ if n is odd. Then s_0 with $|s_0| = 1$ is a zero of p if and only if it is a common zero of h and g.

To check whether $0 \notin \mathcal{V}$, we consider the polynomials h and g for $s = e^{j\omega}, \omega \in [0, \pi]$. It follows that

$$\begin{aligned} e^{-j\frac{m}{2}\omega} h(e^{j\omega}) &= \sum_{k=0}^{n-1} \alpha_k \cos\left[\omega(\frac{m}{2} - k)\right] + \frac{1}{2}\alpha_n =: h^*(\omega), \\ je^{-j\frac{m}{2}\omega} g(e^{j\omega}) &= \sum_{k=0}^{n-1} \beta_k \sin\left[\omega(\frac{m}{2} - k)\right] =: g^*(\omega). \end{aligned}$$

Thus we obtain the representation

$$p(e^{j\omega}) = e^{j\frac{m}{2}\omega}(h^*(\omega) - jg^*(\omega)).$$

To apply the Bernstein expansion we transform the trigonometric polynomials h^* and g^* into algebraic polynomials $\tilde{h}$ and $\tilde{g}$, respectively, in the variable t, say, by projecting the upper half of the unit circle onto the interval $[-1, 1]$ of the real line [31]. This is accomplished by setting $t = \cos\omega$ and using Chebyshev polynomials; for details, see [30].

In the resulting algorithm [30] the cases $\omega = 0$ and $\omega = \pi$, i.e., $t = \pm 1$, have to be treated separately. They can be reduced to the problem of checking a given polynomial $f(\mathbf{q})$ for a zero in $\mathbf{Q}$. This can be accomplished by expanding f into Bernstein polynomials over $\mathbf{Q}$. Let b_I be the associated Bernstein coefficients. If $0 \notin [\min_{I\in S} b_I, \max_{I\in S} b_I]$, then $f(\mathbf{q})$ has no zero in $\mathbf{Q}$ by (5). Else, if $0 \in [\min_{I\in S_0} b_I, \max_{I\in S_0} b_I]$, then $f(\mathbf{q})$ has a zero $\mathbf{q}$ in $\mathbf{Q}$. In the remaining cases we split $\mathbf{Q}$ to obtain two new patches on which we proceed as before. It remains to check the open interval $(-1, 1)$ for common zeros of $\tilde{h}$ and $\tilde{g}$. This interval may be tightened using the procedure described in [24]. The test is similiar to the one described in Subsection 3.1.2.

For details and an application to a stability problem arising in the investigation of asymptotic stability properties of numerical methods for delay differential equations [32], see [30].

Example 1 (continued): Since we did not find large robust Schur stability problems in the literature we modified Example 1 somewhat artificially. By applying the bilinear mapping, we transformed this polynomial family into one having all its zeros in the open unit disc. After 7.1s (on a Hewlett–Packard Workstation 9000/755) and 54 sweeps, our algorithm reports that this family is robustly stable.

3.3. Computation of Stability Radii

A problem closely connected with that of robust $\mathcal{D}$-stability is that of finding the smallest destabilizing perturbation of a $\mathcal{D}$-stable system:

Let the polynomial $p(\mathbf{q}^0)$ be $\mathcal{D}$-stable. We want to find the largest ρ, denoted by ρ^* and called the *stability radius*, such that the polynomials $p(\mathbf{q})$ are $\mathcal{D}$-stable for all $\mathbf{q}$ with $\|\mathbf{q}-\mathbf{q}^0\|_\infty^w < \rho$, where $\|.\|_\infty^w$ denotes the weighted infinity norm, i.e., $\|\mathbf{q}-\mathbf{q}^0\|_\infty^w = \max_i w_i^{-1}|q_i - q_i^0|,\ w_i > 0$.

The stability radius ρ^* can be computed by a bisection search over ρ involving a stability check at each step; for examples, see [33].

3.4. Solving Strict Polynomial Inequalities

The solutions of many interesting problems in control system design and analysis can be recast as those of systems of inequalities involving multivariate polynomials in real variables, cf. [34], which correspond to the following problem:

Let $p_1, \ldots, p_n$ be l-variate polynomials and let a box $\mathbf{Q}$ in $\mathbf{R}^l$ be given. We want to find

$$\mathbf{\Sigma} := \{\mathbf{x} \in \mathbf{Q} : p_i(\mathbf{x}) > 0, i = 1, \ldots, n\}\,; \tag{9}$$

the set $\mathbf{\Sigma}$ is called the *solution set* of the system of polynomial inequalities.

A typical example is the problem of determining the $\mathcal{D}$-stability region of a family of polynomials (6) in a given parameter box $\mathbf{Q}$, i.e., the set

$$\{\mathbf{q} \in \mathbf{Q} : p(s, \mathbf{q}) \neq 0 \ \ \forall s \notin \mathcal{D}\}\,.$$

According to the stability conditions for Hurwitz and Schur stability recalled in Subsections 3.1.1 and 3.2.1, this problem can be formulated as that of solving one and three, respectively, polynomial inequalities. A number of other control problems, such as static output feedback stabilization and simultaneous stabilization, can also be reduced to the solution of systems of polynomial inequalities; for details, see [35]. In practice, stability is but one of many requirements of the closed-loop control systems. Given a nominal plant parameter vector, a nominal controller is usually designed to guarantee closed-loop stability and to meet other specification constraints such as disturbance rejection, time response overshoot, settling time, reference input tracking etc. Often these performance specifications can also be formulated in the frequency domain as polynomial inequalities. If the parameter vector $\mathbf{q}$ is known only to belong to a box $\mathbf{Q}$, a natural question is to ask for which $\mathbf{q} \in \mathbf{Q}$ the nominal controller meets all design specifications. A worked example can be found in [36].

Only in the simplest cases are we able to describe the solution set $\mathbf{\Sigma}$ defined in (9) exactly. In general, we instead seek a good approximation to it. We obtain an inner approximation of $\mathbf{\Sigma}$, denoted by $\mathbf{\Sigma}_i$, by the union of subboxes of $\mathbf{Q}$ on which all polynomials p_i are positive. Similarly, an inner approximation of the exterior of $\mathbf{\Sigma}$ in $\mathbf{Q}$, denoted by $\mathbf{\Sigma}_\mathrm{e}$, is given by the union of subboxes of $\mathbf{Q}$ with the property that on each there is a nonpositive polynomial p_{i^*}. The boundary $\partial\mathbf{\Sigma}$ of $\mathbf{\Sigma}$ is approximated by the union of subboxes of $\mathbf{Q}$ on which all polynomials p_i attain positive values, but on which at least one also attains nonpositive values, cf. Fig. 3. For a fixed positive number ε, let $\mathbf{\Sigma}_b(\varepsilon)$ be the list of subboxes with volume less than ε, the

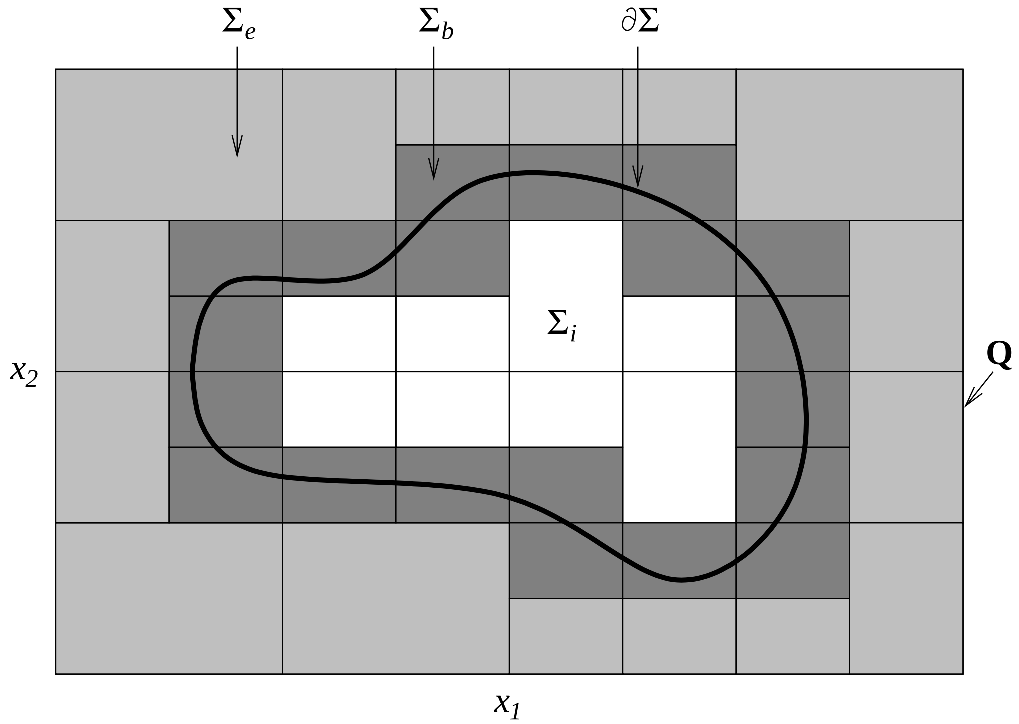

Figure 3. The different approximations of $\mathbf{\Sigma}$ and $\partial\mathbf{\Sigma}$

union of which approximates the boundary $\partial\mathbf{\Sigma}$. In our algorithm we check the (non)positivity of a polynomial by the sign of its Bernstein coefficients using Lemma 1 (iii). Then the set $\mathbf{\Sigma}_i$ consists of all subboxes generated by sweeps on which the Bernstein coefficients of all polynomials $p_i, i = 1, 2, \ldots, n$, are positive; $\mathbf{\Sigma}_e$ comprises all subboxes on which all Bernstein coefficients of a polynomial p_{i^*} are nonpositive; the set $\mathbf{\Sigma}_b(\epsilon)$ contains all subboxes of volume less than ϵ on which each polynomial $p_i, i = 1, 2, \ldots, n$, either possesses positive and nonpositive Bernstein coefficients with indices taken from the set S_0 (under the assumption that such a polynomial exists) or has only positive Bernstein coefficients. For details of the algorithm see [37]. Applications to the computation of $\mathcal{D}$-stability regions can be found in [38].

Alternatively, the solution set $\mathbf{\Sigma}$ can be determined by quantifier elimination, cf. [35] and [39] and the references therein. It seems, however, that Bernstein expansion can handle more complex problems and that it requires less computing time when both methods are applicable [37].

Example 2: The problem taken from [35] is to find a (stable) compensator which simultaneously stabilizes three different plants with the following transfer functions

$$\frac{2-s}{(s^2-1)(s+2)}, \quad \frac{2-s}{s^2(s+2)}, \quad \frac{2-s}{(s^2+1)(s+2)}.$$

In order to reduce the number of parameters to be considered we assume a second-order compensator of the form

$$\frac{A(s+B)^2}{(s+D)^2},$$

with $D > 0$. To achieve stability, we utilize the Liénard-Chipart criterion, see, e.g., [40]. After some simplifications, we obtain the following set of inequalities in the controller parameters A, B, and D :

$$\begin{aligned}
A, B, D &> 0,\\
AB^2 - D^2 &> 0,\\
-AB + A + D^2 - D - 1 &> 0,\\
AB - AD - 2A + D^3 + 4D^2 + 4D &> 0,\\
AB^3 - AB^2D - 4AB^2 + 2ABD + 4AB + 2BD^3 + 5BD^2 + 2BD - D^3 - 4D^2 - 4D &> 0,\\
AB - 2A - BD^2 - 4BD - 4B + 2D^2 + 3D - 2 &> 0.
\end{aligned}$$

We have choosen $A \in [100, 120], B \in [0, 2], D \in [10, 20]$. The software package QEPCAD for *quantifier elimination by partial cylindrical algebraic decomposition* by Hoon Hong from the Research Institute for Symbolic Computation in Linz, Austria, needs already 2 hours of CPU time (on a Sun Workstation) to solve the existence problem, i.e., to show that there is a solution of the above system of inequalities [35]. In only 1.3s (on a PC equipped with a Pentium 133) the algorithm outlined above produces $\mathbf{\Sigma}_i \cup \mathbf{\Sigma}_b$, where the recursion depth for the sweeps is restricted to 15, i.e., the smallest boxes have volume $2^{-15} \times 400 = 0.0122\ldots$. The smallest parameter box in $\mathbf{Q}$ which contains this enclosure for $\mathbf{\Sigma}$ is given by

$$[100, 120] \times [1.15625, 1.60938] \times [12.1875, 17.3438].$$

Fig. 4, taken from [37], shows the set of acceptable values of B and D (white region) for fixed $A = 110$ obtained in 0.5s.

Compared to methods based on quantifier elimination, Bernstein expansion is not so widely applicable:

- Assume that some of the inequalities in (9) are weak, i.e., $p_i(\mathbf{x}) \geq 0$ and let $p_i(\mathbf{x}^*) = 0$ for some $\mathbf{x}^* \in \mathbf{Q}$. If all lower bounds provided by Bernstein expansion are negative and all upper bounds are nonnegative, we could end up with a box which encloses the solution set but we are not in the position to decide whether this box contains a solution at all. So only strict inequalities can be handled efficiently. However, many problems in linear control theory can be reduced to such systems.

- Bernstein expansion requires pre-determined bounds on the parameter range. However, the designer often has a region of special interest.

- Quantifier elimination provides an explicit description of the solution set which is complicated in general. From the point of view of the designer, the description of the entire solution set is often not necessary. What the designer really wants is a good inner approximation of the solution set or even only a large box inside this set, which is precisely what Bernstein expansion provides.

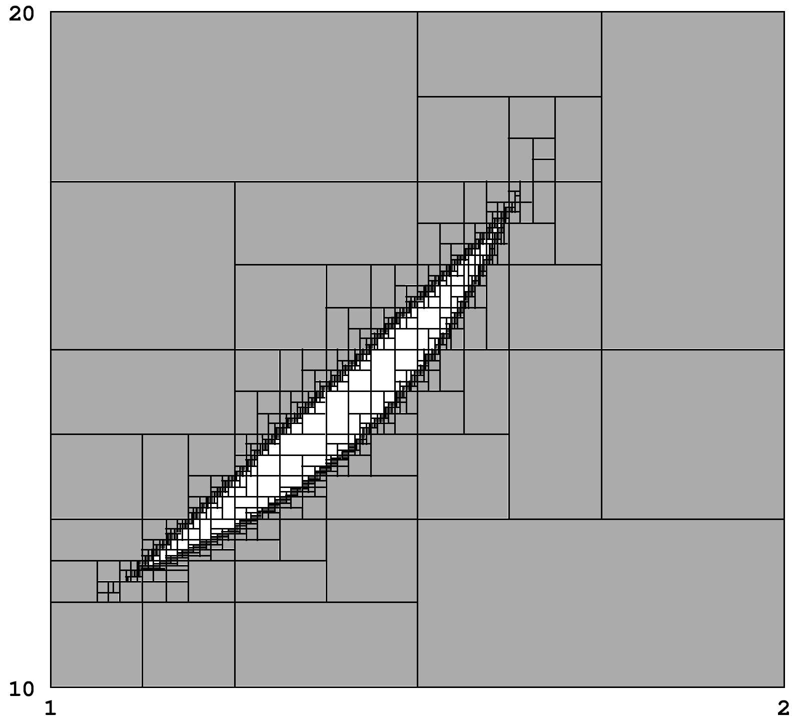

Figure 4. Set of acceptable values of B, D for $A = 110$ in Example 2

4. Conclusions

We have shown that many problems of control can be solved by means of Bernstein expansion. In a forthcoming paper we are applying Bernstein expansion to the solution of systems of algebraic equations. Further research should address the reduction of the storage needed since the application of the algorithms to very large problems is complicated by their large memory requirements. Future research should also be directed to the investigation of the convergence properties of the algorithms based on Bernstein expansion.

Acknowledgement

Support from the Ministry of Science and Research Baden-Württemberg and from the Ministry of Education, Science, Research, and Technology of the Federal Republic of Germany under contract no. 1706998 is gratefully acknowledged. The author would like to thank Professor Éric Walter and Andrew Smith for their careful reading of the manuscript and critical comments.

References

1. R. Hungerbühler and J. Garloff, "Bounds for the range of a bivariate polynomial over a triangle," *Reliable Computing*, vol. 4, 1998, pp. 3-13.
2. J. Rokne, "Bounds for an interval polynomial," *Computing*, vol. 18, 1977, pp. 225-240.
3. J. Rokne, "A Note on the Bernstein algorithm for bounds for interval polynomials," *Computing*, vol. 21, 1979, pp. 159-170.
4. J. Rokne, "The range of values of a complex polynomial over a complex interval," *Computing*, vol. 22, 1979, pp. 153-169.
5. J. Rokne, "Optimal computation of the Bernstein algorithm for the bound of an interval polynomial," *Computing*, vol. 28, 1982, pp. 239-246.
6. H. C. Fischer, "Range computations and applications," In: *Contributions to Computer Arithmetic and Self-Validating Numerical Methods*, C. Ullrich, Ed., Amsterdam: J. C. Baltzer, 1990, pp. 197-211.
7. H. Hong and V. Stahl, "Bernstein form is inclusion monotone," *Computing*, vol. 55, 1995, pp. 43-53.

8. L. Saydy, A. L. Tits, and E. H. Abed, “Guardian maps and the generalized stability of parameterized families of matrices and polynomials,” *Math. Contr., Signals, and Syst.,* vol. 3, 1990, pp. 345-371.
9. B.R. Barmish, *New Tools for Robustness of Linear Systems*, New York: Macmillan, 1994.
10. H. Ratschek and J. Rokne, *Computer Methods for the Range of Functions*, Chichester: Ellis Horwood, 1984.
11. A. Neumaier, *Interval Methods for Systems of Equations*, Cambridge: Cambridge University Press, 1990.
12. E. Walter and L. Jaulin, “Guaranteed characterization of stability domains via set inversion,” *IEEE Trans. Automat. Contr.*, vol. 39, 1994, pp. 886–889.
13. L. Jaulin and E. Walter, “Set inversion via interval analysis for nonlinear bounded-error estimation,” *Automatica*, vol. 29, 1993, pp. 1053–1064.
14. O. Didrit, L. Jaulin, and E. Walter, “Guaranteed analysis and optimization of parametric systems with application to their stability degree,” *European J. Contr.*, vol. 3, 1997, pp. 68–80.
15. S. Malan, M. Milanese, and M. Taragna, “Robust analysis and design of control systems using interval arithmetic,” *Automatica*, vol. 33, 1997, pp. 1363–1372.
16. E. Baumann, “Optimal centred forms,” *BIT*, vol. 28, 1988, pp. 80–87.
17. Y. Ohta, “Nonconvex polygon interval arithmetic as a tool for the analysis and design of robust control systems,” this issue.
18. J. Garloff, “Convergent bounds for the range of multivariate polynomials,” In: *Interval Mathematics 1985,* K. Nickel, Ed. , Lecture Notes in Computer Science, vol. 212, Berlin: Springer, 1986, pp. 37–56.
19. T.J. Rivlin, “Bounds on a polynomial,” *J. Res. Nat. Bur. Standards Sect. B,* vol. 74B, 1970, pp. 47–54.
20. J. Garloff, “The Bernstein algorithm,” *Interval Computations*, vol. 2, 1993, pp. 154-168.
21. M. Zettler and J. Garloff, “Robustness analysis of polynomials with polynomial parameter dependency using Bernstein expansion,” *IEEE Trans. Automat. Contr.*, vol. 43, 1998, pp. 425-431.
22. R.T. Farouki, V.T. Rajan, “Algorithms for polynomials in Bernstein form,” *Computer Aided Geometric Design*, vol. 5, 1988, pp. 1-26.
23. S. Malan, M. Milanese, M. Taragna, and J. Garloff, “B^3 algorithm for robust performances analysis in presence of mixed parametric and dynamic perturbations,” In *Proc. 31st Conf. Decision Contr.*, Tucson, AZ, 1992, pp. 128-133.
24. J. Garloff, B. Graf, and M. Zettler, “Speeding up an algorithm for checking robust stability of polynomials,” In: *Robust Control Design*, Cs. Bányász, Ed., Oxford: Elsevier Sci., 1998, pp. 183-188.
25. R. A. Frazer and W. J. Duncan, “On the criteria for the stability of small motions,” *Proc. Royal Soc. A*, vol. 124, 1929, pp. 642-654.
26. J. Ackermann, *Robust Control, Systems with Uncertain Physical Parameters*, London: Springer, 1993.
27. K. Mulmuley, *Computational Geometry, An Introduction through Randomized Algorithms*, Englewood-Cliffs: Prentice Hall,1994.
28. E.I. Jury and T. Pavlidis, “Stability and aperiodicity constraints for system design,” *IEEE Trans. Circuit Theory*, vol. 10, 1963, pp. 137-141.
29. E. J. Jury, *Inners and Stability of Dynamic Systems*, New York: Wiley, 1974; Krieger, FLA, 2nd edn., 1982.
30. J. Garloff and B. Graf, “Robust Schur stability of polynomials with polynomial parameter dependency,” *Multidimensional Systems and Signal Processing*, vol. 10, 1999, pp. 189-199.
31. N.K. Bose and J.F. Delansky, “Tests for robust Schur interval polynomials,” In: *30th Midwest Symposium on Circuits and Systems*, G. Glasford and K. Jabbour, Eds., Elsevier Sci. Publ., 1988, pp.1357-1361.
32. N. Guglielmi, “Delay dependent stability regions for Θ-methods for delay differential equations,” *IMA Journal of Numerical Analysis*, vol. 18, 1998, pp. 399-418.
33. C. Roguet and J. Garloff, “Computational Experiences with the Bernstein Algorithm,” Techn. Rep. no. 9403, Fachhochschule Konstanz, Fachbereich Informatik, 1994.
34. J. Vehi, J. Rodellar, M. Sainz, and J. Armengol, “Analysis of the robustness of predictive controllers via modal intervals,” this issue.
35. C. Abdallah, P. Dorato, R. Liska, S. Steinberg, and W. Yang, “Applications of quantifier elimination theory to control theory,” Technical Report EECE95-007, University of New Mexico, Department of Electrical and Computer Engineering, Albuquerque, 1995, cf. *Proc. of the 4th IEEE Mediteranean Symposium on Control and Automation,* Maleme, Crete, Greece, 1996, pp. 340-345.
36. G. Fiorio, S. Malan, M. Milanese, and M. Taragna, “Robust performance design of fixed structure controllers for systems with uncertain parameters,” In: *Proc. 32nd Conference on Decision and Control*, San Antonio, Texas, 1993, pp. 3029-3031.
37. J. Garloff and B. Graf, “Solving strict polynomial inequalities by Bernstein expansion,” In: *Symbolic Methods in Control System Analysis and Design*, N. Munro, Ed., London: IEE, 1999, pp. 339-352.
38. B. Graf, “Computation of stability regions,” Techn. Rep. no. 9801, Fachhochschule Konstanz, Fachbereich Informatik, 1998.
39. M. Jirstrand, *Constructive Methods for Inequality Constraints in Control*, Linköping Studies in Science and Technology, Dissertations, no. 527, Linköping University, Department of Electrical Engineering, Linköping, Sweden, 1998.
40. S. Barnett, *Polynomials and Linear Control Systems*, New York: Marcel Dekker, 1983, p. 155.